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(still working on it)

# Borel hierarchies

These are lecture notes from a course I gave in the spring semester of 2014[1]. The following results are new:

- 5.10 It is consistent that the Borel subsets of the plane are not contained in any bounded level of the $\sigma$-algebra generated by the abstract rectangles.
- 5.12 It is consistent that for some $\kappa$ every family of size $\kappa$ of sets of reals is included in a countably generated $\sigma$-algebra but not necessarily at a bounded level.
- 7.1 If $2^{<\mathfrak{c}} = \mathfrak{c}$ and there is a Borel universal map, then there is a map $H : 2^\omega \times 2^\omega \to 2^\omega$ such that for every $\kappa < \mathfrak{c}$ and $G : \kappa \times \kappa \to \kappa$ there are $x_\alpha \in 2^\omega$ for $\alpha < \kappa$ such that for all $\alpha, \beta, \gamma < \kappa \quad G(\alpha, \beta) = \gamma$ iff $H(x_\alpha, x_\beta) = x_\gamma$.
- 9.5 CH implies that for any $\alpha_0$ with $3 \leq \alpha_0 < \omega_1$ there are $X_0, X_1 \subseteq 2^\omega$ with $\mathrm{ord}(X_0) = \alpha_0 = \mathrm{ord}(X_1)$ and $\mathrm{ord}(X_0 \cup X_1) = \alpha_0 + 1$.

Also the proof of Theorem 4.18 is new. I have omitted some of the results proved in lecture but only when the proof I gave is identical to that found in the literature.

# Contents



[1] http://www.math.wisc.edu/~miller/old/m873-14/

# 1 Classical Results

This section contains the Theorem of Lebesgue which uses universal sets and a diagonal argument to show that the length of the Borel hierarchy is as long as possible, $\omega_1$. We also give some results of Bing, Beldsoe, and Mauldin and of Recław which I think of as generalizations of Lebesgue's Theorem.

First we review some standard terminology and results. The Baire space is $\omega^\omega = \{x \mid x : \omega \to \omega\}$ where $\omega = \{0, 1, 2, \ldots\}$. For $s \in \omega^{<\omega} = \bigcup_{n<\omega} \omega^n$ a finite sequence define a basic clopen set:

$$[s] = \{x \in \omega^\omega \;:\; s \subseteq x\}$$

The Baire space is homeomorphic to the irrationals and is a zero dimensional Polish space, i.e., completely metrizable separable with a clopen basis. One complete metric is $d(x, y) = \frac{1}{n}$ where $n$ is the least with $x \upharpoonright n \neq y \upharpoonright n$.

The Cantor space $2^\omega \subseteq \omega^\omega$ is homeomorphic to the middle thirds set

$$\{\sum_{n=0}^{\infty} \frac{2x(n)}{3} \;:\; x \in 2^\omega\}$$

The Borel hierarchy is described as follows:

open $= \underset{\sim}{\mathbf{\Sigma}}^0_1 = G$

closed $= \underset{\sim}{\mathbf{\Pi}}^0_1 = F$

$\underset{\sim}{\mathbf{\Pi}}^0_2 = G_\delta =$ countable intersections of open sets

$\underset{\sim}{\mathbf{\Sigma}}^0_2 = F_\sigma =$ countable unions of closed sets

$\underset{\sim}{\mathbf{\Sigma}}^0_3 = G_{\delta\sigma} =$ countable unions of $G_\delta$ sets

$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha = \{ \bigcup_{n<\omega} A_n \ : \ A_n \in \underset{\sim}{\mathbf{\Pi}}^0_{<\alpha} \}$

$\underset{\sim}{\mathbf{\Pi}}^0_\alpha = \{ \bigcap_{n<\omega} A_n \ : \ A_n \in \underset{\sim}{\mathbf{\Sigma}}^0_{<\alpha} \}$

$G_{\delta\sigma}, G_{\delta\sigma\delta}, \ldots, \qquad F_{\sigma\delta}, F_{\sigma\delta\sigma} \ldots$

The Borel sets are the smallest $\sigma$-algebra containing the open sets. In a metric space $F \subseteq G_\delta$ ($\underset{\sim}{\mathbf{\Pi}}^0_1 \subseteq \underset{\sim}{\mathbf{\Pi}}^0_2$), i.e., every closed set is the countable intersection of open sets.[2] It follows that in a metric space for $1 \leq \alpha < \beta$ that

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha \cup \underset{\sim}{\mathbf{\Pi}}^0_\alpha \ \subseteq \ \underset{\sim}{\mathbf{\Sigma}}^0_\beta \cap \underset{\sim}{\mathbf{\Pi}}^0_\beta \ =^{def} \underset{\sim}{\mathbf{\Delta}}^0_\beta$$

**Example 1.1** *(Willard 1971 [25]) There are nice spaces in which closed sets are $G_{\delta\sigma}$ but not $G_\delta$.*

proof:

Suppose $(X, \tau)$ is a space for which closed sets are $G_\delta$ and $G_{\delta\sigma} \neq G_\delta$. Fix $A \in G_{\delta\sigma} \backslash G_\delta$. Let $\tau_A$ be the smallest topology containing $\tau$ and the complement of $A$, $\sim A =^{def} X \backslash A$. Then the closed sets of $\tau_A$ are a subset of $G_{\delta\sigma}(\tau_A)$ but not of $G_\delta(\tau_A)$.

$U \in \tau_A$ iff there are $V, W \in \tau$ with $U = (V \cap \sim A) \cup W$. $A$ is closed in $\tau_A$ but is not the countable intersection of $\tau_A$ open sets, else

$$A = \bigcap_{n<\omega} (V_n \cap \sim A) \cup W_n$$

But then $A = \bigcap_{n<\omega} W_n$, but $A$ is not $G_\delta$.
□

The equation

$$\bigcup_n P_n \ \cap \ \bigcup_m Q_m \ = \ \bigcup_{n,m} P_n \cap Q_m$$

shows by induction that

- $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ is closed under finite intersections and
- $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ is closed under finite unions.

[2]If $C$ is closed, then $C = \bigcap_n U_n$ where $U_n = \{x \ : \ \exists y \in C \ d(x,y) < \frac{1}{n}\}$.

Another easy proposition is that for $f : X \to Y$ a continuous map:

- If $A \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(Y)$, then $f^{-1}(A) \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X)$.
- If $A \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(Y)$, then $f^{-1}(A) \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(X)$.

Note that $2^\omega$ is naturally homeomorphic to $2^\omega \times 2^\omega$ via the map $x \mapsto (y, z)$ where $y(n) = x(2n)$ and $z(n) = x(2n+1)$. We use $x = \langle y, z \rangle$ to denote the pairing map. Similarly $(2^\omega)^\omega$ is homeomorphic to $2^\omega$ via $x = \langle x_n : n < \omega \rangle$ where $x_n(m) = x(\langle n, m \rangle)$ and $\langle n, m \rangle = 2^n(2m+1) - 1$ is a bijection between $\omega \times \omega$ and $\omega$.

**Theorem 1.2** *(Lebesgue 1905 see [16] Thm 2.5) For every countable $\alpha > 0$*

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(2^\omega) \neq \underset{\sim}{\mathbf{\Pi}}^0_\alpha(2^\omega).$$

This follows from the existence of universal sets and a diagonal argument.

**Universal Sets Lemma.** *For every countable $\alpha > 0$ there exists $U \subseteq 2^\omega \times 2^\omega$ which is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ and universal for $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$-sets, i.e., for every $V \subseteq 2^\omega$ which is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ there exists $x \in 2^\omega$ such that $V = U_x =^{def} \{y \ : \ (x, y) \in U\}$.*
proof:

Note that $U \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ implies $U_x \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ for all $x \in 2^\omega$. To see this let $f : 2^\omega \to 2^\omega \times 2^\omega$ be defined by $f(y) = (x, y)$, then $U_x = f^{-1}(U)$.

Case $\alpha = 1$.

Let $\{C_n : n < \omega\}$ list all clopen subsets of $2^\omega$. Define

$$(x, y) \in U \text{ iff } \exists n\ (x(n) = 1 \text{ and } y \in C_n).$$

Note that $U = \bigcup_{n<\omega}(\{x : x(n) = 1\} \times C_n)$.

Case $\alpha > 1$.

Let $\{\alpha_n : n < \omega\}$ list with infinitely many repetitions the nonzero elements of $\alpha$. Let $U^{\alpha_n}$ be a universal $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_n}$ set. Define $U$ by

$$(x, y) \in U \text{ iff } \exists n\ (x_n, y) \notin U^{\alpha_n}$$

Note that $\sim U^{\alpha_n}$ is a universal $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_n}$ set. For each fixed $n$ the set

$$\{(x, y) \ : \ (x_n, y) \in U^{\alpha_n}\}$$

is $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_n}$ because it is the preimage of $U^{\alpha_n}$ under a continuous map, namely $(x, y) \mapsto (x_n, y)$. Hence it is easy to verify that $U$ is a universal $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set.
□

**The Diagonal Argument**. Suppose $U \subseteq 2^\omega \times 2^\omega$ is a universal $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set. Let

$$D = \{x \in 2^\omega \ : \ (x, x) \notin U\}.$$

Then $D$ is $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ because it is the continuous preimage of $\sim U$ under the map $x \mapsto (x, x)$. However $D \neq U_x$ for any $x \in 2^\omega$ so $D$ is not $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$.

This proves Theorem 1.2.
□

Define ord$(X)$ to the least ordinal such that $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X) = \text{Borel}(X)$. Hence $\text{ord}(2^\omega) = \omega_1$.

**Corollary 1.3** *If $X$ is any topological space which contains a homeomorphic copy of $2^\omega$, then* $\text{ord}(X) = \omega_1$. *More generally, if $Y \subseteq X$ is subspace, then* $\text{ord}(Y) \leq \text{ord}(X)$.

proof:

If $Y \subseteq X$, then

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(Y) = \{B \cap Y \ : \ B \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X)\}$$

and

$$\text{Borel}(Y) = \{B \cap Y \ : \ B \in \text{Borel}(X)\}.$$

□

Suppose $\mathcal{H} \subseteq \mathcal{P}(Y)$ define $\Sigma_\alpha(\mathcal{H})$ as follows:
$\Sigma_0(\mathcal{H}) = \{\sim A \ : \ A \in \mathcal{H}\}$ and for $\alpha > 0$

$$\Sigma_\alpha(\mathcal{H}) = \{\bigcup_{n<\omega} \sim A \ : \ A_n \in \bigcup_{\beta<\alpha} \Sigma_\beta(\mathcal{H})\}$$

We let Borel$(\mathcal{H})$ be the $\sigma$-algebra generated by $\mathcal{H}$ and let ord$(\mathcal{H})$ be the least $\alpha$ with $\Sigma_\alpha(\mathcal{H}) = \text{Borel}(\mathcal{H})$.

**Theorem 1.4** *(Bing, Bledsoe, Mauldin [2] also [16] Thm 3.2 [15] Thm 18) Suppose $\mathcal{H} \subseteq \mathcal{P}(2^\omega)$ is a countable family such that* $\text{Borel}(2^\omega) \subseteq \text{Borel}(\mathcal{H})$. *Then* $\text{ord}(\mathcal{H}) = \omega_1$, *i.e., the $\sigma$-algebra generated by $\mathcal{H}$ has $\omega_1$ many levels.*

To prove this theorem we will need the following two lemmas. Given a countable $\mathcal{H} \subseteq P(2^\omega)$ let

$$\mathcal{F} = \{C \times A \ : \ C \subseteq 2^\omega \text{ is clopen and } A \in \mathcal{H}\} \ \subseteq \ \mathcal{P}(2^\omega \times 2^\omega).$$

**Lemma 1.5** *(Universal sets) For each $\alpha$ with $1 \leq \alpha < \omega_1$ there exists a set $U \in \Sigma_\alpha(\mathcal{F})$ which is universal for $\Sigma_\alpha(\mathcal{H})$ sets, i.e., for every $A \in \Sigma_\alpha(\mathcal{H})$, there exists $x \in 2^\omega$ such that $A = \{y : (x, y) \in U\}$.*

proof:

For $\alpha = 1$: Let $\mathcal{H} = \{A_n : n \in \omega\}$ let

$$U = \bigcup_{n\in\omega} \{x : x(n) = 1\} \times (2^\omega \setminus A_n).$$

For $\alpha > 1$: Let $x \mapsto \langle x_n : n \in \omega\rangle$ be a nice recursive coding taking $2^\omega \to (2^\omega)^\omega$. Let $\beta_n$ for $n \in \omega$ be cofinal in $\alpha$, and $U_n \in \Sigma_{\beta_n}(\mathcal{F})$ be universal for $\Sigma_{\beta_n}(\mathcal{H})$ sets. Define $U'_n$ by $(x, y) \in U'_n$ iff $(x_n, y) \in U_n$. It is easy to check that $U'_n$ is also $\Sigma_{\beta_n}(\mathcal{F})$ and universal for $\Sigma_{\beta_n}(\mathcal{H})$. But now taking

$$U = \bigcup_{n\in\omega} (2^\omega \setminus U'_n)$$

gives us a set in $\Sigma_\alpha(\mathcal{F})$ which is universal for $\Sigma_\alpha(\mathcal{H})$ sets.
□

**Lemma 1.6** *(Diagonalization) Suppose that every clopen set is in Borel($\mathcal{H}$). Then for every $B \in$Borel($\mathcal{F}$) the set $\{x : (x, x) \in B\}$ is in Borel($\mathcal{H}$).*

proof:

For $B = C \times A$ where $A \in \mathcal{H}$ and $C \subseteq 2^\omega$ is clopen, note that

$$\{x : (x, x) \in B\} = C \cap A.$$

Since by assumption $C \in$Borel($\mathcal{H}$), we have the lemma for elements of $\mathcal{F}$. To do Borel($\mathcal{F}$) is an easy induction.
□

Now we give a proof of Theorem 1.4. Suppose Borel($\mathcal{H}$) $= \Sigma_\alpha(\mathcal{H})$. By Lemma 1.5 there exist $U$ in Borel($\mathcal{F}$) which is universal for $\Sigma_\alpha(\mathcal{H})$ and hence Borel($\mathcal{H}$). By Lemma 1.6 the set

$$D = \{x : (x, x) \notin U\}$$

is in Borel($\mathcal{H}$). But this means that for some $x$ that $D = \{y : (x,y) \in U\}$. But then $x \in D$ iff $x \notin D$.
□

**Theorem 1.7** *(Recław 1993 see [16] Thm 3.5 [15] Thm 17) If $X$ is a second countable space and $X$ can be mapped continuously onto any space containing $2^\omega$, then* $\mathrm{ord}(X) = \omega_1$.

proof:

By going to a subspace of $X$ we may that there is an $f : X \to 2^\omega$ which is one-to-one, onto, and continuous. Let $\mathcal{C}$ be a countable open basis for $X$ containing the pre-images under $f$ of the clopen subsets of $2^\omega$. Let

$$\mathcal{H} = \{f(C) \ : \ C \in \mathcal{C}\}.$$

Since it is clear that $\mathcal{H}$ contains all clopen sets, by Theorem 1.4, the $\mathrm{ord}(\mathcal{H}) = \omega_1$. But the map $f$ takes the Borel hierarchy of $X$ directly over to the hierarchy on Borel($\mathcal{H}$), so $\mathrm{ord}(X) = \omega_1$.
□

**Remark 1.8** *Recław's result is also true,* $\mathrm{ord}(X) = \omega_1$*, if we only assume that there is a Borel map onto map $f : X \to 2^\omega$.*

To see this let $\mathcal{C}$ be a countable open basis for $X$ and let $\mathcal{B}$ be the pre-images under $f$ of the clopen subsets of $2^\omega$. Let $\mathcal{G} = \mathcal{C} \cup \mathcal{B}$ and $\mathcal{H} = \{f(C) : C \in \mathcal{G}\}$. Then $\mathrm{ord}(\mathcal{H}) = \omega_1$ by Theorem 1.4. And so $\mathrm{ord}(\mathcal{G}) = \omega_1$. This means that $\mathrm{ord}(X) = \omega_1$, since $\mathrm{Borel}(\mathcal{G}) = \mathrm{Borel}(X)$ and $\underset{\sim}{\mathbf{\Sigma}}^0_1(X) \subseteq \Sigma_1(\mathcal{G})$ implies $\mathrm{ord}(\mathcal{G}) \leq \mathrm{ord}(X)$. To see this note that if $\alpha = \mathrm{ord}(X)$, then

$$\mathrm{Borel}(\mathcal{G}) = \mathrm{Borel}(X) = \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X) \subseteq \Sigma_\alpha(\mathcal{G}) \subseteq \mathrm{Borel}(\mathcal{G}).$$

An extension of Recław's result to Souslin (operation A) sets appears in Miller [14].

**Remark 1.9** *It is relatively consistent to have $X, Y \subseteq 2^\omega$ and $f : X \to Y$ continuous, one-to-one, and onto such that* $\mathrm{ord}(X) = 2$ *and* $\mathrm{ord}(Y) = 3$.

proof:

To see this note that it is relatively consistent to have a $Q$-set $H$ of size $\omega_1$ which is concentrated on a countable set $E$ (Fleissner and Miller [4]).

Let $Y = H \cup E = \{y_\alpha : \alpha < \omega_1\}$ and $H = \{x_\alpha : \alpha < \omega_1\}$ be one-to-one enumerations. Put $X = \{\langle x_\alpha, y_\alpha\rangle : \alpha < \omega_1\}$ and let $f$ be projection onto the second coordinate, i.e., $f(\langle x, y\rangle) = y$. $X$ is a $Q$-set so has order 2 and it easy to check that $Y$ has order 3.

$\Box$

In the iterated Sack real model the continuum is $\omega_2$ and for every $X \subseteq 2^\omega$ of size $\omega_2$ there is a continuous onto map $f : X \to 2^\omega$ (see Miller [12]) and hence $\text{ord}(X) = \omega_1$. So in the Sacks real model every set of reals of size continuum has order $\omega_1$.

**Corollary 1.10** *If $X$ is separable, metric, but not zero-dimensional, then* $\text{ord}(X) = \omega_1$.

proof:

Suppose $\text{ord}(X) < \omega_1$. Let $d$ be any metric on $X$ and $x \in X$ arbitrary. There must be arbitrarily small $\epsilon > 0$ such that there is no $y \in X$ with $d(x, y) = \epsilon$. Otherwise the map $y \mapsto d(x, y)$ has a nontrivial interval in its image.

$\Box$

Any zero-dimensional separable metric space is homeomorphic to a subspace of $2^\omega$. See Kechris [6] page 38.

# 2 The $\omega_1$-Borel hierarchy

Define the levels of the $\omega_1$-Borel hierarchy of subsets of $2^\omega$ as follows:

1. $\mathbf{\Sigma}^*_0 = \mathbf{\Pi}^*_0 =$ clopen subsets of $2^\omega$
2. $\mathbf{\Sigma}^*_\alpha = \{\bigcup_{\beta<\omega_1} A_\beta \ : \ (A_\beta : \beta < \omega_1) \in (\mathbf{\Pi}^*_{<\alpha})^{\omega_1}\}$
3. $\mathbf{\Pi}^*_\alpha = \{2^\omega \setminus A \ : \ A \in \mathbf{\Sigma}^*_\alpha\}$
4. $\mathbf{\Pi}^*_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta \qquad \mathbf{\Sigma}^*_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Sigma}^*_\beta$

The length of this hierarchy is the smallest $\alpha \geq 1$ such that

$$\mathbf{\Pi}^*_\alpha = \mathbf{\Sigma}^*_\alpha.$$

**Theorem 2.1** *(Miller [18]) ($MA_{\omega_1}$) $\mathbf{\Pi}^*_\alpha \neq \mathbf{\Sigma}^*_\alpha$ for every $\alpha < \omega_2$.*

We prove this using the following two lemmas. A well-known consequence of $\mathrm{MA}_{\omega_1}$ is that every subset $Q \subseteq 2^\omega$ of size $\omega_1$ is a Q-set, i.e., for every subset $X \subseteq Q$ there is a $G_\delta$ set $G \subseteq 2^\omega$ with $G \cap Q = X$ (see Fleissner and Miller [4]).

