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ON BOX PRODUCTS

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We prove two theorems about box products. The first theorem says that the box product of countable spaces is pseudonormal, i.e. any two disjoint closed sets one of which is countable can be separated by open sets. The second theorem says that assuming CH a certain uncountable box product is normal (i.e. $<\omega_1-\Box_{\alpha<\omega_1} X_\alpha$ where each X_α is a compact metric space).

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For X_n topological spaces the box product, $\Box_{n<\omega} X_n$, is the topology on the cartesian product generated by sets of the form $\prod_{n<\omega} U_n$ where each U_n is open in X_n. For general background see van Douwen [1].

Theorem 1. *Suppose for each n, X_n is countable and normal. Then $\Box_{n<\omega} X_n$ is pseudonormal.*

Van Douwen [1] shows that this theorem extends to box products of arbitrarily many spaces (Claim 2, 11.1). This theorem is also true if the X_n are only assumed to be compact (Claim 1, 11.1); but it is false if the X_n are only assumed to be metric spaces (12.1). For any Hausdorff space X the following lemma is true.

Lemma 1. *For any countable, closed $K \subseteq X$, the following are equivalent:*

(A) *For every open $V \supseteq K$ there exists an open U such that $K \subseteq U \subseteq \mathrm{cl}(U) \subseteq V$.*

(B) *For every open (in X) cover of K there is an open (in X) locally finite (in X) refinement covering K.*

This is due to Kunen. The proof appears in van Douwen [1, Claim 3, 11.1]. Kunen used it to show that the product of a compact space and a pseudonormal space is pseudonormal. Note that in general the product of pseudonormal spaces need not be pseudonormal, e.g. the rational points on the line $x = -y$ cannot be separated from the irrationals on this line in the square of the Sorgenfrey line.

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Suppose $x_n \in X_n$ and let:

$$E = \left\{ y \in \prod_{n<\omega} X_n : \exists n\ \forall m > n\ y(m) = x_m \right\}.$$

We begin by showing that E satisfies Lemma 1(A) for $K = E$ and thus Lemma 1(B). We use this to show that every countable closed $F \subseteq \square_{n<\omega} X_n$ has the paracompactnesslike property expressed in Lemma 1(B).

Lemma 2. *For all open $V \supseteq E$, there exists an open U such that $E \subseteq U \subseteq \mathrm{cl}(U) \subseteq V$.*

Proof. Assume $X_n = \{x_i^n : i < \omega\}$ and $E = \{y \in \prod_{m<\omega} X_m$: for all but finitely many n, $y(n) = x_0^n\}$. Let $\lhd$ be a well ordering of E of order type ω such that for all $p, q \in E$ if for all n whenever $p(n) = x_i^n$ and $q(n) = x_j^n$ we have $i \leq j$, then $p \unlhd q$. For $p \in E$ let $|p|$ be the order type of p with respect to $\lhd$. Let U_k^n be an open neighborhood of x_k^n in X_n such that for all $i < k$, $x_i^n \notin \mathrm{cl}(U_k^n)$. For each $p \in E$ choose U_p an open neighborhood of p so that:

(1) $U_p = \prod_{n<\omega} U_p^n$ and $\mathrm{cl}(U_p) = \prod_{n<\omega} \mathrm{cl}(U_p^n) \subseteq V$.

(2) If $j < k \leq |p|$, and $p(n) = x_j^n$, then $U_k^n \cap U_p^n = \emptyset$.

(3) If $q \lhd p$, then for all n,

  (a) if $p(n) \notin \mathrm{cl}(U_q^n)$, then $U_p^n \cap U_q^n = \emptyset$,

  (b) if $p(n) \in U_q^n$, then $U_p^n \subseteq U_q^n$.

To pick these U_p first satisfy (1) and (2) ((2) can be done since $x_j^n \notin \mathrm{cl}(U_k^n)$). Then refine them to satisfy (3) by induction on $\lhd$.

Claim. $\mathrm{cl}(\bigcup_{p\in E} U_p) = \bigcup_{p\in E} \mathrm{cl}(U_p)$.

