Carlson Collapse is minimal under MA

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Namba forcing [5] may be regarded as a generalization of Laver forcing [2] to ω_2. The analogous forcing for ω_1 we call the Carlson collapse. We first encountered it when writing our joint paper: Carlson, Kunen, and Miller [1]. In that paper we used the Prikry collapse of ω_1, which is analogous to superperfect tree forcing (e.g., Miller [3]) but with subtrees of $\omega_1^{<\omega}$. We proved in [1] the analogue of Theorem 3 for the Prikry collapse, namely that assuming Martin's axiom the generic extension is minimal. Our paper [1] did not include Theorem 4, although Carlson had already proved it with an easier proof than is given here. Lemma 2 was obtained while giving a topics course [4] on forcing.

One of my colleagues many years ago liked to joke about the referee report that said; "This paper fills a much needed gap in the literature."

Definition 1 *(1) A subtree $p \subseteq \omega_1^{<\omega}$ is Carlson iff there exists $s \in p$ called the root of p such that for all $t \in p$ either $s \subseteq t$ or $t \subseteq s$ and for every $t \in p$ with $s \subseteq t$ there are uncountably many $\alpha < \omega_1$ with $s\hat{}\langle\alpha\rangle \in p$.*

(2) Let $\mathbb{P}$ be the partial order of Carlson trees under inclusion.

(3) We write $p \leq_0 q$ iff $p \leq q$ and $\text{root}(p) = \text{root}(q)$.

(4) For $s \in p$ define $p_s = \{t \in p \ : \ s \subseteq t \text{ or } t \subseteq s\}$.

(5) $B(p) = \{t \in p \ : \ \text{root}(p) \subseteq t\}$ (nodes beyond the root).

Lemma 2 *Suppose MA_{ω_1} and we are given $(p_\alpha \in \mathbb{P} : \alpha < \omega_1)$ and τ a $\mathbb{P}$-name such that for each $\alpha < \omega_1$*

$$p_\alpha \Vdash \tau \in 2^\omega \setminus V.$$

Then there exists $(q_\alpha \leq_0 p_\alpha : \alpha < \omega_1)$ and $(C_\alpha : \alpha < \omega_1)$ pairwise disjoint closed subsets of 2^ω such that for every $\alpha < \omega_1$

$$q_\alpha \Vdash \tau \in C_\alpha.$$

Theorem 3 *If $M \models \text{MA}_{\omega_1}$ and G is $\mathbb{P}$-generic over M, then for every $x \in 2^\omega \cap M[G]$ either $x \in M$ or $G \in M[x]$.*

Proof
Given any p such that $p \Vdash \tau \in 2^\omega \setminus M$ construct $q \leq_0 p$ and closed sets $(C_s : s \in B(q))$ such that

1. $q_s \Vdash \tau \in C_s$ for each $s \in B(q)$,
2. $C_s \subseteq C_t$ if $t \subseteq s$, and
3. $C_{s\hat{}\langle\alpha\rangle} \cap C_{s\hat{}\langle\beta\rangle} = \emptyset$ if $\alpha \neq \beta$.

This is an easy fusion argument combined with Lemma 2. We claim that

$$q \Vdash G \in M[\tau^G].$$

This is because G is determined by the generic collapse map $g \in \omega_1^\omega$ defined by $g = \bigcap G$. Then

$$G = \{p \in \mathbb{P} \ : \ g \in [p]\} \text{ and } g = \bigcup \{s \ : \ \tau^G \in C_s\}.$$

QED

Theorem 4 *(Carlson 1979) Suppose $M \models \mathrm{MA}_{\omega_1}$ and G is $\mathbb{P}$-generic over M. Then for every $f \in \omega^\omega \cap M[G]$ there exists $g \in M \cap \omega^\omega$ such that $\forall n \ \ f(n) < g(n)$.*