**Lemma 2.2** *Suppose there exists a Q-set of size $\omega_1$. Then there exists an onto map $F : 2^\omega \to 2^{\omega_1}$ such for every subbasic clopen set $C \subseteq 2^{\omega_1}$ the set $F^{-1}(C)$ is either $G_\delta$ or $F_\sigma$.*

proof:

Fix $Q = \{u_\alpha \in 2^\omega \ : \ \alpha < \omega_1\}$ a Q-set. Let $G \subseteq 2^\omega \times 2^\omega$ be a universal $G_\delta$ set, i.e., $G$ is $G_\delta$ and for every $G_\delta$ set $H \subseteq 2^\omega$ there exists $x \in 2^\omega$ with $G_x = H$. Define $F$ as follows, given $x \in 2^\omega$ let

$$F(x)(\alpha) = 1 \text{ iff } u_\alpha \in G_x$$

If $C$ is a subbasic clopen set, then for some $\alpha$ and $i = 0$ or $i = 1$

$$C_{\alpha,i} = \{p \in 2^{\omega_1} \ : \ p(\alpha) = i\}.$$

Then for $i = 1$

$$F^{-1}(C_{\alpha,1}) = \{x \ : \ u_\alpha \in G_x\}$$

which is a $G_\delta$ set. Since $C_{\alpha,0}$ is the complement of $C_{\alpha,1}$ we have that $F^{-1}(C_{\alpha,0})$ is an $F_\sigma$-set

Finally, we note that since $Q$ is a Q-set, i.e., every subset is a relative $G_\delta$, it follows that $F$ is onto.

□

The next Lemma is true without any additional assumptions beyond ZFC. Its proof is a generalization of Lebesgue's 1905 proof (see Kechris [6] p.168) for the standard Borel hierarchy.

**Lemma 2.3** *For any $\alpha$ with $0 < \alpha < \omega_2$ there exists a $\mathbf{\Sigma}^*_\alpha$ set $U \subseteq 2^{\omega_1} \times 2^\omega$ which is universal for $\mathbf{\Sigma}^*_\alpha$ subsets of $2^\omega$, i.e., for any $Q \subseteq 2^\omega$ which is $\mathbf{\Sigma}^*_\alpha$ there exists $p \in 2^{\omega_1}$ with $U_p = Q$. Similarly, there is a universal $\mathbf{\Pi}^*_\alpha$ set.*

proof:

The proof is by induction on $\alpha$. Note that the complement of a universal $\mathbf{\Sigma}^*_\alpha$ set is a universal $\mathbf{\Pi}^*_\alpha$-set.

For $\alpha = 1$, $\mathbf{\Sigma}^*_\alpha$ is just the open sets. There is a universal open set $V \subseteq 2^\omega \times 2^\omega$. Put

$$U = \{(p, x) \in 2^{\omega_1} \times 2^\omega \ : \ (p \restriction \omega, x) \in V\}$$

For $\alpha$ such that $2 \leq \alpha < \omega_2$ proceed as follows. Let $(\delta_\beta < \alpha : \beta < \omega_1)$ have the property that for every $\gamma < \alpha$ there are $\omega_1$ many $\delta_\beta \geq \gamma$. It follows that for every $\mathbf{\Sigma}^*_\alpha$ set $Q \subseteq 2^\omega$ there is $(Q_\beta \in \mathbf{\Pi}^*_{\delta_\beta} \ : \ \beta < \omega_1)$ with

$$Q = \bigcup_{\beta < \omega_1} Q_\beta.$$

By induction, there are $U_\beta \subseteq 2^{\omega_1} \times 2^\omega$ universal $\mathbf{\Pi}^*_{\delta_\beta}$ sets. Let $a : \omega_1 \times \omega_1 \to \omega_1$ be a bijection. For each $\beta$ define

$$\pi_\beta : 2^{\omega_1} \times 2^\omega \to 2^{\omega_1} \times 2^\omega, \ \ (p, x) \mapsto (q, x)$$

where $q(\alpha) = p(a(\beta, \alpha))$. Put

$$U = \bigcup_{\beta < \omega_1} \pi_\beta^{-1}(U_\beta)$$

then $U$ will be a universal $\mathbf{\Sigma}^*_\alpha$ set.

□

Now we prove Theorem 2.1. Suppose for contradiction, that every $\omega_1$-Borel set is $\mathbf{\Sigma}^*_\alpha$ for some fixed $\alpha < \omega_2$. Let $U \subseteq 2^{\omega_1} \times 2^\omega$ be a universal $\mathbf{\Sigma}^*_\alpha$ and define

$$V = \{(x, y) \in 2^\omega \times 2^\omega \ : \ (F(x), y) \in U\}.$$

Then $V$ is an $\omega_1$-Borel set (although not necessarily at the $\mathbf{\Sigma}^*_\alpha$) because the preimage of any clopen box $C \times D$ is $\omega_1$-Borel by Lemma 2.2. Define

$$D = \{x : (x, x) \notin V\}.$$

But then $D$ is $\omega_1$-Borel but not $\mathbf{\Sigma}^*_\alpha$. We see this by the usual diagonal argument that if $D = U_p$, then since $F$ is onto there would be $x \in 2^\omega$ such that $F(x) = p$ but then

$$x \in D \text{ iff } (F(x), x) \notin U \text{ iff } x \notin U_p \text{ iff } x \notin D.$$

□

**Remark 2.4** *In the Cohen real model this hierarchy has order either $\omega_1+1$ or $\omega_1+2$, I am not sure which.*

**Remark 2.5** *Note that in the proof $V \subseteq 2^\omega \times 2^\omega$ is a $\mathbf{\Sigma}^*_{2+\alpha}$-set, since the preimage of a clopen set under $F$ is $\mathbf{\Delta}^0_3$. Hence for levels $\alpha \geq \omega$ the set $V$ is a $\mathbf{\Sigma}^*_\alpha$ set which is universal for $\mathbf{\Sigma}^*_\alpha$ sets.*

**Remark 2.6** *Our result easily generalizes to show that MA implies that for any $\kappa$ a cardinal with $\omega \leq \kappa < |2^\omega|$ the $\kappa$-Borel hierarchy has length $\kappa^+$. This implies that assuming MA for any $\kappa_1 < \kappa_2$ there are $\kappa_2$-Borel sets which are not $\kappa_1$-Borel.*[3] *It is also true for the Cohen real model that for $\omega \leq \kappa_1 < \kappa_2 < |2^\omega|$ that there are $\kappa_2$-Borel sets which are not $\kappa_1$-Borel.*

**Proposition 2.7** *([18]) If $\mathcal{P}(2^\omega) = \omega_1$-Borel, then $\mathcal{P}(2^\omega) = \mathbf{\Sigma}^*_\alpha$ for some $\alpha < \omega_2$.*

proof:

Suppose not and let $P_\alpha$ for $\alpha < \omega_2$ be pairwise disjoint homeomorphic copies of $2^\omega$. For each $\alpha$ let $A_\alpha \subseteq P_\alpha$ be such that $A_\alpha \notin \mathbf{\Sigma}^*_\alpha$. Then

$$A =^{def} \bigcup_{\alpha<\omega_2} A_\alpha$$

is not $\omega_1$-Borel.

□

Steprans [23] showed that it is relatively consistent with ZFC that

$$\mathbf{\Pi}^*_3 = \mathbf{\Sigma}^*_3 = \mathcal{P}(2^\omega) \text{ and } \mathbf{\Pi}^*_2 \neq \mathbf{\Sigma}^*_2$$

and the continuum is $\aleph_{\omega_1}$. Carlson [3] showed:

**Theorem 2.8** *(Carlson) If every subset of $2^\omega$ is $\omega_1$-Borel, then the cofinality of the continuum must be $\omega_1$.*

proof:

Let $B_\alpha \subseteq 2^\omega$ for $\alpha < \mathfrak{c}$ list all (ordinary) Borel sets. Let $X_\alpha \in [2^\omega]^\omega$ for $\alpha < \mathfrak{c}$ be a family of pairwise disjoint infinite countable sets. For each $\alpha$ choose $Z_\alpha \subseteq X_\alpha$ such that $B_\beta \cap X_\alpha \neq Z_\alpha$ for every $\beta < \alpha < \mathfrak{c}$. Put

[3] Since $\kappa_2$-Borel sets at level $\kappa_1^+$ or higher cannot be $\kappa_1$-Borel.

$Z = \bigcup_{\alpha<\mathfrak{c}} Z_\alpha$. We claim that if $\mathrm{cof}(\mathfrak{c}) > \omega_1$ then $Z$ is not $\omega_1$-Borel. This will follow easily from the following

**Claim**. For any $C \subseteq 2^\omega$ an $\omega_1$-Borel set, there are (ordinary) Borel sets $C_\alpha$ for $\alpha < \omega_1$ such that for any $x \in 2^\omega$ there is a closed unbounded $Q \subseteq \omega_1$ such that for every $\alpha \in Q$ ($x \in C$ iff $x \in C_\alpha$).
proof:

We code $\omega_1$-Borel sets with well-founded trees. For $T \subseteq \omega_1^{<\omega}$ a well-founded tree let $T^*$ be the rank zero or terminal nodes of $T$. To simplify matters assume that for every $s \in T \setminus T^*$ all immediate extensions of $s$, i.e., $s\hat{\ }\alpha$, are in $T$. Let $\mathcal{C}$ be the clopen subsets of $2^\omega$. Given $c : T^* \to \mathcal{C}$ define $H(s,(T,c)) \subseteq 2^\omega$ for $s \in T$ by the rank of $s$.

- $H(s,(T,c)) = c(s)$ for $s \in T^*$ and otherwise
- $H(s,(T,c)) = \bigcap_{\alpha<\omega_1} \sim H(s\hat{\ }\alpha,(T,c))$

Now let $C = H(\langle\rangle,(T,c))$. For any $\alpha < \omega_1$ let $T_\alpha = \alpha^{<\omega} \cap T$, $c_\alpha = c \restriction T_\alpha^*$, and $C_\alpha = H(\langle\rangle,(T_\alpha,c_\alpha))$. Since $\alpha$ is countable $C_\alpha$ is an ordinary Borel set. Consider any $x \in 2^\omega$. Let $\kappa$ be a large enough regular cardinal. Construct a continuous $\omega_1$ chain $M_\alpha \preceq H_\kappa$ of countable elementary substructures satisfying:

1. $(T,c) \in M_0$ and $x \in M_0$,
2. $M_\alpha \in M_{\alpha+1}$
3. $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for $\lambda < \omega_1$ limit.

Then automatically $Q = \{M_\beta \cap \omega_1 : \beta < \omega_1\}$ will be a closed unbounded set. Now if $\alpha = M_\beta \cap \omega_1$. Since $(T,c)$ and $x$ are both in $M_\beta$ it is easy to see by induction on rank that for any $s \in T_\alpha$ that

$$x \in H(s,(T_\alpha,c_\alpha)) \text{ iff } x \in H(s,(T,c)).$$

This proves the Claim.

To prove the Theorem suppose for contradiction that $Z = C$ where $C$ is $\omega_1$-Borel and let $(C_\alpha : \alpha < \omega_1)$ be given by the Claim. If the cofinality of $\mathfrak{c}$ is greater $\omega_1$ then for some $\beta < \mathfrak{c}$ we have that $\{C_\alpha : \alpha < \omega_1\} \subseteq \{B_\alpha : \alpha < \beta\}$. But $Z_\beta \neq X_\beta \cap C_\alpha$ for every $\alpha < \omega_1$ which is a contradiction.
$\square$

# 3 The order of a separable metric space

**Theorem 3.1** *(Miller [9] Thm 22) It is relatively consistent with ZFC that $ord(X) = \omega_1$ for all uncountable $X \subseteq 2^\omega$.*

It also consistent with any cardinal arithmetic that for every $\alpha$ with $2 \leq \alpha \leq \omega_1$ there is an uncountable $X \subseteq 2^\omega$ with $\text{ord}(X) = \alpha$. One can also have model where these orders are precisely the interval $[\alpha_0 + 1, \omega_1]$.

The model for this theorem is the $\omega_2$ finite support iteration of the direct sum $\mathbb{P}$ of $\Pi_\alpha$-forcings for $\alpha < \omega_1$.

**Definition 3.2** *Nice $\alpha$ tree. For $2 \leq \alpha < \omega_1$ we define $T \subseteq \omega^{<\omega}$ to be a nice $\alpha$-tree iff*

1. *it is well-founded tree of rank $\alpha$, i.e., $rank_T(\langle\rangle) = \alpha$*
2. *for any $s \in T$ with $rank_T(s) > 0$ we have that $s\hat{\ }n \in T$ for all $n < \omega$*
3. *if $rank_T(s) = \beta + 1$, then $rank_T(s\hat{\ }n) = \beta$ for all $n$*
4. *if $rank_T(s) = \lambda$ a limit ordinal, then for any $\beta < \lambda$ there are at most finitely many $n$ with $rank_T(s\hat{\ }n) \leq \beta$.*

**Definition 3.3** *Fix a nice $\alpha$-tree $T$, let $T^*$ be the terminal nodes of $T$, i.e., those of rank zero, and let $T^+ = T \setminus T^*$. $\Pi_\alpha$-forcing is the poset $\mathbb{P}_\alpha$ defined as follows: $p \in \mathbb{P}_\alpha$ iff there are finite $p_0 \subseteq T^* \times 2^{<\omega}$ and $p_1 \subseteq T^+ \times 2^\omega$ such that $p = p_0 \cup p_1$ and $p_0$ is the graph of a partial function and they are consistent. Consistency means that*

- *if $(s, x), (s\hat{\ }n, y) \in p_1$ then $x \neq y$ and*
- *if $(s, x) \in p_1$ and $(s\hat{\ }n, t) \in p_0$, then $x \notin [t]$.*

We think of our conditions as attaching elements of $2^\omega$ to nodes of the tree $T$ subject to the condition that no $x$ is attached to immediately adjacent nodes, i.e., $s$ and $s\hat{\ }n$ are immediately adjacent. We think of $(s\hat{\ }n, t) \in p_0$ (or equivalently $p_0(s\hat{\ }n) = t$ since it is a partial function) as attaching all elements of the clopen set $[t] = \{x \in 2^\omega \ : \ t \subseteq x\}$ to the terminal node $s\hat{\ }n \in T^*$.

**Lemma 3.4** *For any countable ordinal $\alpha_0$, $\Pi_{\alpha_0}$-forcing $\mathbb{P}_{\alpha_0}$ has the countable chain condition.*

proof:

Suppose $A$ is uncountable antichain. Since there are only countably many different $p_0$ without loss we may assume that there exists $r$ such that $p_0 = r$ for all $p \in A$. Consequently for $p, q \in A$ the only thing that can keep $p \cup q$ from being a condition is that there must be an $x \in 2^\omega$ and an $s, s\hat{\ }n \in T^+$ such that

$$(s, x), (s\hat{\ }n, x) \in p \cup q.$$

But now for each $p \in A$ let $H_p : X \to [T^+]^{<\omega}$ be the finite partial function defined by

$$H_p(x) = \{s \in T^+ : (s, x) \in p_1\}$$

where $X = \{x : \exists s \in T^+ \ (s, x) \in p_1\}$. Then $\{H_p : p \in A\}$ is an uncountable antichain in the order of finite partial functions from $2^\omega$ to $[T^+]^{<\omega}$. But this is impossible.

□

It is easy to see that it has property K, in fact, it is $\sigma$-centered, so ccc productive. It follows that the direct sum

$$\mathbb{P} =^{def} \sum \{\mathbb{P}_{\alpha+1} \quad : \quad 2 \leq \alpha < \omega_1\}$$

also has the countable chain condition.

To obtain our model for Theorem 3.1 start with a countable transitive model of ZFC, $M$, and do a finite support iteration of $\mathbb{P}$ of length $\omega_2$, denoted $\mathbb{P}^{\omega_2}$. We let $\mathbb{P}^\alpha$ be the iteration of $\mathbb{P}$ up to length $\alpha$.

Fix $T$ a nice $\alpha$-tree for some $\alpha$. Given $G$ $\mathbb{P}_\alpha$-generic over $M$, for each $s \in T$ define $U_s^G \subseteq 2^\omega$. For $s \in T^*$ define $U_s^G = [t]$ if there exists $p \in G$ such that $p_0(s) = t$. Note that by genericity such a $t$ will always exist, i.e., for any $q$ there exists $p \leq q$ such that $s$ is in the domain of $p$. For $s \in T^+$ define

$$U_s^G = \bigcap_{n<\omega} \sim U_{s\hat{\ }n}^G.$$

Note that $U_s^G$ is a $\underset{\sim}{\mathbf{\Pi}}_\alpha^0$-set where $\alpha = \text{rank}_T(s)$.

**Lemma 3.5** *For any $x \in 2^\omega \cap M$ and $s \in T^+$ and $G$ $\mathbb{P}_\alpha$-generic over $M$*

$$x \in U_s^G \textit{ iff } \exists p \in G \ \ (s, x) \in p$$

proof:

To simplify notation write $(s,x) \in G$ instead of $\{(s,x)\} \in G$ or there exists $p \in G$ with $(s,x) \in p$.

Fix $x$ and $s$. It is easy to see that the following sets are dense:
(if $\text{rank}_T(s) = 1$) $D = \{p \in \mathbb{P}_\alpha \ : \ (s,x) \in p \text{ or } \exists n,t \ \ p_0(s\hat{\ }n) = t \text{ and } x \in [t]\}$
(if $\text{rank}_T(s) > 1$) $E = \{p \in \mathbb{P}_\alpha \ : \ (s,x) \in p \text{ or } \exists n \ (s\hat{\ }n, x) \in p\}$
By definition $x \in U_s^G$ iff $x \notin U_{s\hat{\ }n}^G$ for all $n$.
If $\text{rank}_T(s) > 1$, then by induction:

$x \notin U_{s\hat{\ }n}^G$ for all $n$ iff $(s\hat{\ }n, x) \notin G$ for all $n$

Since $E$ is dense:

$(s\hat{\ }n, x) \notin G$ for all $n$ iff $(s,x) \in G$.

If $\text{rank}_T(s) = 1$, then

$x \notin U_{s\hat{\ }n}^G$ for all $n$ iff for all $p \in G$ if $p_0(s\hat{\ }n) = t$ then $x \notin [t]$.

Since $D$ is dense:

for all $p \in G$ if $p_0(s\hat{\ }n) = t$ then $x \notin [t]$ iff $(s,x) \in G$.

□

**Definition 3.6** *Canonical names. For any poset $\mathbb{P}$ the canonical names $CN(\mathbb{P})$ for an element of $2^\omega$ are defined as follows. $\tau \in CN(\mathbb{P})$ iff there exists $(A_n^0, A_n^1 \ : \ n < \omega)$ such that $A_n^0 \cup A_n^1 \subseteq \mathbb{P}$ is a maximal antichain and*

$$\tau = \{(p, (n,i)\check{}) \ : \ n < \omega, i < 2, \ \text{and } p \in A_n^i\}.$$

**Definition 3.7** *Nice conditions. Let $T_\alpha$ for $2 \leq \alpha < \omega_1$ be the nice $\alpha$-trees used to define $\Pi_\alpha$-forcing $\mathbb{P}_\alpha$. $\mathbb{P}$ For $p \in \mathbb{P}^\alpha$ we say the $p$ is nice iff $p(0) \in \mathbb{P}$ and for all $\gamma$ with $0 < \gamma < \alpha$ $p(\gamma)$ is a name for a finite set $p_0 \cup p_1$ where*

$$p_0 : \bigcup_{2 \leq \alpha < \omega_1} (\{\alpha+1\} \times T_{\alpha+1}^\star) \to 2^{<\omega}$$

*is a finite partial map and*

$$p_1 \subseteq \bigcup_{2 \leq \alpha < \omega_1} \{\alpha+1\} \times (T_{\alpha+1}^+ \times CN(\mathbb{P}_\gamma))$$

*is finite. For every $\tau$ in the range of $p_1$ there is a $t_\tau \in 2^{<\omega}$ such that $p \restriction \gamma \Vdash t_\tau \subseteq \tau$ and for all $\alpha, s, n, \tau, \sigma, r$*

1. *if $(\alpha, (s,\tau)), (\alpha, (sn, \sigma)) \in p_1$ then $t_\tau \perp t_\sigma$*

*2. if $(\alpha,(s,\tau)) \in p_1$ and $p_0(\alpha, sn) = r \in 2^{<\omega}$, then $t_\tau \perp r$.*

The nice conditions are dense so from now on we will assume all conditions are nice.