This, of course, proves the lemma. Suppose $x \notin \bigcup_{p\in E} \mathrm{cl}(U_p)$. For each n if $x(n) = x_k^n$, choose U^n open in X_n with $x_k^n \in U^n$, $U^n \subseteq U_k^n$, and for all $p \in E$ if $|p| \leq k$ and $x_k \notin \mathrm{cl}(U_p^n)$, then $U^n \cap \mathrm{cl}(U_p^n) = \emptyset$. We will show that:

$$\prod_{n<\omega} U^n \cap \bigcup_{p\in E} U_p = \emptyset.$$

This shows $x \notin \mathrm{cl}(\bigcup_{p\in E} U_p)$. Given $p \in E$ define $q \in E$ as follows. Suppose $x(n) = x_k^n$ and $p(n) = x_j^n$. If $j \geq k$ let $q(n) = x(n)$. If $j < k$ let $q(n) = p(n)$. That is, q is the smaller of the two. Since p is eventually equal to x_0^n so is q. Also, $q \unlhd p$. Since $x \notin \mathrm{cl}(U_q)$ there exists an n such that $x(n) \notin \mathrm{cl}(U_q^n)$ and suppose $x(n) = x_k^n$ and $q(n) = x_j^n$. Since $q(n) \neq x(n)$ (since $q(n) \in U_q^n$) it follows that $j < k$ and $p(n) = q(n)$. Since $q \unlhd p$, $U_p^n \subseteq U_q^n$ by (3)(b). If $|q| \leq k$, then by the definition of U^n, $U^n \cap U_q^n = \emptyset$. On the other hand, if $|q| \geq k$, then by (2), $U_k^n \cap U_q^n = \emptyset$ and by definition of U^n, $U^n \subseteq U_k^n$. In either case $U^n \cap U_q^n = \emptyset$ and so $U^n \cap U_p^n = \emptyset$ and therefore $\prod_{n<\omega} U^n \cap U_p = \emptyset$. $\square$

It follows from Lemma 1 and Lemma 2 that E satisfies the paracompactness-like property expressed in Lemma 1(B). We will show that every countable closed $F \subseteq \Box_{n<\omega} X_n$ has this property. It is easy to see that for any closed set C, $C \cap E$ also has this property (i.e. add $\Box_{n<\omega} X_n - C$ to any open cover of $C \cap E$). For $x, y \in \Box_{n<\omega} X_n$ define $x \sim y$ iff $\exists n \, \forall m > n \, x(m) = y(m)$. And let $[x] = \{y \mid x \sim y\}$. Suppose $F \subseteq \Box_{n<\omega} X_n$ is closed and countable and $\mathcal{U}$ is an open cover of F. By a diagonal construction find for each $x \in F$ an open box $U_x = \prod_{n<\omega} U_x^n$ with $x \in U_x$ refining $\mathcal{U}$ such that for all $x, y \in F$ if $x \not\sim y$, then there are infinitely many n such that $\mathrm{cl}(U_x^n) \cap \mathrm{cl}(U_y^n) = \emptyset$. For each $x \in F$ let $\mathcal{V}_x = \{U_y : y \in [x] \cap F\}$ and $V_{[x]} = \bigcup \mathcal{V}_x$.

Claim. $\{V_{[x]} : x \in F\}$ is locally finite.

Let $y \in \Box_{n<\omega} X_n$. For $x \in F$, let $G_x = \{n < \omega \mid y(n) \notin \mathrm{cl}(U_x^n)\}$. If G_{x_0} and G_{x_1} are finite, then $x_0 \sim x_1$. So except for one equivalence class it is easy to build a neighborhood of y disjoint from all U_x.

Now for each $x \in F$ let $\mathcal{F}_x$ be a locally finite refinement of $\mathcal{V}_x$ covering $F \cap [x]$. Then $\mathcal{F} = \bigcup \{\mathcal{F}_x : x \in F\}$ is a locally finite refinement of $\mathcal{U}$ covering F. □

I do not know if it is consistent with ZFC that:

$$\Box_{n<\omega} \mathbf{Q}$$

is normal (where $\mathbf{Q}$ is the space of rational numbers), or if its nonnormality is consistent.

The next theorem generalizes the theorem of Rudin [3] and its generalization in Kunen [2].