Proof
Without loss we may suppose that

$$p \Vdash \overset{\circ}{f} \in \omega^\omega \setminus M.$$

Let $E \subseteq 2^\omega$ be the eventually zero reals. Let $\Phi : 2^\omega \setminus E \to \omega^\omega$ be the natural homeomorphism and let τ be a name for $\Phi^{-1}(f)$. Letting $p_\alpha = p$ for all $\alpha < \omega_1$ we obtain $(q_\alpha \leq_0 p : \alpha < \omega_1)$ and closed pairwise disjoint $(C_\alpha \subseteq 2^\omega : \alpha < \omega_1)$ such that $q_\alpha \Vdash \tau \in C_\alpha$ for all $\alpha < \omega_1$. Since the C_α are pairwise disjoint there must be α with $C_\alpha \cap E = \emptyset$. This implies that $\Phi(C_\alpha) = K_\alpha \subseteq \omega^\omega$ is a compact set and so we may find $g \in \omega^\omega$ such that g dominates every element of K_α. But then

$$q_\alpha \Vdash \forall n \quad \overset{\circ}{f}(n) < \check{g}(n).$$

QED

Proof of Lemma 2.

Claim 5 *Given any sentence θ and condition p there exists $q \leq_0 p$ such that*

$$q \Vdash \theta \text{ or } q \Vdash \neg\theta.$$

Proof
This is the Laver Lemma. It is also true for Namba forcing and many others.
QED

Claim 6 *Suppose $p_0 \Vdash \tau \in 2^\omega \setminus M$. Then there exists $q_1, q_2 \leq_0 p_0$ and pairwise disjoint clopen sets C_1, C_2 such that $q_1 \Vdash \tau \in C_1$ and $q_2 \Vdash \tau \in C_2$.*

Proof
For $p \leq p_0$ define p is good iff there are $q_1, q_2 \leq_0 p$ and pairwise disjoint clopen sets C_1, C_2 such that $q_i \Vdash \tau \in C_i$ for $i = 1, 2$. Note that if p is bad and $s = \text{root}(p)$ then for all but countably many $\alpha < \omega_1$ with $s\hat{}\langle\alpha\rangle \in p$ the condition $p_{s\hat{}\langle\alpha\rangle}$ is bad. This because there are only countably many pairs of disjoint clopen sets C_1, C_2.

By this observation if the Claim fails then we may construct $q \leq_0 p_0$ such that for every $t \in B(q) = \{t \in q : \text{root}(q) \subseteq t\}$ the condition q_t is bad. Using Claim 5, it follows that for every $s \in B(q)$ there exists a unique $x_s \in 2^\omega$ such that for every $n < \omega$ there exists $p \leq_0 q_s$ such that $p \Vdash \tau \restriction n = x_s \restriction n$. If there were two x_s with this property, we could easily get a contradiction to the badness of q_s.

For every $s \in B(q)$ it must be that $x_s = x_{s\hat{}\langle\alpha\rangle}$ for all but countably many $s\hat{}\langle\alpha\rangle \in q$. To see this suppose not and let $Q(s) = \{\alpha < \omega_1 \ : \ s\hat{}\langle\alpha\rangle \in q\}$. Then we would be able to find $t \in 2^{<\omega}$ with the property that uncountably many $\alpha \in Q(s)$ had $t \subseteq x_{s\hat{}\langle\alpha\rangle}$ but $t \neq t' = x_s \restriction |t|$. But this means we can find $p \leq_0 q_s$ such that $p \Vdash t \subseteq \tau$. We can also find $p' \leq_0 q_s$ such that $p' \Vdash t^p r \subseteq \tau$ by the definition of x_s. This contradicts the badness of q_s.

By the above arguments we can find $x \in 2^\omega$ and $q \leq_0 p_0$ such that $x_s = x$ for every $s \in B(q)$. This contradicts the assumption that $p_0 \Vdash \tau \neq \check{x}$.
QED

Claim 7 *Suppose $p_i \Vdash \tau \in 2^\omega \setminus M$ for $i = 1, 2$. Then there exists $q_1 \leq_0 p_1$, $q_2 \leq_0 p_2$ and pairwise disjoint clopen sets C_1, C_2 such that $q_i \Vdash \tau \in C_i$ for $i = 1, 2$.*

Proof
Apply Claim 6 to p_1 and obtain $q_{1,i} \leq_0 p_1$ and disjoint clopen C_1, C_2 such that $q_{1,i} \Vdash \tau \in C_i$ for $i = 1, 2$. We may as well assume C_1, C_2 are complementary. Apply the Laver Lemma to p_2 and get $q_2 \leq_0 p_2$ such that either $q_2 \Vdash \tau \in C_1$ or $q_2 \Vdash \tau \in C_2$. If $q_2 \Vdash \tau \in C_2$, take $q_1 = q_{1,1}$, otherwise take $q_1 = q_{1,2}$.
QED