**Definition 3.8** *Rank. Given $H \subseteq 2^\omega$, nice $p \in \mathbb{P}^\alpha$, and $\tau \in CN(\mathbb{P}^\alpha)$ define rank$(p,H)$ and rank$(\tau,H,p)$ by induction on $\alpha$.*

*1. For $\mathbb{P}_\alpha$ ($\Pi_\alpha$-forcing), if $p = p_0 \cup p_1 \in \mathbb{P}_\alpha$ then*

$$\mathit{rank}(p,H) = \max\{\mathit{rank}_{T_\alpha}(s) \; : \; \exists x \in 2^\omega \setminus H \;\; (s,x) \in p_1\}$$

*2. For $p \in \mathbb{P} = \sum\{\mathbb{P}_{\alpha+1} \;\; : \;\; 2 \leq \alpha < \omega_1\}$*

$$\mathit{rank}(p,H) = \max\{\mathit{rank}(p_\alpha,H) \; : \; \alpha < \omega_1\}$$

*3. For $\gamma$ a limit ordinal and $p \in \mathbb{P}^\gamma$*

$$\mathit{rank}(p,H) = \max\{\mathit{rank}(p \restriction \beta, H) \; : \; \beta < \gamma\}.$$

*4. For $\tau \in CN(\mathbb{P}^\gamma)$ and $p \in \mathbb{P}^\gamma$, rank$(\tau,H,p)$ is the least $\beta$ such that for every $n < \omega$*

$$Q =^{def} \{q \in \mathbb{P}^\gamma \; : \; q \perp p \text{ or } \mathit{rank}(q,H) \leq \beta\} \text{ decides “}\tau(n) = 0\text{”}.$$

*By which we mean*

$$\{r \in \mathbb{P}^\gamma \; : \; \exists q \in Q \quad r \leq q \text{ and } (q \Vdash \tau(n) = 0 \text{ or } q \Vdash \tau(n) = 1)\}$$

*is dense in $\mathbb{P}^\gamma$.*

*5. For $p \in \mathbb{P}^\gamma \star \overset{\circ}{\mathbb{P}}$ $\qquad N_p(\gamma) = \{\tau \; : \; \exists \alpha < \omega_1 \, \exists s \in 2^{<\omega} \; (\alpha,(s,\tau) \in p(\gamma)\}$*

$$\mathit{rank}(p,H) = \max\{\mathit{rank}(p \restriction \gamma, H), \mathit{rank}(\tau,H,p \restriction \gamma) \; : \; \tau \in N_p(\gamma)\}.$$

A set of conditions $Q$ decides $\theta$ iff for every generic $G$ there is a $q \in Q \cap G$ which forces $\theta$ or forces $\neg\theta$.

**Lemma 3.9** *(ccc) For any countable $P \subseteq \mathbb{P}^\gamma$ and countable $N \subseteq CN(\mathbb{P}^\gamma)$ there is a countable $H \subseteq 2^\omega$ such that rank$(p,H) = 0$ for every $p \in P$ and rank$(\tau,H,1) = 0$ for every $\tau \in Q$.*

proof:

Let $\mathbb{P}^\gamma, P, Q \in \mathcal{N} \preceq H_\kappa$ where $\mathcal{N}$ is a countable elementary substructure of the hereditarily of cardinality less than $\kappa$ sets for some sufficiently large regular $\kappa$. Take $H = \mathcal{N} \cap 2^\omega$. Then for any $p \in \mathcal{N} \cap \mathbb{P}^\gamma$ and $\tau \in \mathcal{N} \cap \mathrm{CN}(\mathbb{P}^\gamma)$ we have that $\mathrm{rank}(p, H) = 0$ and $\mathrm{rank}(\tau, H, 1) = 0$.

$\square$

**Definition 3.10** $|\cdot|$ *abbreviations for rank. For the next two lemmas (meet 3.11 and rank 3.12) we will fix a countable $H \subseteq 2^\omega$ and use the abbreviations:*

$$|p| = rank(p, H) \quad and \quad |\tau|(p) = rank(\tau, H, p).$$

**Lemma 3.11** *(Meet lemma) If $G$ is $\mathbb{P}^\alpha$-generic over $M$ and $(q_i \in G : i < N)$ is a finite set with $|q_i| < \beta$, then there exists $q \in G$ with $|q| < \beta$ and $q \leq q_i$ for all $i < N$.*

proof:

Case $\alpha = 0$ hence $\mathbb{P}^0 = \mathbb{P}$. Let $q = \bigcup_{i<N} q_i$. By definition of generic filter $\exists p \in G$ with $p \leq q_i$ each $i < N$. Hence $p \leq q$ and so $q \in G$.

Case $\alpha$ a limit ordinal. There exists $\alpha_0 < \alpha$ with $\mathrm{supp}(q_i) \subseteq \alpha_0$ for $i < N$. By induction hypothesis there is $q \restriction \alpha_0 \in G_{\alpha_0}$ with $q \restriction \alpha_0 \leq q_i \restriction \alpha_0$ for each $i < N$ and $|q \restriction \alpha_0| < \beta$. Let $q \restriction [\alpha_0, \alpha)$ be identically 1.

Case $\alpha + 1$ successor. Suppose $q_i \in G_{\alpha+1} = G_\alpha \star G^\alpha$ which is $\mathbb{P}_\alpha \star \overset{\circ}{\mathbb{P}}$-generic. Let $\Gamma \subseteq G_\alpha$ be finite so that $|r| < \beta$ for all $r \in \Gamma$ and for any $\tau_1, \tau_2, s, n$ if $(s, \tau_1), (sn, \tau_2) \in \bigcup_{i<N} q_i(\alpha)$ then there exists $r \in \Gamma$ such that $r \Vdash \tau_1 \neq \tau_2$ and similarly, if $(s, \tau) \in \bigcup_{i<N} q_i(\alpha)$ and $p_0^{q_i}(sn) = t$ for some $i$, then there exists $r \in \Gamma$ such that $r \Vdash t \not\subseteq \tau$. By inductive hypothesis there is $q \restriction \alpha \in G_\alpha$ with $|q \restriction \alpha| < \beta$ and $q \restriction \alpha \leq q_i \restriction \alpha$ for $i < N$ and $q \restriction \alpha \leq r$ for each $r \in \Gamma$. Then $(q \restriction \alpha, \bigcup_{i<N} q_i(\alpha))$ satisfies the Lemma.

$\square$

**Lemma 3.12** *(Rank Lemma for $\mathbb{P}^\alpha$) For every $\beta \geq 1$ and $p \in \mathbb{P}^\alpha$ there exists $\hat{p} \in \mathbb{P}^\alpha$ compatible with $p$, $|\hat{p}| \leq \beta$, and for every $q \in \mathbb{P}^\alpha$ with $|q| < \beta$, if $q, \hat{p}$ compatible, then $q, p$ compatible.*

proof:

The nontrivial case for this is $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}$.

Assume $p_0 \in \mathbb{P}^\alpha$, $|p_0| \leq \beta$, $p_0 \Vdash$"$\overset{\circ}{C} \subseteq 2^\omega$ closed nonempty" and for every $G$ $\mathbb{P}^\alpha$-generic over $M$ with $p_0 \in G$ for every $s \in 2^{<\omega}$ if $M[G] \models [s] \cap C = \emptyset$

then there is $q \in G$ with $|q| < \beta$ such that $q \Vdash [s] \cap \overset{\circ}{C} = \emptyset$. We prove the following two Claims.

Claim 1. For all $G$ $\mathbb{P}^\alpha$-generic over $M$ with $p_0 \in G$ for every $s \in 2^{<\omega}$ if $M[G] \models [s] \cap C \neq \emptyset$ then there is $p \in G$ with $|p| \leq \beta$ such that $p \Vdash [s] \cap \overset{\circ}{C} \neq \emptyset$.
proof:

Let $p_1 \in G$ with $p_1 \leq p_0$ and $p_1 \Vdash [s] \cap \overset{\circ}{C} \neq \emptyset$. Define

$$D_{p_1} = \{p \in \mathbb{P}^\alpha \ : \ \exists p_2 \quad p \leq p_2 \leq p_1 \text{ and } p \leq \hat{p_2}\}$$

where $\hat{p_2}$ is the condition with $|\hat{p_2}| \leq \beta$ given by the induction hypothesis of the Rank Lemma. $D_{p_1}$ is dense beneath $p_1$ and so it meets $G$. Hence there exists $p_2$ with $p_2, \hat{p_2}$ both in $G$. Apply the Meet Lemma to get $p \in G$ with $|p| \leq \beta$ and $p \leq p_0$ and $p \leq \hat{p_2}$. To see that $p$ works let $G_0$ be $\mathbb{P}^\alpha$ generic with $p \in G_0$. Suppose for contradiction that in $M[G_0]$ that $[s] \cap C = \emptyset$. Then since $p_0 \in G_0$ we have by assumption that for some $q \in G_0$ with $|q| < \beta$ that $q \Vdash [s] \cap \overset{\circ}{C} = \emptyset$. Since $\hat{p_2} \in G_0$ it is compatible with $q$. But by the definition of $\hat{p_2}$ as witness to Meet Lemma for $p_2$, it follows that $q$ and $p_2$ are compatible. But they are not compatible since $p_2 \leq p_1$ and $p_1 \Vdash [s] \cap \overset{\circ}{C} \neq \emptyset$.
$\square$

Claim 2. Let $p_0 \Vdash$"$\overset{\circ}{x}$ is the lexicographically least element of $\overset{\circ}{C}$. Then $| \overset{\circ}{x} | \leq \beta$.
proof:

Let $G$ be $\mathbb{P}^\alpha$-generic over $M$ with $p_0 \in G$. Fix any $N < \omega$ and let $x \upharpoonright N = s$. Hence $[s] \cap C \neq \emptyset$. Furthermore, if $n_i$ for $i < k$ lists all $n < N$ with $s(n) = 1$ then $[t_i] \cap C = \emptyset$ where $t_i(n_i) = 0$ and $t_i \upharpoonright n_i = s \upharpoonright n_i$. By Claim 1, there is $p_1 \in G$ with $|p_1| \leq \beta$ such that $p_1 \Vdash [s] \cap \overset{\circ}{C} \neq \emptyset$ and by assumption $q_i \in G$ for $i < k$ with $|q_i| < \beta$ such that $q_i \Vdash [t_i] \cap \overset{\circ}{C} = \emptyset$. By the Meet Lemma there is $p \in G$ with $|p| \leq \beta$ which extends $p_1$ and $q_i$ for $i < k$. Then $p \Vdash \overset{\circ}{x} \in [s]$. Since $N$ and $G$ were arbitrary $| \overset{\circ}{x} | \leq \beta$.
$\square$

Given $\tau$ a $\mathbb{P}^\alpha$-name for an element of $2^\omega$, $G$ $\mathbb{P}^\alpha$-generic over $M$, and $p \in \mathbb{P}^\alpha$ (not necessarily in $G$) let $C(\tau, p) \subseteq 2^\omega$ be the following closed set in $M[G]$:

$$C(\tau, p) = \bigcap \{K_{\hat{\tau}} \ : \ \exists q \in G \ |q| < \beta, \ |\hat{\tau}|(q) < \beta, \text{ and } p \cup q \Vdash \tau \in K_{\hat{\tau}}\}$$

By $p \cup q \Vdash \theta$ we mean that they are compatible and $r \Vdash \theta$ for every $r \leq p, q$. Here $K \subseteq 2^\omega \times 2^\omega$ is the standard universal closed set and $K_x$ is the cross section.

Given any $p \in \mathbb{P}^\alpha$ let $p_0 = \hat{p}$ be given by the inductive hypothesis for the Rank Lemma and $\tau$ a name for an element of $2^\omega$. Let $\overset{\circ}{C}$ be a name for the closed set $C(\tau, p)$. Then

Claim 3. $p_0, \overset{\circ}{C}$ satisfy the assumption of Claim 1 and 2, namely

(a) $p_0 \Vdash \overset{\circ}{C} \subseteq 2^\omega$ is closed and nonempty

(b) for any $G$ $\mathbb{P}^\alpha$-generic over $M$ with $p_0 \in G$ for any $s \in 2^{<\omega}$ then if $[s] \cap C = \emptyset$ in $M[G]$, then there is $q \in G$ with $|q| < \beta$ such that $q \Vdash [s] \cap \overset{\circ}{C} = \emptyset$.

proof:

We verify (b) first. Suppose $[s] \cap C = \emptyset$ in $M[G]$. By compactness there are finitely many $q_i, \tau_i$ for $i < k$ with $q_i \in G$, $|q_i| < \beta$, $|\tau_i|(q_i) < \beta$, each $q_i$ compatible with $p$, $p \cup q_i \Vdash \tau \in K_{\tau_i}$, and

$$\bigcap_{i<k} K_{\tau_i} \cap [s] = \emptyset$$

We claim that for some $N < \omega$ for all $(x_i \in [\tau_i^G \upharpoonright N] : i < k)$ that

$$\bigcap_{i<k} K_{x_i} \cap [s] = \emptyset$$

Otherwise for each $N$ choose $(x_i^N \in [\tau_i^G \upharpoonright N] : i < k)$ and $y_N \in [s] \bigcap_{i<k} K_{x_i^N} \cap [s]$. But by compactness a subsequence of the $y^N$ converges to some $y \in [s]$ and we get $(y, \tau_i^G) \in K$ for each $i < k$.

Let $s_i = \tau_i^G$. By the Meet Lemma and the definition of $|\tau_i|(q_i) < \beta$ there is a $q \in G$ with $|q| < \beta$ such that $q \Vdash \tau_i \upharpoonright N = s_i$ for all $i < k$. It follows that $q \Vdash [s] \cap \overset{\circ}{C} = \emptyset$.

To prove (a) that $C$ is nonempty, suppose for contradiction that for some $G$ $\mathbb{P}^\alpha$-generic over $M$ with $p_0 \in G$ we have that $C$ is empty. Then apply part (b) for $s = \langle\rangle$ the empty sequence. Then there is $q \in G$ with $|q| < \beta$ and

$q \Vdash \overset{\circ}{C} = \emptyset$. But $p_0 = \hat{p}$ so since $\hat{p}, q$ are compatible, $p, q$ are compatible. But this is impossible because $p \Vdash \tau \in \overset{\circ}{C}$ so it cannot be empty.
□

Before getting to the proof of the Rank Lemma for $\mathbb{P}^{\alpha+1}$ we note some properties of the universal $\Pi^0_1$ set $K \subseteq 2^\omega \times 2^\omega$. First of all it is easier to think in terms of its complement $U = \sim K$ which is universal for open sets. Let $\{s_n : n < \omega\} = 2^{<\omega}$ be a recursive listing and put

$$y \in U_x \text{ iff } \exists n \ (x(n) = 1 \text{ and } s_n \subseteq y)$$

For each $n < \omega$ there is a recursive level preserving map $f : (2^\omega)^n \to 2^\omega$ such that for any sequence $(x_i \in 2^\omega : i < n)$ if $f(x_i \in 2^\omega : i < n) = y$ then $U_y = \cap_{i<n} U_{x_i}$ and hence $K_y = \cup_{i<n} K_{x_i}$. Simply define $f(\vec{x}) = y$ by

$$y(m) = 1 \text{ iff } \quad \forall i < n \ \exists l < m \ \ (x_i(l) = 1 \text{ and } s_l \subseteq s_m)$$

Note also that $(2^\omega)^n$ is natural homeomorphic to $2^\omega$ via a recursive join operator and we use $\vec{x}$ to denote this element of $2^\omega$.

So given $p \in \mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \overset{\circ}{\mathbb{P}}$ and $\beta \geq 1$ we let $p = (p \restriction \alpha, p(\alpha))$. We may assume that $p(\alpha) = p_0 \cup p_1$ where $p_0 : T^* \to 2^{<\omega}$ is a finite partial map and $p_1$ is a finite subset of $T^0 \times \mathcal{N}$ where $\mathcal{N}$ are $\mathbb{P}^\alpha$ names for elements of $2^\omega$ and $T^*$ are the terminal nodes of $T$ and $T^0$ are the nonterminal nonroot nodes. In addition we may assume for each $(s, \tau) \in p_1$ there is a $t \in 2^{<\omega}$ such that $p \Vdash t \subseteq \tau$ and the $t$ witness that $p_1$ is a condition, namely

- if $(s, \tau_1), (sn, \tau_2) \in p_1$ then $t_1 \perp t_2$
- if $(s, \tau) \in p_1$ and $p_0(sn) = r$ then $t \perp r$

We write $p_1 = p_1(\vec{\tau})$ where $\vec{\tau}$ is an $n$-tuple list all $\tau$ mentioned in $p_1$. To get $\hat{p}$ for the Rank Lemma for $\mathbb{P}^{\alpha+1}$ let

$$\hat{p} =^{def} ( \ \widehat{p \restriction \alpha} \, , \, p_0 \cup p_1(\vec{\tau}_{lex}) \ )$$

where $\vec{\tau}_{lex}$ is a $\mathbb{P}^\alpha$ name for the lexicographically least element of $\overset{\circ}{C} (p \restriction \alpha, \vec{\tau})$.

By the claims $|\hat{p}| \leq \beta$. Note that $C \subseteq \prod_{i<n}[t_i]$ and so $\vec{\tau}_{lex} \in \prod_{i<n}[t_i]$ and so $\hat{p}$ and $p$ are compatible. Finally we need to show that if $|q| < \beta$ and $p, q$ incompatible, then $\hat{p}, q$ incompatible. Suppose $p, q$ incompatible. If

$p \restriction \alpha,\ q \restriction \alpha$ incompatible we are done by inductive choice of $\widehat{p \restriction \alpha}$. So we may assume that they are compatible but

$$p \restriction \alpha \cup q \restriction \alpha \Vdash p(\alpha) \perp q(\alpha)$$

Let $q(\alpha) = q_0 \cup q_1$ where the names occurring in $q_1$ have rank $< \beta$ with respect to $q \restriction \alpha$. Now we detail how the incompatibility $p(\alpha) \perp q(\alpha)$ translates into closed sets. We may construct $\Sigma$ a finite set of names for elements of $2^\omega$ such that

- $|\rho|(q \restriction \alpha) < \beta$ for each $\rho \in \Sigma$
- if $(s, \tau_i) \in p_1$ and $q_0(sn) = r$, then for some $\rho \in \Sigma$
$$K_\rho = \{\vec{x} \ : \ x_i \in [r]\}$$
- if $(s, \sigma) \in q_1$ and $p_0(sn) = r$, then some $\rho \in \Sigma$ is name such that for any generic $G$
$$K_{\rho^G} = \begin{cases} \emptyset & \text{if } \sigma^G \notin [r] \\ (2^\omega)^n & \text{otherwise} \end{cases}$$
- if $((s, \sigma) \in q_1$ and $(sn, \tau_i) \in p_1)$ or $((sn, \sigma) \in q_1$ and $(s, \tau_i) \in p_1)$, then some $\rho \in \Sigma$ is name such that for any generic $G$
$$K_{\rho^G} = \{\vec{x} \ : \ x_i = \sigma^G\}$$

We assume that all $\rho$ in $\Sigma$ arise from the above requirements and let $\Sigma = \{\rho_i \ : \ i < N\}$. Then we have that

$$(p \restriction \alpha \cup q \restriction \alpha) \Vdash p(\alpha) \perp q(\alpha) \quad \text{iff} \quad (p \restriction \alpha \cup q \restriction \alpha) \Vdash \vec{\tau} \in \bigcup_{i<N} K_{\rho_i}$$

Letting $\rho = f(\vec{\rho})$ we get that $|\rho|(q \restriction \alpha) < \beta$ and

$$(\widehat{p \restriction \alpha} \cup q \restriction \alpha) \Vdash \vec{\tau}_{lex} \in C(\vec{\tau}, p \restriction \alpha) \subseteq K_\rho = \bigcup_{i<N} K_{\rho_i}.$$

It follows that $\hat{p}, q$ are incompatible and the Rank-Lemma successor case is proved.
$\square$

Here is the main point of the Rank Lemma:

**Proposition 3.13** *Suppose $\tau \in CN(\mathbb{P}^{\omega_2})$ with $|\tau|(1) = 0$, $p \in \mathbb{P}^{\omega_2}$, and $B(v)$ a $\underset{\sim}{\mathbf{\Pi}}^0_\beta$-set coded in the ground model $M$ such that $p \Vdash B(\tau)$. If $\hat{p}$ is given by the Rank Lemma ($\hat{p}$ compatible with $p$, $|\hat{p}| \leq \beta$, and for every $q \in \mathbb{P}^{\omega_2}$ with $|q| < \beta$, if $q, \hat{p}$ compatible, then $q, p$ compatible), then $\hat{p} \Vdash B(\tau)$.*

proof:

Case $\beta = 0$. This is true by the definition of $|\tau|(1) = 0$.