Given X_α for $\alpha < \omega_1$ the box product, $<\omega_1-\Box_{\alpha<\omega_1} X_\alpha$ is the topology on the cartesian product generated by sets of the form $\prod_{\alpha<\omega_1} U_\alpha$ where each U_α is open in X_α and for all but countably many α, $U_\alpha = X_\alpha$.

Theorem 2. *Assume the continuum hypothesis. If for each α, X_α is a compact Hausdorff space of weight $\leq \mathfrak{c}$, then $<\omega_1-\Box_{\alpha<\omega_1} X_\alpha$ is paracompact.*

Proof. As before, for $x, y \in \prod_{\alpha<\omega_1} X_\alpha$ define $x \sim y$ iff $\{\alpha : x(\alpha) \neq y(\alpha)\}$ is finite, and define $[x] = \{y \mid x \sim y\}$. We use $<\omega_1-\nabla_{\alpha<\omega_1} X_\alpha$ (the pinched product) to denote the space of equivalence classes with the topology given by the quotient map σ ($\sigma(x) = [x]$). For any basic open set $U = \prod_{\alpha<\omega_1} U_\alpha$, the countable set $\{\alpha < \omega_1 \mid U_\alpha \neq X_\alpha\}$ is called the support of U.

Lemma 3. *$<\omega_1-\nabla_{\alpha<\omega_1} X_\alpha$ is a P-space (i.e. the countable intersection of open sets is open).*

Proof. Suppose $\sigma(x) \in \sigma(U^n)$ for $n < \omega$ where $U^n = \prod_{\alpha<\omega_1} U_\alpha^n$ are basic open sets. Let $\{\alpha_m : m < \omega\}$ cover the support of each U^n. Let $V_{\alpha_n} = \bigcap_{m<n} U_{\alpha_n}^m$ for each n and

for $\alpha \notin \{\alpha_m : m < \omega\}$ let $V_\alpha = X_\alpha$. Then

$$\sigma(x) \in \sigma\Big(\prod_{\alpha<\omega_1} V_\alpha\Big) \subseteq \bigcap_{n<\omega} \sigma(U^n). \qquad \Box$$

Since the weight of $<\omega_1 - \nabla_{\alpha<\omega_1} X_\alpha$ is ω_1 we have immediately that $<\omega_1 - \nabla_{\alpha<\omega_1} X_\alpha$ is paracompact (see Kunen [2, Lemma 1.3]).

Lemma 4. *σ is a closed, continuous map.*

Proof. σ is continuous by definition. Suppose K is a closed subset of $<\omega_1 - \Box_{\alpha<\omega_1} X_\alpha$ and $\sigma(x) \notin \sigma(K)$.

Claim. For every $F \in [\omega_1]^{<\omega}$ there exists a basic open set $U_F = \prod_{\alpha<\omega_1} U_\alpha^F$ with $x \in U_F$, $U_F \cap K = \emptyset$, and for all $\alpha \in F$, $U_\alpha^F = X_\alpha$.

Proof. Let $z = x \restriction (\omega_1 - F)$. For any $y \in \prod_{\alpha\in F} X_\alpha$ we know that $y \cup z \notin K$ (since $\sigma(x) \notin \sigma(K)$). For each such y let $U_y = \prod_{\alpha<\omega_1} U_\alpha^y$ be a basic open set with $y \cup z \in U_y$ and $U_y \cap K = \emptyset$. By the compactness of $\prod_{\alpha\in F} X_\alpha$, there are $y_0, y_1, y_2, \ldots, y_{n-1}$ such that $\{\prod_{\alpha\in F} U_\alpha^{y_i} : i < n\}$ covers $\prod_{\alpha\in F} X_\alpha$. Define $U_\alpha^F = X_\alpha$ for $\alpha \in F$ and $U_\alpha^F = \bigcap_{i<n} U_\alpha^{y_i}$ otherwise. $\Box$

Now we continue proving the lemma.

We seek an $\alpha < \omega_1$ such that for every $F \in [\alpha]^{<\omega}$, there is a U_F with the support of U_F contained in α and disjoint from F. Such an α can be gotten by a Lowenheim–Skolem argument. Let κ be a large enough regular cardinal such that H_κ, the sets hereditarily of cardinality less than κ, contains $<\omega_1 - \Box_{\alpha<\omega_1} X_\alpha$. Let (M, ε) be a countable elementary substructure of (H_κ, ε) with x, K, and $<\omega_1 - \Box_{\alpha<\omega_1} X_\alpha$ elements of M.