Claim 8 *Suppose $n < \omega$ and $p_i \Vdash \tau \in 2^\omega \setminus M$ for $i < n$. Then there exists $(q_i \leq_0 p_i : i < n)$ and pairwise disjoint clopen sets $(C_i : i < n)$ such that $q_i \Vdash \tau \in C_i$ for $i < n$.*

Proof
Iteratively apply Claim 7 to all pairs $i < j < n$.
QED

Definition 9 *For T a finite subtree of $B(q)$ define*
(1) $p \leq_T q$ iff $p \leq_0 q$ and $T \subseteq p$.
(2) For each $t \in T$ define

$$q_{t,T} = \{s \in q \ : \ s \subseteq t \text{ or } (t \subset s \text{ and } s \restriction (|t| + 1) \notin T\}$$

Note that $\{q_{t,T} \ : \ t \in T\}$ is a finite maximal antichain beneath q. This is analogous to Laver's $q \leq_n p$ except there are uncountably many T.

Claim 10 *Suppose $p \Vdash \tau \in 2^\omega \setminus M$ and $p' \Vdash \tau \in 2^\omega \setminus M$ and $T \subseteq B(p)$ and $T' \subseteq B(p')$ are finite subtrees. Then there are $q \leq_T p$ and $q' \leq_{T'} p'$ and pairwise disjoint clopen sets C and C' such that $q \Vdash \tau \in C$ and $q' \Vdash \tau \in C'$.*

Proof
Let p_i for $i < m$ list all $p_{T,t}$ for $t \in T$ and let p_i for $m \leq i < n$ list all $p'_{T',t'}$ for $t' \in T'$. Apply Claim 8 to obtain q_i and C_i. Let $q = \bigcup\{q_i : i < m\}$ and $C = \bigcup\{C_i : i < m\}$. Similarly put $q' = \bigcup\{q_i : m \leq i < n\}$ and $C' = \bigcup\{C_i : m \leq i < n\}$.
QED

Claim 11 *Suppose $p \Vdash \tau \in 2^\omega$, then there exists $q \leq_0 p$ such and $\pi : q \to 2^{<\omega}$ such that for every $s \in q \quad |\pi(s)| = |s|$ and $q_s \Vdash \pi(s) \subseteq \tau$.*

Proof
Use the Laver lemma (Claim 5) and fusion to get the result by considering the sequence of sentences "$\tau(n) = 0$".
QED

Definition 12 *For q as in Claim 11 define the poset $\mathbb{Q}(q)$ as follows:*

$(T, C) \in \mathbb{Q}(q)$ iff T is a finite subtree of $B(q)$, C is clopen subset of 2^ω, and there exists $p \leq_T q$ such that $p \Vdash \tau \in C$.

Define $(T_1, C_1) \leq (T_2, C_2)$ iff $T_1 \supseteq T_2$ and $C_1 \subseteq C_2$.

From now on for the poset $\mathbb{Q}(q)$ the condition q will always have the property of Claim 11 and hence for any $p \leq q$ and clopen set C we have that $p \Vdash \tau \in C$ iff the range of the induced continuous map $\pi : [p] \to 2^\omega$ is a subset of C.

Claim 13 *$\mathbb{Q}(q)$ has the ccc.*

Proof
Since there are only countably many clopen sets it is enough to see that any pair with the same clopen set, (T_1, C) and (T_2, C) is compatible. We claim that $(T_1 \cup T_2, C) \in \mathbb{Q}(q)$. Note that for $p \leq q$ that $p \Vdash \tau \in C$ iff $[\pi(s)] \cap C \neq \emptyset$ for every $s \in p$. It follows that if $p_1 \leq_{T_1} q$ and $p_2 \leq_{T_2} q$ and each force $\tau \in C$, then $(p_1 \cup p_2) \leq_{T_1 \cup T_2} q$ and $p_1 \cup p_2 \Vdash \tau \in C$.
QED