Case $\beta > 0$. Let $B(v) = \bigwedge_{n<\omega} B_n(v)$ where $B_n(v)$ $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta_n}$ for some $\beta_n < \beta$. If for contradiction $\hat{p}$ does not force $B(\tau)$, then there exists $q \leq \hat{p}$ and $n < \omega$ such that $q \Vdash \neg B_n$. By induction there exists $\hat{q}$ compatible with $q$, $|\hat{q}| \leq \beta_n$ and $\hat{q} \Vdash \neg B_n$. Since $q$ extends $\hat{p}$ it follows that $\hat{q}, \hat{p}$ are compatible. Since $|\hat{q}| \leq \beta_n < \beta$ we have that $\hat{q}$ is compatible with $p$. This is a contradiction since $p \Vdash B_n(\tau)$.

□

**Proposition 3.14** *Suppose $G$ $\mathbb{P}^{\omega_1}$-generic over $M$. Then for any $Y \in M$ and $\alpha < \omega_1^M$ if $M \models Y \subseteq 2^\omega$ and $|Y| = \omega_1$, then*

$$M[G] \models \forall \alpha < \omega \; \forall V \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha \quad V \cap Y \neq U^{G_0}_{\alpha,\langle\rangle} \cap Y.$$

*Here $U^{G_0}_{\alpha,\langle\rangle}$ is the generic $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set added by the first coordinate's $\Pi_\alpha$-forcing, namely*

$$U^{G_0}_{\alpha,\langle\rangle} = \{x \in 2^\omega \cap M \;:\; \exists p \in G \; (\alpha, (\langle\rangle, x)) \in p(0)\}.$$

proof:

Let $V$ be a universal $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$-set coded in $M$. Suppose

$$p_0 \Vdash \forall y \in Y \;\; (y \in V_\tau \text{ iff } y \in U^{G_0}_{\alpha,\langle\rangle})$$

Using ccc Lemma 3.9 choose $H \subseteq 2^\omega$ countable so that $|p| = \text{rank}(p, H) = 0$ and $|\tau| = \text{rank}(\tau, H, 1) = 0$. Take any $y \in Y \setminus H$. Let $p_1(0) = p(0) \cup (\alpha, (\langle\rangle, y))$ and $p_1 \upharpoonright [1, \omega_2) = p_0 \upharpoonright [1, \omega_2)$, Note that $p_1 \Vdash y \in U^{G_0}_{\alpha,\langle\rangle}$ and hence $p_1 \Vdash y \in V_\tau$. Now $B(v) =^{def} y \in V_v$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ predicate coded in $M$. Let $B(v) = \bigvee_{n<\omega} B_n(v)$ where $B_n(v)$ is a $\underset{\sim}{\mathbf{\Pi}}^0_{\beta_n}$ predicate with $\beta_n < \beta$. Find $p \leq p_1$ and $n < \omega$ such that $p \Vdash B_n(\tau)$. By Proposition 3.13 there is a condition $\hat{p}$ compatible with $p$ such that $\hat{p} \Vdash B_n(\tau)$ and $\text{rank}(\hat{p}, H) \leq \beta_n < \beta$. By using the meet Lemma 3.11 we may assume $\hat{p} \leq p_0$. Hence it follows $\hat{p} \Vdash yy \in U^{G_0}_{\alpha,\langle\rangle}$. But by the definition of rank since $y \notin H$, there is some sufficiently large $m < \omega$ such that $r(0) = \hat{p}(0) \cup (\alpha, (\langle m \rangle, y))$ is consistent. Letting $r \restriction [1, \omega_2) = \hat{p} \restriction [1, \omega_2)$ leads to a contradiction:

$$r \Vdash B_n(\tau)$$

$$r \Vdash (\bigvee_{n<\omega} B_n(\tau)) \text{ iff } y \in U^{G_0}_{\alpha,\langle\rangle}$$

$$r \Vdash y \notin U^{G_0}_{\alpha,\langle\rangle}$$

□

**Claim**. $M[G] \models \forall Y \in [2^\omega]^{\omega_1} \ \ \mathrm{ord}(Y) = \omega_1$.

This follows from two standard facts:

1. $\forall Z \in \mathcal{P}(\omega_1)^{M[G]} \ \ \exists \alpha_0 < \omega_2^M \ \ Z \in M[G_{\alpha_0}]$
2. $\forall G \ \ (\mathbb{P}^{\omega_2})^M$ generic over $M \quad \forall \alpha_0 < \omega_2^M \ \exists H \ \ (\mathbb{P}^{\omega_2})^{M[G_{\alpha_0}]}$ generic over $M[G_{\alpha_0}]$ such that $M[G] = M[G_{\alpha_0}][H]$.

This concludes the proof of Theorem 3.1.

# 4 The sigma-algebra of abstract rectangles

**Theorem 4.1** *(Rao 1968 [21], Kunen [7]) Assume the continuum hypothesis then every subset of the plane is in the $\sigma$-algebra generated by the abstract rectangles. In fact, at level two.*

proof:

It is enough to see that $\mathcal{P}(\omega_1 \times \omega_1) = \sigma\{A \times B \ : \ A, B \subseteq \omega_1\}$, i.e. every subset of $\omega_1 \times \omega_1$ is in the $\sigma$-algebra generated by the abstract rectangles.

**Definition 4.2**

- $\mathcal{R} = \{A \times B \ : \ A, B \subseteq \omega_1\}$
- $\Sigma_0(\mathcal{R}) = \Pi_0(\mathcal{R}) = \mathcal{R}$
- $\Pi_\alpha(\mathcal{R}) = \{\omega_1 \times \omega_1 \backslash P \ : \ P \in \Sigma_\alpha(\mathcal{R})\}$
- $\Sigma_\alpha(\mathcal{R}) = \{\bigcup_{n<\omega} P_n \ : \ P_n \in \bigcup_{\beta<\alpha} \Pi_\beta(\mathcal{R})\}$
- $\sigma\{A \times B \ : \ A, B \subseteq \omega_1\} = \sigma\mathcal{R} = \bigcup_{\alpha<\omega_1} \Sigma_\alpha(\mathcal{R}) = \bigcup_{\alpha<\omega_1} \Pi_\alpha(\mathcal{R})$

- *bool($\mathcal{R}$) = smallest family containing $\mathcal{R}$ and closed under finite union and complementation*

Note that bool($\mathcal{R}$) $\subseteq \Sigma_1(\mathcal{R}) \cap \Pi_1(\mathcal{R})$. Also $\Sigma_\alpha(\mathcal{R})$ for $\alpha > 0$ is closed under countable union and finite intersection.

**Lemma 4.3** *For $f : 2^\omega \to 2^\omega$, the graph($f$) $\in \Pi_1(\{A \times B \ : \ A, B \subseteq 2^\omega\})$.*

proof:
For any $s \in 2^{<\omega}$ let $D_s = f^{-1}([s])$. Then the following are equivalent for any $x, y \in 2^\omega$:

- $f(x) = y$
- $\forall s(s \subseteq f(x)$ iff $s \subseteq y)$
- $\forall s(x \in A_s$ iff $y \in [s]$
- $(x, y) \in \bigcap_{s \in 2^{<\omega}} (D_s \times [s]) \cup (\sim D_s \times \sim [s])$

□

Note that the Lemma is also true for any partial function $f : D \to 2^\omega$ for some $D \subseteq 2^\omega$. Since if $\hat{f} \supseteq f$ is total, then

$$\text{graph}(f) = \text{graph}(\hat{f}) \cap (D \times 2^\omega) \in \Pi_1(\mathcal{R}).$$

Now we prove Theorem 4.1 that $\mathcal{P}(\omega_1 \times \omega_1) = \Sigma_2(\mathcal{R})$. Suppose $A$ is a subset of $\omega_1 \times \omega_1$ with the property that $\beta \leq \alpha$ for every $(\alpha, \beta) \in A$. Let $f_n : \omega_1 \to \omega_1$ be partial functions for $n < \omega$ so that for any $\alpha < \omega_1$

$$\{\beta \ : \ (\alpha, \beta) \in A\} = \{f_n(\alpha) \ : \ n < \omega\}.$$

It follows that $A = \bigcup_{n<\omega} \text{graph}(f_n)$ is $\Sigma_2(\mathcal{R})$. Now any subset of $\omega_1 \times \omega_1$ can be written as a union $A \cup B$ where $B$ has the property that $\alpha \leq \beta$ for any $(\alpha, \beta) \in B$. By symmetry $B \in \Sigma_2(\mathcal{R})$ and so $(A \cup B) \in \Sigma_2(\mathcal{R})$.
□

**Theorem 4.4** *(Kunen [7] 1968) Assume Martin's axiom, then every subset of the plane is in the $\sigma$-algebra generated by the abstract rectangles at level two. In the Cohen real model or the random real model the well-ordering on the continuum is not in the $\sigma$-algebra generated by the abstract rectangles.*

**Theorem 4.5** *(Rothberger [22] 1952) Suppose that* $2^\omega = \omega_2$ *and* $2^{\omega_1} = \omega_{\omega_2}$ *then the* $\sigma$*-algebra generated by the abstract rectangles in the plane is not the power set of the plane.*

proof:

Let $H_\alpha$ for $\alpha < \aleph_{\omega_2}$ list all countable subsets of $\mathcal{P}(\omega_1)$. Let $\sigma H_\alpha$ be the $\sigma$-algebra generated by $H_\alpha$. Note that $|\sigma H_\alpha| \leq |2^\omega| = \omega_2$ since $H_\alpha$ is countable. For each $\beta < \omega_2$ choose $X_\alpha \subseteq \omega_1$ with $X_\alpha \notin \bigcup_{\alpha<\aleph_\beta} \sigma H_\alpha$. Let $X = \bigcup_{\beta<\omega_2}\{\beta\} \times X_\beta$.

**Claim**. $X \notin \sigma\{A \times B \ : \ A \subseteq \omega_2 \text{ and } B \subseteq \omega_1\}$.

Suppose for contradiction that

$$X \in \Sigma_\gamma(\{A_n \times B_n \subseteq \omega_2 \times \omega_1 \ : \ n < \omega\}).$$

It is easy to see that the cross sections satisfy:

$$\forall\beta < \omega_2 \ \ X_\beta \in \Sigma_\gamma(\{B_n \ : \ n < \omega\}).$$

But if $H_{\alpha_0} = \{B_n \ : \ n < \omega\}$ where $\alpha_0 < \omega_{\beta_0}$, then $X_{\beta_0} \notin \sigma H_{\alpha_0}$, which is a contradiction.

□

Note Rothberger states this result in more generality, this is the simplest case.

**Theorem 4.6** *(Bing, Bledsoe, Mauldin [2] 1974) If every subset of the plane is in the* $\sigma$*-algebra generated by the abstract rectangles, then for some countable* $\alpha$ *every subset of the plane is in the* $\sigma$*-algebra generated by the abstract rectangles by level* $\alpha$*.*

proof:

This is more general:

**Claim**. For any cardinal $\kappa$ if $\sigma\{A \times B \ : \ A, B \subseteq \kappa\} = \mathcal{P}(\kappa \times \kappa)$, then there exists $\alpha < \omega_1$ such that $\Pi_\alpha(\{A \times B \ : \ A, B \subseteq \kappa\}) = \mathcal{P}(\kappa \times \kappa)$.

Suppose not. Take $P_\alpha \subseteq \kappa$ for $\alpha < \omega_1$ pairwise disjoint and cardinality $\kappa$. For each $\alpha < \omega_1$ take $A_\alpha \subseteq P_\alpha \times P_\alpha$ such that

$$A_\alpha \notin \Pi_\alpha(\{A \times B \ : \ A, B \subseteq \kappa\}).$$

Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Note that for any $\alpha < \omega_1$ $(P_\alpha \times P_\alpha) \in \Pi_1$. It follows that if $A \in \Pi_{\alpha_0}$, then for any $\alpha < \omega_1$ $A_\alpha = A \cap (P_\alpha \times P_\alpha) \in \Pi_{\alpha_0}$, which is a contradiction.
□

**Theorem 4.7** *(Miller [9] [4]) If every subset of a separable metric space $X$ is Borel in $X$, then for some countable $\alpha$ every subset of $X$ is $\Sigma^0_\alpha$ in $X$.*

For the proof, we will need the following two Lemmas.

**Lemma 4.8** *Suppose there exists $X \subseteq 2^\omega$, $X = \{x_\alpha : \alpha < \kappa\}$, and there exists $\alpha < \omega_1$ such that for every $\gamma < \kappa$ every $Y \subseteq \{x_\beta : \beta < \gamma\}$ is $\utilde{\mathbf{\Sigma}}^0_\alpha$ in $X$. Then*

$$\Sigma_\alpha\{A \times B \;:\; A, B \subseteq \kappa\} = \mathcal{P}(\kappa \times \kappa).$$

proof:
Consider any $A \subseteq \kappa \times \kappa$ such that $\beta \leq \alpha$ for any $(\alpha, \beta) \in A$. Let $X = \{x_\alpha \;:\; \alpha < \kappa\}$ and let $V \subseteq 2^\omega \times 2^\omega$ be a universal $\utilde{\mathbf{\Sigma}}^0_\alpha$-set. For each $\alpha < \kappa$ choose $y_\alpha \in 2^\omega$ distinct such that

$$\forall \beta < \kappa \quad (\alpha, \beta) \in A \text{ iff } (y_\alpha, x_\beta) \in V \text{ iff } x_\beta \in V_{y_\alpha}.$$

Define $F : \kappa \times \kappa \to X \times Y$ by $F(\alpha, \beta) = (x_\alpha, y_\beta)$. Note that $F$ is a rectangle preserving bijection such that $F(A) = V \cap (X \times Y)$. Note that

$$V \in \utilde{\mathbf{\Sigma}}^0_\alpha\{C_n \times D_n \;:\; n < \omega\}$$

where $C_n, D_n$ are clopen subsets of $2^\omega$. It follows that

$$A \in \utilde{\mathbf{\Sigma}}^0_\alpha(\{\alpha : y_\alpha \in C_n\} \times \{\beta : x_\beta \in D_n\}).$$

By a symmetrical argument we can handle any $B \subseteq \kappa \times \kappa$ where $\beta \geq \alpha$ for any $(\alpha, \beta) \in B$, and hence any set of the form $A \cup B$.
□

[4] I proved this on the plane trip back to Berkeley from the January 1977 AMS-ASL meeting in St. Louis. It was so cold that year the AMS vowed never to have their January meetings anywhere but warm places. It was late at night; the plane was pretty much empty and was delayed due to excessive ice on the wings, so they opened up the bar cart as we sat on the tarmac.

**Lemma 4.9** *If $X \subseteq 2^\omega$, every subset of $X$ is Borel in $X$, $\omega < \kappa = |X|$, and $\sigma\{A \times B \,:\, A \times B \subseteq \omega_1 \times \kappa\} = \mathcal{P}(\omega_1 \times \kappa)$, then* $\mathrm{ord}(X) < \omega_1$.

proof:

Every rectangle $A \times B \subseteq X \times X$ is Borel in $X \times X$, so every subset of $X \times X$ is Borel in $X$. If $\mathrm{ord}(X) = \omega_1$, then for every $\alpha < \omega_1$ choose $H_\alpha \subseteq X$ such that $H_\alpha \notin \utilde{\mathbf{\Sigma}}^0_\alpha(X)$. Choose $x_\alpha \in X$ for $\alpha < \omega_1$ distinct. Let $H = \bigcup_{\alpha<\omega_1} \{x_\alpha\} \times H_\alpha$. If $H$ is $\utilde{\mathbf{\Sigma}}^0_{\alpha_0}$ in $X \times X$, then $H_\alpha$ is $\utilde{\mathbf{\Sigma}}^0_{\alpha_0}$ in $X$ for all $\alpha < \omega_1$. This is a contradiction.

□

Proof of Theorem 4.7:

Let $X = \{x_\alpha \,:\, \alpha < \kappa\}$ and prove the Theorem by induction on $\kappa$.

**Case**. $\kappa = \omega_1$. We are done by Lemma 4.9 and Theorem 4.1 since

$$\Sigma_2\{A \times B \,:\, A, B \subseteq \omega_1 \times \omega_1\} = \mathcal{P}(\omega_1 \times \omega_1).$$

**Case**. $\mathrm{cof}(\kappa) = \omega$. Let $X = \bigcup_{n<\omega} X_n$ pairwise disjoint and each $|X_n| < \kappa$. By induction $\mathrm{ord}(X_n) < \omega_1$ for each $n < \omega$. Choose countable $\alpha$ so that $\mathrm{ord}(X_n) \le \alpha$ and each $X_n$ is $\utilde{\mathbf{\Sigma}}^0_\alpha$ in $X$. For any $A \subseteq X$ we have that $A \cap X_n$ is $\utilde{\mathbf{\Sigma}}^0_\alpha$ in $X$ and so $X \cap A = \bigcup_{n<\omega} A \cap X_n$ is $\utilde{\mathbf{\Sigma}}^0_\alpha$ in $X$.

**Case**. $\mathrm{cof}(\kappa) > \omega_1$. Define $X_\alpha = \{x_\beta : \beta < \alpha\}$. Note that $\mathrm{ord}(X_\alpha)$ for $\alpha < \kappa$ is a nondecreasing function and so there is a $\beta < \omega_1$ such that $\mathrm{ord}(X_\alpha) \le \beta$ all $\alpha < \kappa$. Similarly there is a countable $\gamma \ge \beta$ such that every $X_\alpha$ is $\utilde{\mathbf{\Sigma}}^0_\gamma$ in $X$. To see this note that for $\alpha < \beta$, if $X_\alpha$ is $\utilde{\mathbf{\Sigma}}^0_\gamma$ in $X_\beta$ and $X_\beta$ is $\utilde{\mathbf{\Sigma}}^0_\gamma$ in $X$, then $X_\alpha$ is $\utilde{\mathbf{\Sigma}}^0_\gamma$ in $X$. It follows that every $Y \subseteq X$ with $Y \subseteq X_\alpha$ for some $\alpha < \kappa$ is $\utilde{\mathbf{\Sigma}}^0_\gamma$ and so by Lemma 4.8 we have that $\Sigma_\alpha\{A \times B \,:\, A, B \subseteq \kappa\} = \mathcal{P}(\kappa \times \kappa)$. Hence we are done by Lemma 4.9.

**Case**. $\mathrm{cof}(\kappa) = \omega_1$. Let $\kappa_\alpha$ for $\alpha < \omega_1$ be a cofinal in $\kappa$ increasing sequence.

**Claim**. There exists $\alpha_0 < \omega_1$ such that $X_{\kappa_\alpha}$ is $\utilde{\mathbf{\Pi}}^0_{\alpha_0}$ in $X$ for all $\alpha < \omega_1$. pf: Let $Q = \{(\alpha, \beta) \,:\, \alpha < \omega_1 \text{ and } \beta \le \kappa_\alpha\} \subseteq \omega_1 \times \kappa$. Then its complement $\sim Q = \{(\alpha, \beta) \,:\, \kappa_\alpha < \beta\}$ has countable cross sections, so there exists partial functions $f_n : \kappa \to \omega_1$ for $n < \omega$ such that for any $\beta < \kappa$

$$\{\alpha : (\alpha, \beta) \in \sim Q\} = \{f_n(\beta) : n < \omega\}$$

equivalently

$$\sim Q = \bigcup_{n<\omega} \text{graph}(f_n).$$

Similar to the proof of Theorem 4.1 we get that

$$\sim Q \in \Sigma_2(\{A \times B : A \subseteq \omega_1, B \subseteq \kappa\}.$$

And therefor $Q \in \Pi_2(\{A \times B : A \subseteq \omega_1, B \subseteq \kappa\}$. Now since every rectangle in $X \times X$ is Borel in $X \times X$ we get that $\{(x_\alpha, x_\beta) \ : \ \beta \leq \kappa_\alpha\}$ is Borel in $X \times X$. If it is $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_0}$ in $X \times X$, then so are all its cross sections and so we are done.

**Claim**. Suppose $\text{ord}(X_{\kappa_\alpha}) = \beta_\alpha$ which is countable by induction, then $\sup_{\alpha<\omega_1} \beta_\alpha < \omega_1$. pf: Suppose not and choose $\alpha_i < \omega_1$ for $0 < i < \omega_1$ strictly increasing, so that $\beta_{\alpha_i} > \alpha_0$ and $\beta_{\alpha_{i+1}} > \beta_{\alpha_i} + \omega$. For each $i < \omega_1$ let $Y_i = X_{\kappa_{i+1}} \backslash X_{\kappa_i}$. The $Y_i$ are pairwise disjoint, $\text{ord}(Y_i) = \beta_{i+1}$, and $Y_i$ is $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_0+1}$ in $X$. Choose $A_i \subseteq Y_i$ which is not $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_i}$ in $Y_i$. But then $A = \bigcup_{i<\omega_1} A_i$ is not Borel in $X$, since $A_i = A \cap Y_i$.