Although M is not transitive it can be seen that for any $X \in M$ if $M \models$ "X is countable", then X is contained in M (ω is contained in M and if $M \models$ "$f: \omega \to^{\text{onto}} X$", then $X = \{f(n) : n < \omega\} \subseteq M$).

Let $\alpha = M \cap \omega_1$. Working in M, for each $F \in [\alpha]^{<\omega}$ find $U_F \in M$ with $x \in U_F$, $U_F \cap K = \emptyset$, and F disjoint from the support of U_F. Let F_n for $n < \omega$ be an increasing sequence of finite subsets of α such that $\alpha = \bigcup_{n<\omega} F_n$. Now for every $\beta < \omega_1$ let $V_\beta = \bigcap_{\beta \notin F_n} U_{F_n}^\beta$. For $\beta < \alpha$ this is a finite intersection and for $\beta \geq \alpha$, $V_\beta = X_\beta$ since the support of U_{F_n} is contained in M. Let $V = \prod_{\beta<\omega_1} V_\beta$, then clearly $x \in V$.

Claim. $\sigma(V) \cap \sigma(K) = \emptyset$.

Proof. Suppose $y_0 \sim y_1$, $y_0 \in V$, and $y_1 \in K$. Choose $n < \omega$ such that:

$$\{\beta < \alpha \mid y_0(\beta) \neq y_1(\beta)\} \subseteq F_n.$$

Then by definition of V, $y_0 \in U_{F_n}$, and by the definition of U_{F_n}, $y_1 \in U_{F_n}$, contradicting the fact that $U_{F_n} \cap K = \emptyset$. □

Since x and K were arbitrary we have that σ is a closed map; thus proving Lemma 4. □

Lemma 5. *For each $x \in \prod_{\alpha<\omega_1} X_\alpha$, $[x]$ is Lindelof (as a subspace of $<\omega_1-\square_{<\omega_1} X_\alpha$).*

Proof. Let $\mathcal{U}$ be an open cover by basic open boxes of $[x]$. As before let (M, ε) be a countable elementary substructure of (H_κ, ε) such that x and $\mathcal{U}$ are elements of M and let $\alpha = M \cap \omega_1$. I claim that $\mathcal{U} \cap M$ covers $[x]$. Given any $y \sim x$, let $z = y \restriction \alpha \cup x \restriction (\omega_1 - \alpha)$. Since $z \in M$ there exists $V \in \mathcal{U} \cap M$ such that $z \in V$. Since the support of V is contained in α, we have that $y \in V$. □

Lemma 4, Lemma 5 and the paracompactness of $<\omega_1-\nabla_{\alpha<\omega_1} X_\alpha$ imply that $<\omega_1-\square_{\alpha<\omega_1} X_\alpha$ is paracompact (see Kunen [2, Lemma 1.4]).

Remarks. Theorem 2 cannot be extended to $<\omega_1-\square_{\alpha<\omega_2} X_\alpha$, since if X_α is the two element space. this product is 2^{ω_2} with the G_δ-topology, which is not normal (see van Douwen [1]).

Kunen has noted that the existence of an ω_1-scale in ω^ω implies that $<\omega_1-\nabla_{\alpha<\omega_1} X_\alpha$ is paracompact where each X_α is a compact metric space (and therefore $<\omega_1-\square_{\alpha<\omega_1} X_\alpha$ is paracompact). However, I do not know whether MA is enough to imply $<\omega_1-\square_{\alpha<\omega_1} X_\alpha$ is normal where the X_α are compact metric spaces.

Of course, Lemmas 4 and 5 can be proven without using the Lowenheim–Skolem Theorem. In Lemma 4 all that is needed is an $\alpha < \omega_1$ such that for every $F \in [\alpha]^{<\omega}$, the support of U_F is contained in α. For Lemma 5 find $\alpha < \omega_1$ and $\mathcal{F} \subseteq \mathcal{U}$ countable such that for every $V \in \mathcal{F}$ the support of V is contained in α and for every $y \sim x$ if for every $\beta \geq \alpha$ $y(\beta) = x(\beta)$, then there exists $V \in \mathcal{F}$ such that $y \in V$.

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