Definition 14 *For G and $\mathbb{Q}(q)$-filter define*

$$q^G = \bigcup \{T \,:\, \exists C \; (T, C) \in G\}$$

and

$$C^G = \bigcap \{C \,:\, \exists T \; (T, C) \in G\}.$$

Claim 15 *We can find $\mathcal{D}$ a family of dense subsets of $\mathbb{Q}(q)$ with $|\mathcal{D}| = \omega_1$ such that for every G a $\mathbb{Q}(q)$ filter which meets each element of $\mathcal{D}$ we have that $q^G \leq_0 p$ and $q^G \Vdash \tau \in C^G$.*

Proof
Note that the trivial condition $(\{root(q)\}, 2^\omega)$ is always in G. For $s \in B(q)$ and $\alpha < \omega_1$ define

$$D_{s,\alpha} = \{(T,C) \in \mathbb{Q}(q) \ : \ (T,C) \Vdash s \notin q^G \text{ or } \exists \beta > \alpha \ \ s\hat{\ }\langle\beta\rangle \in T\}$$

To see that it is dense note that any condition can be extended to a condition (T,C) such that either $(T,C) \Vdash s \notin q^G$ or $(T,C) \Vdash s \in q^G$. In the first case $(T,C) \in D_{s,\alpha}$ and we are done. In the second case we must be able to find $(T',C') \leq (T,C)$ with $s \in T'$. By the definition of $\mathbb{Q}(q)$ there exists $p \leq_{T'} q$ such that $p \Vdash \tau \in C'$. Choose any $\beta > \alpha$ with $s\hat{\ }\langle\beta\rangle \in p$. Then p witnesses that $(T' \cup \{s\hat{\ }\langle\beta\rangle\}, C')$ is in $\mathbb{Q}(q)$ and $D_{s,\alpha}$.

Meeting all the $D_{s,\alpha}$ guarantees that $q^G \in \mathbb{P}$ and $q^G \leq_0 q$.

To see that $q^G \Vdash \tau \in C^G$ it is enough to show that for every $(T,C) \in G$ that

$$q^G \Vdash \tau \in C.$$

Choose n large enough so that there exists $\Gamma \subseteq 2^n$ such that we may write $C = \bigcup\{[s] \ : \ s \in \Gamma\}$. We claim that (T,C) forces that for every $s \in q^G \cap \omega_1^n$ that $\pi(s) \in \Gamma$. If this were not the case, then for some $(T',C') \leq (T,C)$ and $s \in T' \cap \omega_1^n$ we would have that $\pi(s) \notin \Gamma$. But this means that $q_s \Vdash \tau \in [\pi(s)]$ and therefor $q_s \Vdash \tau \notin C$ contradicting the definition of $\mathbb{Q}(q)$ that $C' \subseteq C$ and there exists $p \leq_{T'} q$ (so $p_s \leq_0 q_s$) and $p \Vdash \tau \in C'$.
QED

Finally we prove Lemma 2. We assume that the $q_\alpha \leq_0 p_\alpha$ have the property as in Claim 11. We let

$$\mathbb{Q} = \sum\{\mathbb{Q}(q_\alpha) \ : \ \alpha < \omega_1\}$$

be the direct sum which has the ccc by MA_{ω_1}. For any $\alpha < \beta < \omega_1$ let

$$D_{\alpha,\beta} = \{p \in \mathbb{Q} \ : \ p_\alpha = (T_\alpha, C_\alpha), \ \ p_\beta = (T_\beta, C_\beta), \text{ and } C_\alpha \cap C_\beta = \emptyset\}$$

By Claim 10 this set is dense. By Claim 15 we may find a family $\mathcal{D}$ of dense subsets of $\mathbb{Q}$ with $|\mathcal{D}| = \omega_1$ such that if G is a $\mathbb{Q}$-filter meeting each element of $\mathcal{D}$ then each $q_\alpha^G \leq_0 q_\alpha$ has the property that $q_\alpha^G \Vdash \tau \in C_\alpha^G$. If G meets all the $D_{\alpha,\beta}$ then the C_α^G will be pairwise disjoint. Applying MA_{ω_1} gives us the sequences $(q_\alpha^G \leq_0 q_\alpha \leq_0 p_\alpha : \alpha < \omega_1)$ and $(C_\alpha^G : \alpha < \omega_1)$ to prove Lemma 2.
QED

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