It follows from the second Claim and Lemmas 4.8 and 4.9 that $\text{ord}(X) < \omega_1$. This proves Theorem 4.7.
$\square$

**Theorem 4.10** *([9]) For any countable $\alpha$ it is consistent to have a separable metric space $X$ in which every subset is Borel and the order of $X$ is $\alpha$. Furthermore in this model for successor $\alpha = \alpha_0 + 1 \geq 3$ for any $Z \subseteq 2^\omega$ if every subset of is $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$ in $Z$, then $Z$ is countable.*

We just prove this for countable successor ordinals $\alpha_0 + 1$ greater than two. For limit $\alpha$ see [9].

**Definition 4.11** $\mathbb{P}(T, Y, X)$. *Fix countable $\alpha_0 \geq 2$. This forcing is similar to $\Pi_{\alpha_0+1}$ forcing (Definition 3.3). Assume $Y \subseteq X \subseteq 2^\omega$. Recall that definition uses a nice $\alpha_0 + 1$ tree $T$ which will remain fixed throughout the proof. We denote terminal nodes of $T$ by $T^*$ and interior nodes $T^0 = T \backslash (\{\langle\rangle\} \cup T^*)$. Then $p \in \mathbb{P}(T, Y, X)$ iff $p = p_0 \cup p_1$ finite with $p_0 : T^* \to 2^{<\omega}$ finite partial and finite $p_1 \subseteq T^0 \times X$ subject to the consistency demands:*

- *if $((n), x) \in p_1$, then $x \notin Y$*

- *if* $(s,x) \in p_1$ *and* $(sn,y) \in p_1$*, then* $x \neq y$
- *if* $(s,x) \in p_1$ *and* $p_0(sn) = r$*, then* $x \notin [r]$

Note that we only attach elements of $X$ to the interior nodes of $T$, we do not attach any reals to the top node $\langle\rangle$ of $T$, and we only attach reals from $X \backslash Y$ to the rank $\alpha_0$ nodes of $T$, i.e., those of the form $(n)$. We remark that $\mathbb{P}(T, \emptyset, 2^\omega)$ is the same as the direct sum of countably many copies of $\Pi_{\alpha_0}$-forcing. We could think of the first condition as equivalent to putting all $(\langle\rangle, y)$ for $y \in Y$ into $p_1$.

**Definition 4.12** $U_s^G$. *Similar to before for a generic* $G$ *a* $\mathbb{P}(T,Y,X)$*-filter, define* $U_s^G \subseteq 2^\omega$ *for* $s \in T$*. For* $s \in T^*$ *define* $U_s^G = [r]$ *iff* $p_0(s) = r$ *for some* $p \in G$*. For* $s \in T \backslash T^*$ *define* $U_s^G = \bigcap_{n<\omega} \sim U_{sn}^G$*.*

**Lemma 4.13** *For* $G$ *any generic* $\mathbb{P}(T,Y,X)$*-filter:*

1. *For any* $s \in T^0$ *and* $x \in X$ $x \in U_s^G$ *iff* $\{(s,x)\} \in G$*.*
2. $U_{\langle\rangle}^G \cap X = Y$*.*

proof:

For any $s \in T^0$ and $x \in X$ define $D_{s,x} \subseteq \mathbb{P}_{\alpha_0+1}(Y,X)$ by $p \in D_{s,x}$ iff

- $(s,x) \in p$ or
- $\exists n < \omega \;\; (sn,x) \in p_1$ or
- $\exists r \in 2^{<\omega} \;\; p_0(sn) = r$ and $x \in [r]$.

Then $D_{s,x}$ is dense. Note for $n < \omega$ and $y \in Y$ you can never add $((n), y)$ however you will be add $((n,m), y)$ for some sufficiently large $m$.

(1) Suppose $x \in U_s^G = \bigcap_{n<\omega} \sim U_{sn}^G$, then $x \notin U_{sn}^G$ for all $n$. Hence by induction for all $n$ for all $p \in G$ ($(sn,x) \notin p$ or $x \notin [p_0(sn)]$) if $\text{rank}_T(s) = 1$. Since $D_{s,x}$ is dense we have that $\{(s,x)\} \in G$.

Conversely, suppose $\{(s,x)\} \in G$. Then for every $n < \omega$ and $p \in G$ ($(sn,x) \notin p$ or $x \notin [p_0(sn)]$) if $\text{rank}_T(s) = 1$. So by induction $x \notin U_{sn}^G$ for all $n$ and by definition of $U_s^G$ we have that $x \in U_s^G$.

(2) Suppose $y \in Y$. Fix $n$. By the definition of $\mathbb{P}(T,Y,X)$ $((n), y)$ is not in any condition. Hence for any $p$ there exists $m$ sufficiently large so that $p \cup \{((n,m), y)\}$ is consistent. It follows that $y \notin U_{(n)}^G$ and so $y \in U_{\langle\rangle}^G$.

Conversely, suppose $x \in X \setminus Y$. Then $\{p : \exists n \ ((n), x) \in p\}$ is dense, so $x \in U^G_{(n)}$ for some $n$ and hence $y \notin U^G_{\langle\rangle}$.
$\square$

Let $M$ be a countable standard model of ZFC+GCH. Fix $X = 2^\omega \cap M$ (or any uncountable subset of $2^\omega$ in $M$) and $T$ a nice $\alpha_0 + 1$-tree in $M$. We will iterate $\mathbb{P}(T, Y_\alpha, X)$ for $\alpha_{<}\omega_2$ with finite support, diagonalizing to get all $Y \subseteq X$. An explicit description of this model is as follows.

Let $\mathbb{P}^0 = \mathbb{P}(T, \emptyset, X)$.

Inductively assume that $\mathbb{P}^\alpha \subseteq \Sigma_{\beta<\alpha}\mathbb{P}(T, \emptyset, X)$ (the direct sum). Suppose $\overset{\circ}{Y}_\alpha$ is a nice $\mathbb{P}^\alpha$-name for a subset of $X$, i.e., there exists $(A^\alpha_x : x \in X)$ where $A^\alpha_x \subseteq \mathbb{P}^\alpha$ is countable and

$$\overset{\circ}{Y}_\alpha = \{(p, \check{x}) : x \in X \text{ and } p \in A^\alpha_x\}.$$

Then $p \in \mathbb{P}^{\alpha+1}$ iff $p \upharpoonright \alpha \in \mathbb{P}^\alpha$, $p(\alpha) \in \mathbb{P}(T, \emptyset, X)$, and if $p(\alpha) = p_0 \cup p_1$, then whenever $((n), x) \in p_1$ for some $n < \omega$ and $x \in X$, then

$$p \upharpoonright \alpha \Vdash x \notin \overset{\circ}{Y}_\alpha$$

or equivalently $(p \upharpoonright \alpha) \perp q$ for all $q \in A^\alpha_x$.

For $\lambda \leq \omega_2$ a limit ordinal

$$\mathbb{P}^\lambda = \{p : \forall \alpha < \lambda \ \ p \upharpoonright \alpha \in \mathbb{P}^\alpha \text{ and } \mathrm{supp}(p) \text{ is finite } \}\}.$$

where $\mathrm{supp}(p) =^{def} \{\alpha < \lambda : p(\alpha) \neq 1\}$ is the support of $p$. Note that for any $\alpha < \omega_2$, $\Sigma_{\beta<\alpha}\mathbb{P}(T, \emptyset, X)$ has cardinality $\omega_1$. It also has ccc (in fact property K). Note that $p, q \in \mathbb{P}^\alpha$ are compatible iff $p \cup q \in \mathbb{P}^\alpha$ iff $p \cup q \in \Sigma_{\beta<\alpha}\mathbb{P}(T, \emptyset, X)$. For $\alpha < \omega_2$ we may regard

$$\mathbb{P}^\alpha = \{p \in \mathbb{P}^{\omega_2} : p \upharpoonright [\alpha, \omega_2) \equiv 1\}.$$

By a standard dovetailing argument in $M$ we may choose the sequence of names $\overset{\circ}{Y}_\alpha$ so that for any $G$ $\mathbb{P}^{\omega_2}$-generic over $M$ for any $Y \subseteq X$ in $M[G]$ there is an $\alpha < \omega_2^M$ such that $Y^G_\alpha = Y$.

By Lemma 4.13 we have that in $M[G]$ every subset of $X$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha_0+1}$ in $X$. Recall that $\mathbb{P}^0 = \mathbb{P}(T, \emptyset, X)$. Let $U^G_{(0)}$ be the generic $\underset{\sim}{\boldsymbol{\Pi}}^0_{\alpha_0}$-set added by $\mathbb{P}^0$. (Any of the other $U^G_{(n)}$ would do as well.) We will show that there is no $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha_0}$-set $V \subseteq 2^\omega$ in $M[G]$ such that $U^G_{(0)} \cap X = V \cap X$.

Given $H \subseteq X$ define

$$\text{rank}(p, H) = \max\{\text{rank}_T(s) \ : \ \exists x \in X \setminus H \ \exists \alpha < \omega_2 \ \ (s, x) \in p(\alpha)\}.$$

Working in $M$ suppose we are given $\Gamma \in \mathbb{P}^{\omega_2}$ countable and $\tau$ a nice $\mathbb{P}^{\omega_2}$ for an element of $2^\omega$. Then by the ccc we can find a countable $H \subseteq X$ and countable $K \subseteq \omega_2$ with the following properties:

1. $\text{rank}(p, H) = 0$ for all $p \in \Gamma$
2. $\forall n \in \omega \quad \{p \in \mathbb{P}^{\omega_2} \ : \ \text{supp}(p) \subseteq K$ and $\text{rank}(p, H) = 0\}$ decides[5] "$\tau(n) = 0$".
3. $\forall x \in H \ \forall \alpha \in K \ \{p \in \mathbb{P}^\alpha \ : \ \text{supp}(p) \subseteq K$ and $\text{rank}(p, H) = 0\}$ decides "$x \in \overset{\circ}{Y}_\alpha$".

The analogue of the meet lemma for this forcing is trivial.

**Lemma 4.14** *Meet Lemma. For any $p, q \in \mathbb{P}^{\omega_2}$ we have:*
*$p$ and $q$ are compatible iff $p \cup q \in \mathbb{P}^{\omega_2}$.*
*The union is defined by $(p \cup q)(\alpha) =^{def} p(\alpha) \cup q(\alpha)$ for each $\alpha < \omega_2$.*

proof:
Prove by induction that $(p \cup q) \upharpoonright \alpha \in \mathbb{P}^\alpha$ and extends both $p \upharpoonright \alpha$ and $q \upharpoonright \alpha$.
□

The union operation preserves rank and support.

**Lemma 4.15** *Rank Lemma for $H, K$. Assume $H, K$ satisfy condition 3 above, i.e., $\forall x \in H \ \forall \alpha \in K \ \{p \in \mathbb{P}^\alpha \ : \ supp(p) \subseteq K$ and $rank(p, H) = 0\}$ decides "$x \in \overset{\circ}{Y}_\alpha$". Suppose $p \in \mathbb{P}^{\omega_2}$ and $1 \leq \beta < \alpha_0$. Then there exists $\hat{p}$ compatible with $p$, $rank(\hat{p}, H) \leq \beta$, $supp(\hat{p}) \subseteq K$, and for any $q \in \mathbb{P}^{\omega_2}$ with $rank(q, H) < \beta$ and $supp(q) \subseteq K$, $(p \perp q \Rightarrow \hat{p} \perp q)$.*

proof:
Extend $p$ to $\tilde{p}$ so that for any $\alpha, s, x, \lambda$ if $(s, x) \in p(\alpha)$ and $\text{rank}_T(s) = \lambda$ is a limit ordinal, then for every $i < \omega$ with $\text{rank}_T(si) \leq \beta + 1 < \lambda$ there exists $j$ with $(sij, x) \in \tilde{p}(\alpha)$. The definition of nice tree tells us there are at most finitely many such $i$.

Let $G$ be $\mathbb{P}^{\omega_2}$-generic with $\tilde{p} \in G$. Choose $\Gamma \subseteq G$ finite so that

[5] Recall that a set conditions $Q$ decides a sentence $\theta$ iff every generic filter contains a condition in $Q$ which forces either $\theta$ or its negation.

(a) $\forall q \in \Gamma\ \ \mathrm{rank}(q, H) = 0$ and $\mathrm{supp}(q) \subseteq K$

(b) if $((n), x) \in p(\alpha)$ for some $\alpha \in K$ and $x \in H$, then there exists $q \in \Gamma$ such that $q \restriction \alpha \Vdash x \notin \overset{\circ}{Y}_\alpha$.

Note that in (b) it must be that $p \restriction \alpha \Vdash x \notin \overset{\circ}{Y}_\alpha$.

Working in $M$ define $\hat{p}$ as follows: for $\alpha \in K$

$$\hat{p}(\alpha) = \bigcup\{q(\alpha)\ :\ q \in \Gamma\} \cup \{(s, x) \in \tilde{p}_1(\alpha)\ :\ \mathrm{rank}_T(s) \leq \beta \text{ or } x \in H\} \cup \tilde{p}_0(\alpha)$$

and $\hat{p}(\alpha) = 1$ for $\alpha \notin K$.

We prove that $\hat{p} \leq q$ for each $q \in \Gamma$ and $\mathrm{rank}_T(\hat{p}) \leq \beta$. Note that $\beta < \alpha_0 = \mathrm{rank}_T((n))$ so we have retained no conditions of the form $((n), x)$ for $x \notin H$, i.e., if $((n), x) \in \hat{p}$, then $\alpha \in K$ and $x \in H$. So $\hat{p} \restriction \alpha \leq q \restriction \alpha$ for some $q \in \Gamma$ such that $q \restriction \alpha \Vdash x \notin \overset{\circ}{Y}_\alpha$ and so $\hat{p} \restriction \alpha \Vdash x \notin \overset{\circ}{Y}_\alpha$. This shows that $\hat{p}$ is a condition.

We check that it satisfies the Lemma. Since $\tilde{p} \leq p$ and $\tilde{p}, \hat{p}$ are both in $G$, we have that $p$ and $\hat{p}$ are compatible. Suppose $\mathrm{rank}(q, H) < \beta$ and $\mathrm{supp}(q) \subseteq K$. We need to show that $p \perp q \to \hat{p} \perp q$. Assume $p \perp q$, hence $p \cup q$ is not a condition so there must be $\alpha \in \mathrm{supp}(p) \cap \mathrm{supp}(q)$ (so $\alpha \in K$) $(p \cup q) \restriction \alpha \in \mathbb{P}^\alpha$, but $p(\alpha) \cup q(\alpha) \notin \mathbb{P}(T, \overset{\circ}{Y}_\alpha, X)$. Therefor at least one of the following cases occurs:

Case 1. For some $s \in T^*$ we have $p_0(\alpha)(s) \neq q_0(\alpha)(s)$.

But $\hat{p}_0(\alpha)(s) = p_0(\alpha)(s)$, so $\hat{p}(\alpha) \perp q(\alpha)$.

Case 2. For some $s, r, n$ with $s \in T^0$ and $sn \in T^*$ and $x \in X$: $(p_0(\alpha)(sn) = r$ or $q_0(\alpha)(sn) = r)$ and $(s, x) \in p_1(\alpha) \cup q_1(\alpha)$, but $x \in [r]$.

In this case, $\mathrm{rank}_T(s) = 1 \leq \beta$ and hence $(s, x) \in \hat{p}_1(\alpha)$ if $(s, x) \in p_1(\alpha)$. So if $q_0(\alpha)(sn) = r$, then $\hat{p}(\alpha) \perp q(\alpha)$. The other possibility is that $p_0(sn) = r$ and $(s, x) \in q_1(\alpha)$. Then since $\hat{p}_0(sn) = p_0(sn)$, we also have $\hat{p}(\alpha) \perp q(\alpha)$.

Case 3. For some $s, sn \in T^0$ and $x \in X \quad (s, x), (sn, x) \in p_1(\alpha) \cup q_1(\alpha)$.

In this case, if $x \in H$, then $(s, x), (sn, x) \in \hat{p}_1(\alpha) \cup q_1(\alpha)$ and so $\hat{p}(\alpha) \perp q(\alpha)$. Suppose $x \notin H$, then one of the following occurs:

1. $(sn, x) \in p_1(\alpha)$ and $(s, x) \in q_1(\alpha)$
2. $(s, x) \in p_1(\alpha)$ and $(sn, x) \in q_1(\alpha)$

For (1) since $\text{rank}(q, H) < \beta$ we have that $\text{rank}_T(sn) < \text{rank}_T(s) < \beta$ and so by the definition of $\hat{p}$ we have that $(sn, x) \in \hat{p}_1(\alpha)$ so $\hat{p}(\alpha) \perp q(\alpha)$. For (2) we have that $\text{rank}_T(sn) < \beta$ because $\text{rank}(q, H) < \beta$. If $\text{rank}_T(s) \leq \beta$, then $(s, x) \in \hat{p}_1(\alpha)$ so $\hat{p}(\alpha) \perp q(\alpha)$. Finally we have the possibility that $\text{rank}_T(s) = \lambda > \beta$ a limit ordinal. In this case we choose $\tilde{p}(\alpha)$ so that for some $m$ we have that $(snm, x) \in \tilde{p}_1(\alpha)$. Since $\text{rank}_T(snm) < \beta$, $(snm, x) \in \hat{p}_1(\alpha)$, and therefor $\hat{p}(\alpha) \perp q(\alpha)$.
□

**Lemma 4.16** *Suppose $\tau$ a nice $\mathbb{P}^{\omega_2}$ for an element of $2^\omega$ and $H \subseteq X$ and $K \subseteq \omega_2$ are countable satisfying*

1. *$\forall n \in \omega \quad \{p \in \mathbb{P}^{\omega_2} : supp(p) \subseteq K, rank(p, H) = 0\}$ decides "$\tau(n) = 0$".*
2. *$\forall x \in H \ \forall \alpha \in K \ \{p \in \mathbb{P}^\alpha : supp(p) \subseteq K, rank(p, H) = 0\}$ decides "$x \in \overset{\circ}{Y}_\alpha$".*

*Suppose $B(v)$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_\beta$ predicate for $1 \leq \beta \leq \alpha_0$ with parameters from $M$ and $p \in \mathbb{P}^{\omega_2}$ satisfies: $p \Vdash B(\tau)$. Then there exists $\hat{p}$ compatible with $p$, $rank(\hat{p}, H) < \beta$, $supp(\hat{p}) \subseteq K$, and $\hat{p} \Vdash B(\tau)$.*

Case $\beta = 1$.

Suppose $R \subseteq 2^{<\omega}$ is in $M$ and $p \Vdash \exists n \in \omega \ \ R(\tau \upharpoonright n)$. Find $q \leq p$, $n \in \omega$, $t \in 2^n$ such that $R(t)$ and $q \Vdash \tau \upharpoonright n = \check{t}$. Take $G$ $\mathbb{P}^{\omega_2}$-generic with $q \in G$. By (1) we can choose finite $\Gamma \subseteq G$ such that

for all $m < n$ there is a $r \in \Gamma$ $r \Vdash$"$\tau(m) = t(n)$" and

$\text{supp}(r) \subseteq K$ and $\text{rank}(r, H) = 0$ for all $r \in \Gamma$.

Then $\hat{p} = \bigcup \Gamma$ satisfies the Lemma.

Case $\beta \leq \alpha_0$ a limit ordinal.

Suppose $p \Vdash$ "$\exists n \in \omega \ B_n(\tau)$" where for each $n$ $B_n(v)$ is a $\underset{\sim}{\mathbf{\Pi}}^0_{\beta_n}$ predicate coded in $M$ with $\beta_n < \beta$. Extend $p$ by $q \leq p$ such that for some $k$ $q \Vdash$ "$B_k(\tau)$". Since $\underset{\sim}{\mathbf{\Pi}}^0_{\beta_k}$ predicates are $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta_{k+1}}$ and $\beta_{k+1} < \beta$, we have by induction $\hat{p}$ compatible with $q$ (hence $p$) with $\text{rank}(\hat{p}, H) \leq \beta_{k+1} < \beta$, $\text{supp}(\hat{p}) \subseteq K$, and $\hat{p} \Vdash$ "$B_k(\tau)$".

Case $1 < \beta + 1 \leq \alpha_0$ a successor ordinal.

Suppose $p \Vdash$ "$\exists n \in \omega \ B(n, \tau)$" where $B(n, v)$ is a $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ predicate coded in $M$. We may extend $p$ to $p_0 \leq p$ so that for some $n$ $p_0 \Vdash B(n, \tau)$. By the

Rank Lemma since $\beta < \alpha_0$ there is some $\hat{p_0}$ compatible with $p_0$ such that $\text{rank}(\hat{p_0}, H) \leq \beta$, $\text{supp}(\hat{p_0}) \subseteq K$, and for any $q \in \mathbb{P}^{\omega_2}$ with $\text{rank}(q, H) < \beta$ and $\text{supp}(q) \subseteq K$, $(p_0 \perp q \Rightarrow \hat{p_0} \perp q)$. But then $\hat{p_0} \Vdash B(n, \tau)$. Because if not, by inductive hypothesis there would be $q$ compatible with $\hat{p_0}$, $\text{rank}(q, H) < \beta$, $\text{supp}(q) \subseteq K$, and $q \Vdash \neg B(n, \tau)$. But such a $q$ is incompatible with $p_0$ which is a contradiction.
$\Box$

**Lemma 4.17** *Suppose $X = \{x_\alpha : \alpha < \omega_1\}$ and $Z = \{z_\alpha : \alpha < \omega_1\} \subseteq 2^\omega$ be an arbitrary set of reals in $M$. For any $G$ $\mathbb{P}^{\omega_2}$-generic over $M$, then*

$$(U^G_{(0)} \times 2^\omega) \cap \{(x_\alpha, z_\alpha) \ : \ \alpha < \omega_1\} \neq V \cap \{(x_\alpha, z_\alpha) \ : \ \alpha < \omega_1\}$$

*for any $V \subseteq 2^\omega \times 2^\omega$ a $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$-set coded in $M[G]$*

proof:

Recall that $U^G_{(0)}$ (4.12) is one of the "generic" $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_0}$ sets determined by the first coordinate, i.e., for $x \in X$ we have that $x \in U^G_{(0)}$ iff $((0), x) \in p_1(0)$ for some $p \in G$.

Work in $M$. Let $V \subseteq 2^\omega \times (2^\omega \times 2^\omega)$ be a universal $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$-set. For contradiction, suppose there exists $q \in \mathbb{P}^{\omega_2}$ and $\tau$ is a nice name for an element of $2^\omega$ such that

$$q \Vdash \forall \alpha < \omega_1 \ \ (x_\alpha \in \overset{\circ}{U}_{(0)} \ \text{ iff } (\tau, (x_\alpha, z_\alpha) \in V).$$

Choose $H \subseteq X, K \subseteq \omega_2$ countable such that $\text{rank}(q, H) = 0$, $\text{supp}(q) \subseteq K$, and satisfying the conditions of Lemma 4.16 for $\tau$.

Fix any $\alpha \in \omega_1$ with $x_\alpha \notin H$ and define the $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$ predicate $B(v)$ by

$$B(v) \equiv (v, (x_\alpha, z_\alpha)) \in V.$$

Let $p \leq q$ be defined by only adding $((0), x_\alpha)$ to the first coordinate of $q$, i.e., $p(0) = q(0) \cup ((0), x_\alpha)$ and $p \upharpoonright [1, \omega_2) = q \upharpoonright [1, \omega_2)$. This is possible because $q(0)$ cannot mention $x_\alpha$ because $\text{rank}(q, H) = 0$. Note that $p \Vdash B(\tau)$. By Lemma 4.16 there is a $\hat{p}$ compatible with $p$ such that $\text{rank}(\hat{p}, H) < \alpha_0$ and $\hat{p} \Vdash B(\tau)$. By replacing $\hat{p}$ by $\hat{p} \cup p$ we may assume $\hat{p} \leq p$. Since $\text{rank}(\hat{p}, H) < \alpha_0$ we have that $((0), x_\alpha)$ is not in $\hat{p}_1(0)$. It follows by taking a large enough $k$ that $\hat{p}_1(0) \cup \{((0, k), x_\alpha)\}$ is consistent, i.e., an element of $\mathbb{P}(T, \emptyset, X)$. If we

define $r$ by $r(0) = \hat{p}(0) \cup \{((0,k), x_\alpha)\}$ and $r \upharpoonright [1, \omega_2) = \hat{p} \upharpoonright [1, \omega_2)$ we get a contradiction: $r \Vdash$ "$x_\alpha \in \overset{\circ}{U}_{(0)}$ iff $B(\tau)$", $r \Vdash B(\tau)$, and $r \Vdash x_\alpha \in \overset{\circ}{U}_{(0,k)}$.
□

Finally we prove Theorem 4.10. Lemma 4.13 shows that $\mathrm{ord}(X) \leq \alpha_0 + 1$ and Lemma 4.17 shows that $\mathrm{ord}(X) > \alpha_0$. Given any $Z \subseteq 2^\omega$ of size $\omega_1$ in $M[G]$ there will be $\delta < \omega_2$ with $Z \in M[G_\delta]$. We can assume unbounded many $Y_\delta$ code the empty set, so by replacing $M$ by the ground model $M[G_\delta]$ and forcing with $\mathbb{P}^{[\delta, \omega_2)}$, Lemma 4.17 shows that $\mathrm{ord}(Z) > \alpha_0$.
□

**Theorem 4.18** *([9]) For any countable $\alpha \geq 2$ it is consistent that every subset of the plane is in the $\sigma$-algebra generated by the abstract rectangles at level $\alpha$ but for every $\beta < \alpha$ not every subset is at level $\beta$.*

proof:

We sketch the proof only for the case of a countable successor $\alpha = \alpha_0 + 1 \geq 3$. Start with a countable transitive model $M$ of ZFC+$2^\omega = 2^{\omega_1} = \omega_2$. Let $X = 2^\omega \cap M$. Hence $|X| = \omega_2$. Do a finite support iteration (as in the proof of Theorem 4.10) of length $\omega_2$ of $\mathbb{P}(T, \overset{\circ}{Y}_\alpha, X)$ for $\alpha < \omega_2$ making sure to have names for all potential subsets of $X$ of size $\leq \omega_1$. It follows from Lemma 4.8 that $\mathcal{P}(\omega_2 \times \omega_2) = \Sigma_{\alpha_0+1}(\{A \times B \ : \ A, B \subseteq \omega_2\}$.

It also follows by a similar proof to Theorem 4.10 that in $M[G]$ there is no $Z \subseteq 2^\omega$ with $|Z| = \omega_1$ such that every subset of $Z$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$ in $Z$. So we are done by the following:

**Lemma 4.19** *(Bing, Bledsoe, Mauldin [2]) Suppose $\alpha < \omega_1$, $\omega < \kappa$, $|2^\kappa| = \mathfrak{c}$, and $\mathcal{P}(\kappa \times \mathfrak{c}) = \Sigma_\alpha(\{A \times B \ : \ A \subseteq \kappa, \ B \subseteq \mathfrak{c}\}$. Then there exists $Z \subseteq 2^\omega$ with $|Z| = \kappa$ and every subset of $Z$ is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ in $Z$.*

proof:

Let $Y_\alpha \subseteq \kappa$ for $\alpha < \mathfrak{c}$ list all subsets of $\kappa$. Define

$$Y = \{(\beta, \alpha) : \beta \in Y_\alpha, \alpha < \mathfrak{c}\}.$$

Let $\{A_n \times B_n : n < \omega\}$ be rectangles with $Y \in \Sigma_\alpha(\{A_n \times B_n : n < \omega\})$. Take $\psi : \kappa \to 2^\omega$ to be the Marczewski characteristic function:

$$\psi(\alpha)(n) = 1 \text{ iff } \alpha \in A_n.$$

Then $Z = \{\psi(\alpha) \ : \ \alpha < \kappa\}$ has the required property. Note that the cross sections of a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$-set are $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$.
□

# 5 Universal functions

**Theorem 5.1** *(Larson, Miller, Steprans, Weiss [20]) Suppose* $2^{<\mathfrak{c}} = \mathfrak{c}$ *then the following are equivalent:*

*(1) There is a Borel universal function, i.e, a Borel function* $F : 2^\omega \times 2^\omega \to 2^\omega$ *such that for every abstract* $G : 2^\omega \times 2^\omega \to 2^\omega$ *there are* $h : 2^\omega \to 2^\omega$ *and* $k : 2^\omega \to 2^\omega$ *such that for every* $x, y \in 2^\omega \quad G(x, y) = F(h(x), k(y))$*.*

*(2) Every subset of the plane is in the* $\sigma$*-algebra generated by the abstract rectangles.*

*Furthermore the universal function has level* $\alpha$ *iff every subset of the plane is in the* $\sigma$*-algebra generated by the abstract rectangles at level* $\alpha$*.*

**Theorem 5.2** *([20]) If* $2^{<\kappa} = \kappa$*, then there is an abstract universal function* $F : \kappa \times \kappa \to \kappa$*.*

**Theorem 5.3** *([20]) It is relatively consistent with ZFC, that there is no abstract universal function* $F : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$*.*

**Theorem 5.4** *([20]) There does not exist a Borel function* $F : 2^\omega \times 2^\omega \to 2^\omega$ *such that for every Borel* $G : 2^\omega \times 2^\omega \to 2^\omega$ *there are* $h, k : 2^\omega \to 2^\omega$ *such that* $k$ *is Borel and for every* $x, y \in 2^\omega$

$$G(x, y) = F(h(x), k(y))$$

The following two theorems are proved just like Theorems 8 and 9 of Bing, Bledsoe, and Mauldin [2] who only stated it for the square case, e.g., $\kappa \times \kappa$.

**Theorem 5.5** *(Bing, Bledsoe, and Mauldin Theorem 8 for* $\kappa = \lambda$*)*

1. *The following are equivalent:*
   - (a) $\underset{\sim}{\Sigma}^0_{\omega_1}(\{A \times B : A \subseteq \kappa, B \subseteq \lambda\}) = \mathcal{P}(\kappa \times \lambda)$
   - (b) *for every* $\mathcal{A} \in [\mathcal{P}(\lambda)]^\kappa$ *there is* $\mathcal{B} \in [\mathcal{P}(\lambda)]^\omega$ *and* $\alpha < \omega_1$ *such that* $\mathcal{A} \subseteq \underset{\sim}{\Sigma}^0_\alpha(\mathcal{B})$
   - (c) *for every* $\mathcal{A} \in [\mathcal{P}(\kappa)]^\lambda$ *there is* $\mathcal{B} \in [\mathcal{P}(\kappa)]^\omega$ *and* $\alpha < \omega_1$ *such that* $\mathcal{A} \subseteq \underset{\sim}{\Sigma}^0_\alpha(\mathcal{B})$
2. *Suppose* $\alpha < \omega_1$*, then the following are equivalent:*

(a) $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\{A \times B : A \subseteq \kappa, B \subseteq \lambda\}) = \mathcal{P}(\kappa \times \lambda)$.

(b) for every $\mathcal{A} \in [\mathcal{P}(\lambda)]^\kappa$ there is $\mathcal{B} \in [\mathcal{P}(\lambda)]^\omega$ such that $\mathcal{A} \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\mathcal{B})$

(c) for every $\mathcal{A} \in [\mathcal{P}(\kappa)]^\lambda$ there is $\mathcal{B} \in [\mathcal{P}(\kappa)]^\omega$ such that $\mathcal{A} \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\mathcal{B})$

**Theorem 5.6** *(Bing, Bledsoe, and Mauldin Theorem 9 for $\kappa = \lambda$) The following are equivalent:*

1. $\underset{\sim}{\mathbf{\Sigma}}^0_{\omega_1}(\{A \times B : A \subseteq \kappa, B \subseteq \lambda\}) = \mathcal{P}(\kappa \times \lambda)$
2. *for every $\mathcal{A} \in [\mathcal{P}(\lambda)]^\kappa$ there is $\mathcal{B} \in [\mathcal{P}(\lambda)]^\omega$ and $\alpha < \omega_1$ such that $\mathcal{A} \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\mathcal{B})$*
3. *there is $\alpha < \omega_1$ such that for every $\mathcal{A} \in [\mathcal{P}(\lambda)]^\kappa$ there is $\mathcal{B} \in [\mathcal{P}(\lambda)]^\omega$ such that $\mathcal{A} \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\mathcal{B})$*
4. *there exists $\alpha < \omega_1$ such that $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\{A \times B : A \subseteq \kappa, B \subseteq \lambda\}) = \mathcal{P}(\kappa \times \lambda)$.*

**Definition 5.7** *$X \subseteq 2^\omega$ is a strong $Q_\alpha$-set iff letting $\mathcal{B}_0 =^{def} \{X \cap C \ : \ C \subseteq 2^\omega$ clopen $\}$ then*

- *$\mathcal{P}(X) = \underset{\sim}{\mathbf{\Sigma}}^0_{\omega_1}(\mathcal{B}_0)$ ( $\sigma$-algebra generated by $\mathcal{B}_0$ ) and*
- *$\forall \mathcal{B} \supseteq \mathcal{B}_0$ countable $\mathrm{ord}(\mathcal{B}) = \alpha$, i.e. $\alpha$ is the least ordinal such that $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\mathcal{B}) = \underset{\sim}{\mathbf{\Sigma}}^0_{\omega_1}(\mathcal{B})$.*

The above definition is the one we use to verify the existence of a strong $Q_\alpha$-set in a generic extension. When we force a generic $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set it is not $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ even when we add sets from the ground model as new "clopen" sets.

**Theorem 5.8** *The following are equivalent for a cardinal $\kappa$ such that $2^\kappa = \mathfrak{c}$ and $\alpha < \omega_1$:*

1. *there exists a strong $Q_\alpha$-set of cardinality $\kappa$*
2. *$\alpha$ is the smallest ordinal such that*
$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\{A \times B : A \subseteq \kappa, B \subseteq \mathfrak{c}\}) = \mathcal{P}(\kappa \times \mathfrak{c})$$
3. *there is a $Q_\alpha$-set of size $\kappa$ but no $Q_\beta$-set of size $\kappa$ for any $\beta < \alpha$*

4. *$\alpha$ is the minimal ordinal for which there is a countable $\mathcal{B} \subseteq \mathcal{P}(\kappa)$ with $\utilde{\mathbf{\Sigma}}^0_\alpha(\mathcal{B}) = \mathcal{P}(\kappa)$*

proof:

Assume (1). First note that $\Sigma_\alpha\{A \times B : A \subseteq \kappa, B \subseteq \mathfrak{c}\} = \mathcal{P}(\kappa \times \mathfrak{c}\}$. This is proved just like Lemma 4.8. Namely, let $\{x_\beta : \beta < \kappa\} \subseteq 2^\omega$ be a $Q_\alpha$-set.

Fix $U \subseteq 2^\omega \times 2^\omega$ a universal $\utilde{\mathbf{\Sigma}}^0_\alpha$ set. Then for any subset $C \subseteq \kappa \times \mathfrak{c}$ choose $\{y_\beta \in 2^\omega : \beta < \mathfrak{c}\}$ so that for each $\gamma < \kappa$ and $\beta < \mathfrak{c}$, $(x_\gamma, y_\beta) \in U$ iff $(\gamma, \beta) \in C$. Since $U$ is $\utilde{\mathbf{\Sigma}}^0_\alpha$ in the clopen rectangles in $2^\omega \times 2^\omega$, it follows that $C$ is $\utilde{\mathbf{\Sigma}}^0_\alpha$ in the abstract rectangles on $\kappa \times \mathfrak{c}$.

Now suppose for contraction that for some $\beta < \alpha$

$$\Sigma_\beta\{A \times B : A \subseteq \kappa, B \subseteq \mathfrak{c}\} = \mathcal{P}(\kappa \times \mathfrak{c}\}.$$

Let $C \subseteq \kappa \times \mathfrak{c}$ be such that the cross sections of $C$ list $\mathcal{P}(\kappa)$. Suppose $C \in \utilde{\mathbf{\Sigma}}^0_\beta(\{A_n \times A_m : n, m < \omega\})$. Consider the Marczewski characteristic function of the sequence of sets $A_n$, i.e., $f : \mathfrak{c} \to 2^\omega$ defined by $f(\beta)(n) = 1$ iff $\beta \in A_n$. We can assume that the $A_n$ separate points (by adding a countable sequence of sets if necessary) so that $f$ is a 1-1 function. Let $z_\beta = f(\beta)$. The function $f$ maps the abstract sets $A_n$ into relatively clopen sets in $Z = \{z_\gamma : \gamma < \mathfrak{c}\}$. It follows that $\{z_\gamma : \gamma < \kappa\}$ is a $Q_\delta$-set for some $\delta \leq \beta < \alpha$ which contradicts the definition of "strong" $Q_\alpha$-set.

The proof of (2) implies (1) is virtually the same. The fact that

$$\utilde{\mathbf{\Sigma}}^0_\alpha(\{A \times B : A \subseteq \kappa, B \subseteq \mathfrak{c}\}) = \mathcal{P}(\kappa \times \mathfrak{c})$$

via the Marczewski function gives us a $Q_\alpha$-set of cardinality $\kappa$. The minimality of $\alpha$ gives us that there is no $Q_\beta$-set of size $\kappa$ for any $\beta < \alpha$.

(3) and (4) are equivalent by using the Marczewski characteristic function.

(1) implies (3): Let $\{x_\gamma : \gamma < \kappa\}$ be a strong $Q_\alpha$-set and suppose $\{y_\gamma : \gamma < \kappa\}$ is a $Q_\beta$-set for some $\beta < \alpha$. For any clopen set $C \subseteq 2^\omega$ let $C' = \{x_\alpha : y_\alpha \in C\}$, then $\mathcal{B} = \mathcal{B}_0 \cup \{C' : C \text{ clopen }\}$ has order $\leq \beta$ contradicting the definition of strong $Q_\alpha$-set.

(3) implies (1): Any $Q_\alpha$-set of size $\kappa$ must be strong, otherwise by using the Marczewski characteristic function we could produce for some $\beta < \alpha$ a $Q_\beta$-set of cardinality $\kappa$.

□

**Theorem 5.9** *It is consistent that for every countable $\alpha \geq 2$ there is a strong $Q_\alpha$-set.*

proof:

This holds in a model mentioned in [9] see Theorem 55 and 52. Theorem 55 [9] states that it is consistent that for every countable $\alpha \geq 2$ there is a $Q_\alpha$-set. The proof is similar to Theorem 52 in that these sets are all of different cardinality. By an analogous argument to Theorem 4.10, in fact, those $Q_\alpha$-sets are strong $Q_\alpha$-sets.

$\square$

**Theorem 5.10** *Remark 4.6 [20]. It is consistent that the Borel subsets of the plane are not contained in any bounded level of the $\sigma$-algebra generated by the abstract rectangles. The proof of Theorem 5.1 shows that in this situation, there does not exist a Borel function*

$$F : 2^\omega \times 2^\omega \to 2^\omega$$

*such that for every Borel $H : 2^\omega \times 2^\omega \to 2^\omega$ there exist functions $g$ and $h$ from $2^\omega$ to $2^\omega$ such that*

$$H(x, y) = F(g(x), h(y))$$

*for all $x, y \in 2^\omega$. Hence, if we drop the condition that $k$ is Borel in Theorem 5.4 it is consistent that there be no such Borel $F$.*

We show that:

**Theorem 5.11** *If for unboundedly many $\alpha < \omega_1$ there is a strong $Q_\alpha$-set, then there is no countable $\alpha_0$ such that*

$$\mathit{Borel}(2^\omega \times 2^\omega) \subseteq \Sigma_{\alpha_0}(\{A \times B \ : \ A, B \subseteq 2^\omega\}).$$

*proof:*

*Let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0+1}$-set and $X$ a strong $Q_{\alpha_0+1}$-set. Choose $A_n \times B_n \subseteq 2^\omega \times 2^\omega$ for $n < \omega$ so that $U \in \Sigma_{\alpha_0}(\{A_n \times B_n \ : \ n < \omega\})$.*

Define

$$\mathcal{H} = \{X \cap C \ : \ C \text{ clopen } \subseteq 2^\omega\} \cup \{B_n \cap X : n < \omega\}.$$

**Claim**. $\mathrm{ord}(\mathcal{H}) \leq \alpha_0$, so $X$ is not a strong $Q_{\alpha_0+1}$-set.

proof:

For any $Y \subseteq X$ there exists $z \in 2^\omega$ such that $U_z \cap X = Y$, because $U$ is a universal $\Sigma_{\alpha_0+1}$-set and $X$ is a $Q_{\alpha_0+1}$-set. But any cross section of set in $\Sigma_{\alpha_0}(\{A_n \times B_n : n < \omega\})$ is a set in $\Sigma_{\alpha_0}(\{B_n : n < \omega\})$. It follows that $Y \in \Sigma_{\alpha_0}(\mathcal{H})$. Hence $\mathcal{P}(X) = \Sigma_{\alpha_0}(\mathcal{H})$ and so $\text{ord}(\mathcal{H}) \leq \alpha_0$.

This proves the Claim and hence the Theorem.

□

The following theorem was proved in May 2017. It answers negatively the rectangular form of a question asked by Bing, Bledsoe, and Mauldin [2] in the paragraph just before Theorem 10, namely, in Theorem 5.6 (2,3) can we replace "$\mathcal{A} \subseteq \undertilde{\Sigma}^0_\alpha(\mathcal{B})$" with "$\mathcal{A} \subseteq \undertilde{\Sigma}^0_{\omega_1}(\mathcal{B})$"? I want to thank Ashutosh Kumar for bringing the problem to my attention. I do not know the answer for the square version of this question.

**Theorem 5.12** *Suppose for unboundedly many $\alpha < \omega_1$ there exists a strong $Q_\alpha$-set, then there exists an uncountable cardinal $\kappa < \mathfrak{c}$ such that*

1. *every family of size $\kappa$ of sets of reals is included in a countably generated $\sigma$-algebra, $\forall \mathcal{A} \in [\mathcal{P}(\mathfrak{c})]^\kappa \quad \exists \mathcal{B} \in [\mathcal{P}(\mathfrak{c})]^\omega \quad \mathcal{A} \subseteq \undertilde{\Sigma}^0_{\omega_1}(\mathcal{B})$.*

2. *there is a family of size $\kappa$ of sets of reals which is not include in a bounded level of any countably generated $\sigma$-algebra. $\exists \mathcal{A} \in [\mathcal{P}(\mathfrak{c})]^\kappa$ $\forall \mathcal{B} \in [\mathcal{P}(\mathfrak{c})]^\omega \ \forall \alpha < \omega_1 \quad \mathcal{A} \not\subseteq \undertilde{\Sigma}^0_\alpha(\mathcal{B})$.*

proof:

Let $\Gamma \subseteq \omega_1$ be unbounded so that for every $\alpha \in \Gamma$ there exists a strong $Q_\alpha$-set $X_\alpha \subseteq 2^\omega$. Note that for $\alpha < \beta$ elements of $\Gamma$ that $|X_\alpha| < |X_\beta|$. Let $\kappa_\alpha = |X_\alpha|$ and put $\kappa = \sup_{\alpha \in \Gamma} \kappa_\alpha$. Given any family $\mathcal{A} \subseteq \mathcal{P}(\mathfrak{c})$ of size $\kappa$ of write $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$ where $|\mathcal{A}_\alpha| = \kappa_\alpha$. By Theorems 5.6,5.8 $\mathcal{A}_\alpha$ is included in the $\sigma$-algebra generated by a countable set $\mathcal{B}_\alpha$, i.e., $\mathcal{A}_\alpha \subseteq \undertilde{\Sigma}^0_\alpha(\mathcal{B}_\alpha)$. but then $\bigcup\{\mathcal{B}_\alpha : \alpha \in \Gamma\}$ is included in a countably generated $\sigma$-algebra and hence so is $\mathcal{A}$. This proves item (1).

To prove (2) note that for each $\alpha \in \Gamma$ there is $\mathcal{A}_\alpha \subseteq \mathcal{P}(\mathfrak{c})$ of cardinality $\kappa_\alpha$ which is not in any countably generated $\sigma$-algebra at a level before $\alpha$, i.e., $\mathcal{A}_\alpha \not\subseteq \undertilde{\Sigma}^0_\beta(\mathcal{B})$ for any countable $\mathcal{B} \subseteq \mathcal{P}(\mathfrak{c})$ and $\beta < \alpha$. It follows that $\mathcal{A} = \bigcup_{\alpha \in \Gamma} \mathcal{A}_\alpha$ satisfies (2).

□

# 6 Universal Functions of Higher Dimension

The main result of this section is from ([20]). Universal functions $F$ of higher dimensions reduce to countably many cases where the only thing that matters is the arity of the parameter functions, e.g., one of the forms:

$G(x, y) = F(h(x), k(y))$
$G(x, y, z) = F(h(x, y), k(y, z), l(x, z))$
$G(x_1, x_2, x_3, x_4) = F(h(x_2, x_3, x_4), k(x_1, x_3, x_4), l(x_1, x_2, x_4), i(x_1, x_2, x_3))$
$\ldots$
$G(x_0, \ldots, x_n) = F(\vec{x}_s : s \in [n+1]^n)$
$\ldots$

Furthermore, each of these forms is consistently weaker than the preceding one.

**Definition 6.1** *A $k$-dimensional universal function is a function*

$$F : (2^\omega)^k \to 2^\omega$$

*such that for every function $G : (2^\omega)^k \to 2^\omega$ there is $h : 2^\omega \to 2^\omega$ such that*

$$G(x_1, x_2, \ldots, x_k) = F(h(x_1), h(x_2), \ldots, h(x_k))$$

*for all $(x_1, x_2, \ldots, x_k) \in (2^\omega)^k$.*

**Proposition 6.2** *Suppose $F(x, y)$ is a universal function, then $F(F(x, y), z)$ is a 3-dimensional universal function. Similarly the existence of a universal function in dimension 2 is equivalent to the existence of a universal function in dimension $k$ for any $k > 1$.*

proof:

Given $G(x, y, z)$ define $G_0(u, z) = G(u_0, u_1, z)$ using unpairing, $u = \langle u_0, u_1 \rangle$. By universality of $F$ there are $g, h$ with $G_0(u, z) = F(g(u), h(z))$. Again by universality of $F$ there are $g_0, g_1$ with $g(\langle u_0, u_1 \rangle) = F(g_0(u_0), g_1(u_1))$ and hence $G(x, y, z) = F(F(g_0(x), g_1(y)), h(z))$.

To prove a 3-dimensional implies a 2-dimensional use unpairing, i.e., put $\hat{F}(u, y) = F(u_1, y, u_2)$ so if $G(x, y) = F(h(x), k(y), j(z))$, then putting $\hat{h}(x) = \langle h(x), j(0) \rangle$ we have $G(x, y) = F(h(x), k(y), j(0)) = \hat{F}(\hat{h}(x), k(y))$.
$\square$

This proposition is true for either Borel or abstract universal functions, but note however that the Baire complexity of $F(F(x, y), z)$ is higher than that of $F$. The question "Is it consistent that the Borel rank in different dimensions is different ?", is open.

Juris Steprans recently pointed out (Feb 2017) that the obvious attempt to prove that a 3-dimensional universal function implies the existence of a 2-dimensional universal function, namely freezing a coordinate, may not work; i.e., putting $\hat{F}(x,y) = F(x,y,z_0)$, because different $G$ might require different $z_0$. However, it almost works. Here is his proof: Suppose $F : \kappa^3 \to \kappa$ is universal, i.e., for every $G : \kappa^3 \to \kappa$ there are $h, k, j : \kappa \to \kappa$ such that $G(x,y,z) = F(h(x), k(y), j(z))$ for all $x, y, z \in \kappa$. Let $A_z \subseteq \kappa$ for $z \in \kappa$ partition $\kappa$ into sets of size $\kappa$. Define $F_z(x,y) = F(x,y,z)$. Then we claim that for some $z$ the map $F_z$ restricted to $A_z \times A_z$ is universal for all maps from $A_z \times A_z$ to $\kappa$. Suppose not. For each $z$ let $G_z : A_z \times A_z \to \kappa$ witness that it is not universal. Take any $G : \kappa \times \kappa \to \kappa$ which extends all $G_z$. Since $F$ is universal there are $h, k, j$ with $G(x,y) = F(h(x), k(y), j(z))$ all $x, y, z \in \kappa$. Letting $z_0 = j(0)$ gives us that

$$G_{z_0}(x,y) = G(x,y) = F(h(x), k(y), z_0) = F_{z_0}(h(x), k(y)) \text{ for all } x, y \in A_{z_0}$$

which contradicts the choice of $G_{z_0}$.

This argument requires that we restrict to a subset $A_z$ of $\kappa$, we don't know if there could be a 3-dimensional universal $F$ such no $F_z$ is 2-dimensional universal with respect to maps on all of $\kappa^2$.

We may also consider universal functions $F$ where the parameters functions are functions of more than one variable, for example:

$$\forall G\, \exists g, h, k\, \forall x, y, z \quad G(x,y,z) = F(g(x,y), h(y,z), k(z,x)).$$

This form easily follows from the existence of a dimension 3 universal. Note that by using pairing functions we can always combine parameter functions which have the same sequence of variables. The reader can imagine many variants. For example,

$G(x,y,z) = F(g(x,y), h(y,z))$

$G(x_1, x_2, x_3, x_4) = F(g_1(x_1,x_2), g_2(x_2,x_3), g_3(x_3,x_4), g_4(x_4,x_1))$

where we have omitted quantifiers for clarity. These two variants are equivalent to the existence of 2-dimensional universal function. To see this in the first example put $y = 0$ and get

$G(x,z) = F(g(x,0), h(0,z))$.

In the second example put $x_2 = x_4 = 0$ and get

$G(x_1, x_3) = F(g_1(x_1,0), g_2(0,x_3), g_3(x_3,0), g_4(0,x_1))$.

More generally, suppose $F$ and $\vec{x_k}$'s have the property that for every $G$ there are $g_k$'s such that for all $\vec{x}$

$$G(\vec{x}) = F(g_1(\vec{x}_1), \ldots, g_n(\vec{x}_n)).$$

Suppose there are two variables $x$ and $y$ from $\vec{x}$ which do not simultaneously belong to any $\vec{x}_k$. Then we get a universal 2-dimensional function simply by putting all of the other variables equal to zero.

**Proposition 6.3** *If there is a $(3,2)$-dimensional universal function, i.e., an $F(x,y,z)$ such that for every $G$ there is $h$ with*

$$G(x,y,z) = F(h(x,y), h(y,z), h(z,x)) \text{ all } x,y,z$$

*then for every $n > 3$ there is a $(n,2)$-dimensional universal function $F$, i.e., for every $G$ $n$-ary there is a binary $h$ with*

$$G(x_1, x_2, \ldots, x_n) = F(\langle h(x_i, x_j) : 1 \leq i < j \leq n \rangle) \text{ all } \vec{x}.$$

*$F$ is $\binom{n}{2}$-ary. Conversely, if there is a $(n,2)$-dimensional universal function for some $n > 3$, then there is a $(3,2)$-dimensional universal function.*

proof:
Consider the case for $n = 4$.
Suppose that $F$ is $(3,2)$-dimensional universal function. Given a 4-ary function $G(x,y,z,w)$ for each fixed $w$ we get a function $h_w(u,v)$ with

$$G(x,y,z,w) = F(h_w(x,y), h_w(y,z), h_w(z,x)) \text{ for all } x,y,z.$$

But now considering $h(u,v,w) = h_w(u,v)$ we get a function $k(s,t)$ with $h(u,v,w) = F(k(u,v), k(v,w), k(w,u))$. Note that

$$\begin{aligned}
&G(x,y,z,w) = \\
&\quad F(F(k(x,y), k(y,w), k(w,x)), \\
&\qquad F(k(y,z), k(z,w), k(w,y)), \\
&\qquad\quad F(k(z,x), k(x,w), k(w,z))).
\end{aligned}$$

Note that $k(s,t)$ and $k(t,s)$ can be combined by pairing and unpairing into a single function $k_1(s,t)$. From this one can define a $(4,2)$-dimensional universal function.

For the converse, if $F$ is a $(4,2)$-dimensional universal function, then for every $G$ 3-ary, there exists $h$ binary with

$$G(x,y,z) = F(h(x,y), h(y,z), h(x,z), h(x,0), h(y,0), h(z,0)).$$

But note that, for example, $h(x,y)$ and $h(x,0)$ can be combined into a single function of $h_1(x,y)$. Hence we can get a $(3,2)$-dimensional universal function.

□

To state the generalization of these ideas

**Definition 6.4** *Let $U(\kappa,m,n)$ mean that we have a $(m,n)$-dimensional universal function on $\kappa$. This means for $k = \binom{n}{m}$ there exists $F : \kappa^k \to \kappa$ such that for every $G : \kappa^m \to \kappa$ there is $h : \kappa^n \to \kappa$ such that*

$$G(x_0, x_1, \ldots, x_{m-1}) = F\left(h(x_j : j \in Q) : Q \in [m]^n\right) \text{ for all } \vec{x} \in \kappa^m.$$

Then the last two propositions can be generalized to show:

**Proposition 6.5** *For any infinite cardinal $\kappa$ and positive integer $n$*

1. *$U(\kappa, n+1, n)$ implies $\forall m > n \; U(\kappa, m, n)$.*
2. *$(\exists m > n \; U(\kappa, m, n))$ implies $U(\kappa, n+1, n)$.*
3. *$U(\kappa, n+1, n)$ implies $U(\kappa, n+2, n+1)$*

We show that $U(\kappa, n+1, n)$ are the only generalized multi-dimensional universal functions properties.

**Definition 6.6** *Suppose $\Sigma \subseteq \mathcal{P}(\{0,1,2,\ldots,n-1\}) = \mathcal{P}(n)$ (the power set of $n$). Define $U(\kappa, n, \Sigma)$ to mean that there exists $F : \kappa^\Sigma \to \kappa$ such that for every $G : \kappa^n \to \kappa$ there are $h_Q : \kappa^{|Q|} \to \kappa$ for $Q \in \Sigma$ such that*

$$G(x_0, x_1, \ldots, x_{n-1}) = F\left(h_Q(x_j : j \in Q) : Q \in \Sigma\right) \text{ for all } \vec{x} \in \kappa^n.$$

**Proposition 6.7** *Let $\kappa$ be an infinite cardinal, $n \geq 2$, and $\Sigma, \Sigma_0, \Sigma_1$ subsets of $\mathcal{P}(n)$.*

1. *If $\Sigma_0 \subseteq \Sigma_1$ , then $U(\kappa, n, \Sigma_0)$ implies $U(\kappa, n, \Sigma_1)$.*

2. *If* $Q_0 \subseteq Q_1 \in \Sigma$*, then* $U(\kappa, n, \Sigma)$ *is equivalent to* $U(\kappa, n, \Sigma \cup \{Q_0\})$*.*

3. *Suppose* $\Sigma$ *is closed under taking subsets, every* $k < n$ *is in some element of* $\Sigma$*, and* $\{0, 1, 2, \ldots, n-1\} \notin \Sigma$*. Let* $k+1$ *be the size of the smallest subset of* $\{0, 1, 2, \ldots, n-1\}$ *not in* $\Sigma$*. Then* $U(\kappa, n, \Sigma)$ *is equivalent to* $U(\kappa, k+1, k)$*.*

proof:

(1) This is true because the $F$ which works for $\Sigma_0$ also works for $\Sigma_1$ by ignoring the values of $h_Q$ for $Q \in \Sigma_1 \setminus \Sigma_0$.

(2) This is true because given $h_{Q_0}, h_{Q_1}$ we may define a new $\hat{h}_{Q_1}$ by outputting the pairing

$$\hat{h}_{Q_1}(x_j : j \in Q_1) = \langle (h_{Q_0}(x_j : j \in Q_0), (h_{Q_1}(x_j : j \in Q_1) \rangle$$

(3) First note that by (2) we may as well assume that $\Sigma$ is closed under taking subsets. If some $k$ does not appear in any element of $\Sigma$, then $U(\kappa, n, \Sigma)$ is trivially false. If $\{0, 1, 2, \ldots, n-1\}$ is in $\Sigma$, then $U(\kappa, n, \Sigma)$ is trivially true.

So let $R \subseteq \{0, 1, \ldots n-1\}$ not in $\Sigma$ with $|R| = k+1$. By choice of $k+1$ all subsets of $R$ of size $k$ are in $\Sigma$. By setting $x_i = 0$ for $i \notin R$, we see that $U(\kappa, k+1, k)$ is true.

Now assume $U(\kappa, k+1, k)$ is true. By Proposition 6.5 we have that $U(\kappa, n, k)$ is true and hence if $\Sigma_0 = [n]^k$ then $U(\kappa, n, \Sigma_0)$ is true. But $\Sigma_0 \subseteq \Sigma$ and so by (1), $U(\kappa, n, \Sigma)$ is true.

$\square$

**Proposition 6.8** *The following are true in ZFC.*

1. $U(\omega, 2, 1)$

2. $U(\omega_1, 3, 2)$

3. $U(\kappa, 2, 1)$ *implies* $U(\kappa^+, 3, 2)$

4. $U(\kappa, n+1, n)$ *implies* $U(\kappa^+, n+2, n+1)$

5. $U(\omega_n, n+2, n+1)$ *every* $n \geq 0$*.*

proof:

For (1) see Theorem 5.2. We prove (2) and leave 3-5 to the reader.

Suppose that $f : \omega^2 \to \omega$ witnesses $U(\omega, 2, 1)$. For any countable ordinal $\delta > 0$ let $\delta = \{\delta_i : i < \omega\}$. Define

$$F_0(\delta, n, m) = \delta_{f(n,m)}.$$

Now suppose $G : \omega_1^3 \to \omega_1$. Define

$$k(\delta) = \sup\{G(\alpha, \beta, \gamma) \ : \ \alpha, \beta, \gamma \leq \delta\} + 1$$

For any $\gamma < \omega_1$ let $\gamma^* = k(\gamma)$. Define $g : \omega^2 \to \omega$ by

$$G((\gamma + 1)_n, (\gamma + 1)_m, \gamma)) = \gamma^*_{g(n,m)}.$$

By the universality property of $f$ there exists $h : \omega \to \omega$ with

$$g(n, m) = f(h(n), h(m)) \text{ for every } n, m \in \omega.$$

For $\delta \leq \gamma$ define $h_1(\delta, \gamma) = h(k)$ where $\delta = (\gamma + 1)_k$. Then we have that

$$\forall \alpha, \beta \leq \gamma < \omega_1 \quad G(\alpha, \beta, \gamma) = F_0(k(\gamma), h_1(\alpha, \gamma), h_1(\beta, \gamma)).$$

Define $F$ as follows:

$F(\alpha, \beta, \gamma, \alpha^*, \beta^*, \gamma^*, n_1, m_1, n_2, m_2, n_3, m_3) =$

$$\begin{cases} F_0(\gamma^*, n_1, m_1) & \text{if } \alpha, \beta \leq \gamma \\ F_0(\beta^*, n_2, m_2) & \text{if } \gamma < \beta \text{ and } \alpha \leq \beta \\ F_0(\alpha^*, n_3, m_3) & \text{if } \beta, \gamma < \alpha \end{cases}$$

Then given $G$ we can find $k, h_1, h_2, h_3$ so that
$G(\alpha, \beta, \gamma)) =$
$F(\alpha, \beta, \gamma, k(\alpha), k(\beta), k(\gamma),$
$h_1(\alpha, \gamma), h_1(\beta, \gamma), \ \ h_2(\alpha, \beta), h_2(\gamma, \beta), \ \ h_3(\beta, \alpha), h_3(\gamma, \alpha)).$
$\square$

The $\kappa$-Cohen real model is any model of ZFC obtained by forcing with the poset of finite partial functions from $\kappa$ to 2 over a countable transitive ground model satisfying ZFC.

**Proposition 6.9** *In the $\omega_2$-Cohen real model we have that $U(\omega_1, 2, 1)$ fails. Similarly, $U(\omega_2, 3, 2)$ fails in the $\omega_3$-Cohen real model. More generally, we have that $U(\gamma, n + 1, n)$ fails in the $\kappa$-Cohen real model when $\kappa > \gamma \geq \omega_n$.*

proof:

We show that $U(\omega_2, 3, 2)$ fails in the $\omega_3$-Cohen real model, leaving the rest to the reader.

Let $M$ be a countable transitive model of ZFC and in $M$ define $\mathbb{P}$ to be the poset of finite partial maps from $\omega_3 \times \omega_3 \times \omega_3$ into 2. We claim that if $G$ is $\mathbb{P}$-generic over $M$, then there is no map $F : \omega_2 \times \omega_2 \times \omega_2 \to \omega_2$ which is (3,2)-universal for maps of the form $H : \omega \times \omega_1 \times \omega_2 \to 2$.

Suppose for contradiction that $F$ is such a map. By the ccc we may find $\gamma_0 < \omega_3$ with $F \in M[G \restriction \gamma_0^3]$. Hence we may find maps $h_1 : \omega \times \omega_1 \to \omega_3$, $h_2 : \omega \times \omega_2 \to \omega_3$, and $h_3 : \omega_1 \times \omega_2 \to \omega_3$ such that

$$H(n, \beta, \gamma) =^{def} G(n, \beta, \gamma_0 + \gamma) = F(h_1(n, \beta), h_2(n, \gamma), h_3(\beta, \gamma)).$$

for every $n < \omega, \beta < \omega_1, \gamma < \omega_2$. By ccc we can choose $\gamma_1 < \omega_2$ such that $h_1 \in M[G^*]$ where $G^*$ is $G$ restricted $\{(\alpha, \beta, \rho) \in \omega^3 \ : \ \rho \neq \gamma_0 + \gamma_1\}$. Define $g : \omega \times \omega_1 \to 2$ by

$$g(n, \alpha) = G(n, \alpha, \gamma_0 + \gamma_1)$$

Note that we have that $F, h_1 \in M[G^*]$, $g$ is Cohen generic over $M[G^*]$, and

$$g(n, \alpha) = F(h_1(n, \alpha), h_2(n, \gamma_0 + \gamma_1), h_3(\alpha, \gamma_0 + \gamma_1)).$$

Since the extension by $g$ is ccc, we may find $\alpha_0 < \omega_1$ such that

$$h_2 \in M[G^*][g \restriction (\omega \times \alpha_0)] =^{def} N.$$

But this is a contradiction because $g_{\alpha_0}$ defined by $g_{\alpha_0}(n) = g(n, \alpha_0)$ is Cohen generic over $N$. But $F, h_1, h_2 \in N$ and for any $\gamma_2 < \omega_2$ the map $k$ defined by

$$k(n) = F(h_1(n, \alpha_0), h_2(n, \gamma_0 + \gamma_1), \gamma_2) \text{ for all } n < \omega$$

is in $N$ and so can never be equal to $g_{\alpha_0}$. Thus $h_3(\alpha_0, \gamma_0 + \gamma_1) = \gamma_2$ cannot be defined.

□

**Corollary 6.10** *([20]) Let $\aleph_\omega \leq \gamma < \kappa$. In the $\kappa$-Cohen real model we have that*

$$U(\omega_n, n+2, n+1) + \neg U(\omega_n, n+1, n) \text{ for all } n > 0,$$

*and*

$$\neg U(\gamma, n+1, n) \text{ for all } n > 0.$$

*Hence, in the Cohen real model for every $n \geq 1$ there is a universal function on $\omega_n$ where the parameter functions have arity $n+1$ but no universal function where the parameters functions have arity $n$.*

# 7 Model Theoretic Universal

**Theorem 7.1** *(Remark 7.13 [20].) If $2^{<\mathfrak{c}} = \mathfrak{c}$ and there is a Borel universal map, i.e. Borel $F : 2^\omega \times 2^\omega \to 2^\omega$ such that for every $G : 2^\omega \times 2^\omega \to 2^\omega$ there is $h$ such that $G(x, y) = F(h(x), h(y))$ for all $x \in 2^\omega$, then there is a Borel map $H$ such that for every cardinal $\kappa < \mathfrak{c}$ for every $G : \kappa \times \kappa \to \kappa$ there are $x_\alpha \in 2^\omega$ for $\alpha < \kappa$ such that for $\alpha, \beta, \gamma < \kappa$*

$$G(\alpha, \beta) = \gamma \textit{ iff } H(x_\alpha, x_\beta) = x_\gamma$$

proof:

By Theorems 4.6, Lemma 4.19, and Theorem 5.1 there exists $\alpha < \omega_1$, $Z \subseteq 2^\omega$ with $|Z| = \mathfrak{c}$ and every $Y \in [Z]^{<\mathfrak{c}}$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha$ in $Z$. Note that by Lemma 4.8 $\mathcal{P}(\mathfrak{c} \times \mathfrak{c}) = \sigma(\{A, B : A, B \subseteq \mathfrak{c}\})$, so if $X \subseteq Z$ of size $\kappa$ then every subset of $X^2$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha_0}$ in $X^2$ where $\alpha_0 = \alpha + \alpha$.

Let $X = \{x_\alpha : \alpha < \kappa\}$. Given any map $G : \kappa \times \kappa \to \kappa$ define

$$Y_n = \{(x_\alpha, x_\beta) \ : \ G(\alpha, \beta) = \delta \text{ and } x_\delta(n) = 1\}$$

for each $n$. Let $B_n \subseteq 2^\omega \times 2^\omega$ be $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha_0}$ so that $Y_n = B_n \cap (X \times X)$. Define the Borel map $K$ by $K(u, v)(n) = 1$ iff $(u, v) \in B_n$. Note that

$$G(\alpha, \beta) = \delta \text{ iff } K(x_\alpha, x_\beta) = x_\delta.$$

Let $L$ be Borel and universal for all such maps $K$, i.e., For all Borel $K$ of rank less than $\alpha_0 + 1$ there is a $y$ such that $\forall u, v\ K(u, v) = L(y, u, v)$. Now define $H((y, u), (y, v)) = (y, L(y, u, v))$. Putting $\hat{x}_\alpha = (y, x_\alpha)$ we have that for every $\alpha, \beta, \delta$

$$H(\hat{x}_\alpha, \hat{x}_\beta) = \hat{x}_\delta \text{ iff } G(\alpha, \beta) = \delta.$$

□

# 8 Generic Souslin sets

**Theorem 8.1** *(Marczewski see Miller[17]) If $I$ is a ccc $\sigma$-ideal in the Borel sets then the family of $I$-measurable sets is closed under the Souslin operation.*

**Theorem 8.2** *([16]) (CH) For any $\alpha$ with $2 \leq \alpha \leq \omega_1$ there is exists an uncountable $X \subseteq 2^\omega$ such that* $\text{ord}(X) = \alpha$ *and every Souslin set in $X$ is Borel in $X$.*

**Theorem 8.3** *(Miller [10]) It is consistent to have $X \subseteq 2^\omega$ such that every subset of $X$ is Souslin in $X$ and the Borel order of $X$ is $\omega_1$.*

The set $X$ also has the property that $\mathcal{P}(X)$ is not a countably generated $\sigma$-algebra.

**Theorem 8.4** *([10]) It is relatively consistent with ZFC that for every subset $A \subseteq 2^\omega \times 2^\omega$ there are abstract rectangles $B_s \times C_s$ with*

$$A = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} (B_{f\restriction n} \times C_{f\restriction n})$$

*but not every subset of $2^\omega \times 2^\omega$ is in the $\sigma$-algebra generated by the abstract rectangles.*

# 9 Products and Unions

**Theorem 9.1** *(Sierpinski 1935) Assume CH. There Luzin sets and Sierpinski sets whose square can be continuously mapped onto $2^\omega$.*

**Corollary 9.2** *(CH) For any $\alpha$ with $2 \leq \alpha < \omega_1$ there is $X \subseteq 2^\omega$ such that*

$$\text{ord}(X) = \alpha \textit{ and } \text{ord}(X^2) = \omega_1$$

proof:

Let $S \subseteq 2^\omega$ be a Sierpinski set whose square continuously maps onto $2^\omega$. Let $X_\alpha$ have order $\alpha$ (which exists by CH [9]), then the clopen separated union $X = S \oplus X_\alpha$ has order $\alpha$ and its square has order $\omega_1$ by Reclaw 1.7.

**Theorem 9.3** *(Miller [13]) (CH) There is an uncountable $\sigma$-set $X \subseteq 2^\omega$ which is concentrated on a countable set. ($\sigma$-set means $\text{ord}(X) = 2$.)*

**Theorem 9.4** *(Fleissner, Miller [4]) It is relatively consistent with ZFC to have an uncountable Q-set which is concentrated on a countable set.*

**Theorem 9.5** *(CH) For any $\alpha_0$ with $3 \leq \alpha_0 < \omega_1$ there are $X_0, X_1 \subseteq 2^\omega$ with $\text{ord}(X_0) = \alpha_0 = \text{ord}(X_1)$ and $\text{ord}(X_0 \cup X_1) = \alpha_0 + 1$.*

proof:

Let $T$ be a nice $\alpha_0$-tree (see Definition 3.2).

**Definition 9.6** *Define $\mathbb{P}_T$ by $p \in \mathbb{P}_T$ iff $p : T \to 2$ is a finite partial function such that for all $s \in T^0$ and $n < \omega$ if $s, sn \in dom(p)$ and $p(s) = 1$, then $p(sn) = 0$. Define the rank of $p$: $|p| = \max\{rank_T(s) \ : \ s \in dom(p)\}$.*

**Lemma 9.7** *Rank Lemma. For all countable $\beta \geq 1$ for all $p \in \mathbb{P}_T$ there exists $\hat{p} \in \mathbb{P}_T$ compatible with $p$, $|\hat{p}| \leq \beta$, and for all $q \in \mathbb{P}_T$ with $|q| < \beta$, $(p \perp q) \to (\hat{p} \perp q)$.*

proof:

As usual $\perp$ stands for incompatible. First extend $p$ to $p^*$ which has the property that for any $s \in \text{dom}(p)$ with $p(s) = 1$ and $\text{rank}_T(s) = \lambda$ a limit ordinal greater than $\beta$ and $n$ such that $\text{rank}_T(sn) < \beta$ there exists $m$ such that $p^*(snm) = 1$. Note that by the definition of nice tree there are at most finitely many $sn$.

Now define $\hat{p} = p* \restriction \{s \in \text{dom}p^* : \text{rank}_T(s) \leq \beta\}$. Suppose $|q| < \beta$ and $p \perp q$. So one of the following must be true:

1. There exists $s \in \text{dom}(q) \cap \text{dom}(p)$ with $p(s) \neq q(s)$.
2. There exists $s \in \text{dom}(q)$ and $sn \in \text{dom}(p)$ with $q(s) = 1$ and $p(sn) = 1$.
3. There exists $s \in \text{dom}(p)$ and $sn \in \text{dom}(q)$ with $p(s) = 1$ and $q(sn) = 1$.

In the first case since $|q| < \beta$ we have that $s \in \text{dom}(\hat{p})$ so $\hat{p} \perp q$. In the second case $\text{rank}_T(sn) < \text{rank}_T(s) < \beta$ so again $\hat{p} \perp q$. In the third case $\text{rank}_T(sn) < \beta$ so either $\text{rank}_T(s) \leq \beta$ (so $\hat{p} \perp q$) or $\text{rank}_T(s) = \lambda > \beta$ a limit ordinal. By the construction of $p^*$ there is some $m$ with $p^*(snm) = 1$. Since $\hat{p}(snm) = 1$ and $q(sn) = 1$ it follows that $\hat{p} \perp q$.
$\square$

**Definition 9.8** *For $G$ sufficiently $\mathbb{P}_T$-generic, its union, $\bigcup G$, will be a map from $T$ into 2. Let $g$ be the restriction of $\bigcup G$ to $T^*$, the terminal nodes of $T$. So $g : T^* \to 2$ is defined by $g(s) = i$ iff $\exists p \in G$ with $p(s) = i$.*

**Lemma 9.9** *Suppose $\alpha \geq 1$, $B(v)$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ predicate on $2^{T^*}$ coded in $M$, and $p \in \mathbb{P}_T$. If $p \Vdash B(\overset{\circ}{g})$, then there exists $\hat{p}$ compatible with $p$, $|\hat{p}| < \alpha$, and $\hat{p} \Vdash B(\overset{\circ}{g})$.*

proof:

Case $\alpha = 1$. Suppose $(B(v) \iff \exists n\ C_n(v))$ where $C_n$ are clopen. Take $q \le p$ and $n$ so that $q \Vdash C_n(\overset{\circ}{g})$. By extending $q$ we may assume that $C_n(f)$ holds for all $f \in T^*$ with $q \upharpoonright T^* \subseteq f$. Then $\hat{p} =^{def} q \upharpoonright T^*$ is as required.

Case $\alpha > 1$. Suppose $(B(v) \iff \exists n\ B_n(v))$ where $B_n$ is $\underset{\sim}{\mathbf{\Pi}}^0_{\beta_n}$ for some $\beta_n < \alpha$. Let $p_1 \le p$ and $n < \omega$ be such that $p_1 \Vdash B_n(\overset{\circ}{g})$. Let $\hat{p_1}$ be obtained from the Rank Lemma for $\beta = \beta_n$. Then it must be that $\hat{p_1} \Vdash B_n(\overset{\circ}{g})$. Otherwise there exists $p_2 \le \hat{p}_1$ such that $p_2 \Vdash \neg B_n(\overset{\circ}{g})$. By induction there $q$ compatible with $p_2$ (and hence with $\hat{p_1}$), $|q| < \beta_n$ and $q \Vdash \neg B_n(\overset{\circ}{g})$. But by the Rank Lemma such a $q$ would be compatible with $p_1$, contradiction.

□

**Lemma 9.10** *If $B(v)$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$ predicate on $2^{T^*}$ coded in $M$, then there exists $G$ $\mathbb{P}_T$-generic over $M$ such that $B(g)$ iff $G(\langle\rangle) = 0$.*

proof:

If not, $1 \Vdash$ "$B(\overset{\circ}{g})$ iff $G(\langle\rangle) = 1$". Let $p = (\langle\rangle, 1)$, so $p \Vdash G(\langle\rangle) = 1$ and therefor $p \Vdash B(\overset{\circ}{g})$. By the Lemma 9.9 there is $q$ compatible with $p$, $|q| < \alpha_0$ and $q \Vdash B(\overset{\circ}{g})$. But note that $q^* = q \cup \{(\langle\rangle, 0)\}$ is a condition because $\langle\rangle$ is not in the domain of $q$ since it has rank $\alpha_0$. But $q^* \Vdash$" $G(\langle\rangle) = 0$ and $B(\overset{\circ}{g})$" which is a contradiction.

□

Now we prove Theorem 9.5. Let $M_\beta \preceq H_\kappa$ for $\beta < \omega_1$ be countable elementary substructures of $H_\kappa$ for some sufficiently large regular $\kappa$, so that $\beta < \gamma$ implies $M_\beta \preceq M_\gamma$ and $\mathcal{P}(\omega) \subseteq \bigcup_{\beta<\omega_1} M_\beta$. Choose $G_\alpha$ $\mathbb{P}_T$-generic over $M_\alpha$ with the property that for any $B(v)$ a $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$ predicate on $2^{T^*}$ for some $\alpha$ $B(g_\alpha)$ iff $G_\alpha(\langle\rangle) = 0$. Let $X = \{g_\alpha \in 2^{T^*} : \alpha < \omega_1\}$ and for $i = 0, 1$ let $X_i = \{g_\alpha : G_\alpha(\langle\rangle) = i\}$. Define $U_s \subseteq 2^{T^*}$ for $s \in T$ as follows. $U_s = \{x \in 2^{T^*} : x(s) = 1\}$. For $s \in T^0$ $\quad U_s = \bigcap_{n<\omega} \sim U_{sn}$. Note that if $\text{rank}_T(s) = \beta$ then $U_s$ is $\underset{\sim}{\mathbf{\Pi}}^0_\beta$. Therefor since $U_{\langle\rangle} \cap X = X_1$ we have that $X_1$ is a $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_0}$ subset of $X$ which by construction is not $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha_0}$. So $\text{ord}(X) \ge \alpha_0 + 1$.

Define for $p \in \mathbb{P}_T$ $[p] = \bigcup\{U_s : p(s) = 1\} \cup \bigcup\{\sim U_s : p(s) = 0\}$. Note that $[p]$ is $\underset{\sim}{\mathbf{\Delta}}^0_{\alpha_0+1}$. For any Borel $B \subseteq 2^{T^*}$ there is an $\alpha < \omega_1$ with $B$ coded in $M_\alpha$. For all $\gamma \ge \alpha$ $G_\gamma$ is $\mathbb{P}_T$-generic over $M_\gamma$ and hence

$$g_\gamma \in B \text{ iff } \exists p \in G_\gamma \ \ p \Vdash g_\gamma \in B.$$

Note that by Borel absoluteness $g_\gamma \in B$ iff $M_\gamma[g_\gamma] \models g_\gamma \in B$. Let

$$\Sigma =^{def} \{p \in \mathbb{P}_T \ : \ p \Vdash \overset{\circ}{g} \in B\}.$$

Let $X^{\geq\alpha} = \{g_\gamma \ : \ \gamma \geq \alpha\}$. Then

$$X^{\geq\alpha} \cap B = X^{\geq\alpha} \cap \bigcup_{p\in\Sigma} [p].$$

Since we add and subtract a countable set from any $\utilde{\mathbf{\Delta}}^0_{\alpha_0+1}$ set and remain $\utilde{\mathbf{\Delta}}^0_{\alpha_0+1}$, we see that $\mathrm{ord}(X) \leq \alpha_0 + 1$.

For any $p \in \mathbb{P}_T$ and $i = 0, 1$ we have that $[p] \cap X_i$ is $\utilde{\mathbf{\Delta}}^0_{\alpha_0}$ since it is either empty (if $p(\langle\rangle) = 1 - i$) or equal to $[p^*] \cap X_i$ where $p^* = p \setminus \{(\langle\rangle, i)\}$. It follows that $\mathrm{ord}(X_i) \leq \alpha_0$.

To get an example with order exactly $\alpha_0$: Either by modifying the above construction or using the Luzin set argument from [9] Thm 18, we can get $Y \subseteq 2^\omega$ with order exactly $\alpha_0$. Let $X_i^+ =^{def} X_i \oplus Y$ where $\oplus$ means to take a clopen separated union. Then $\mathrm{ord}(X_i^+) = \alpha_0$ and $X_0^+ \cup X_1^+ = X \oplus Y$ has order $\alpha_0 + 1$.

This proves Theorem 9.5.

Note that in case $\alpha_0$ is a successor order we can use the above proof to get $Z_0, Z_1$ such that $\mathrm{ord}(Z_0) = \alpha_0$, $\mathrm{ord}(Z_1) = \alpha_0 - 1$, and $\mathrm{ord}(Z_0 \cup Z_1) = \alpha_0 + 1$. To see this note that since $p(\langle\rangle) = 1$ implies $p(\langle n\rangle) = 0$ for all $\langle n\rangle \in \mathrm{dom}(p)$ we have that $\mathrm{ord}(X_1) \leq \alpha_0 - 1$. Take $Y'$ with $\mathrm{ord}(Y') = \alpha_0 - 1$ and put $Z_0 = X_0 \oplus Y \oplus \emptyset$ and $Z_1 = X_1 \oplus \emptyset \oplus Y'$.

# 10 Invariant Descriptive Set Theory

For $\rho$ a countable similarity type let $X_\rho$ be the Polish space of $\rho$-structures with universe $\omega$. For example if $\rho = \{R, f, U, c\}$ where $R$ is a binary relation symbol, $f$ a binary operation symbol, $U$ a unary operation symbol and $c$ a constant symbols then

$$X_\rho = 2^{\omega\times\omega} \times \omega^{\omega\times\omega} \times 2^\omega \times \omega.$$

The language $L_{\omega_1,\omega}(\rho)$ is obtained by adding countably infinite conjunctions and disjunctions to the usual first order logical axioms. For example:

$$\forall y \bigvee_{n<\omega} \exists x_1, x_2, \ldots x_n \, \forall z (R(z, y) \to \vee_{i=1}^n z = x_i)$$

which says that for every $y$ there are at most finitely many $z$ with $R(z, y)$.

We consider only formulas with at most finitely many free variables. Inductively define $\Sigma_\alpha$ and $\Pi_\alpha$ formulas as follows: $\Sigma_0 = \Pi_0$ formulas are the ordinary first-order quantifier free finite formulas. For $\alpha > 0$ a formula $\phi(\vec{y})$ is $\Sigma_\alpha$ iff there are $\phi_n(\vec{x_n}, \vec{y})$ each a $\Pi_{<\alpha}$ formula and

$$\phi(\vec{y}) = \bigvee_{n<\omega} \exists \vec{x_n} \phi_n(\vec{x_n}, \vec{y}).$$

A formula $\phi(\vec{y})$ is $\Pi_\alpha$ iff there are $\phi_n(\vec{x_n}, \vec{y})$ each a $\Sigma_{<\alpha}$ formula and

$$\phi(\vec{y}) = \bigwedge_{n<\omega} \forall \vec{x_n} \phi_n(\vec{x_n}, \vec{y})$$

**Theorem 10.1** *(Mostowski see Kuratowski [8] page ???) If $\theta$ is a $\Sigma_\alpha$ sentence of $L_{\omega_1,\omega}(\rho)$, then the set of models of $\theta$ is a $\Sigma^0_\alpha$ Borel subset of $X_\rho$.*

proof:

For any $n$ and formula $\theta(\vec{x})$ where $\vec{x} = x_0, \ldots, x_{n-1}$ includes all free variable of $\theta$ and any $s \in \omega^n$ consider the models of $\theta(s)$:

$$\{M \in X_\rho \; : \; M \models \theta(s(0), s(1), \ldots, s(n-1))\}$$

Details left to reader.
□

**Theorem 10.2** *(Scott 1964 see Barwise [1]) For any countable structure $A$ in a countable similarity type $\rho$, there is a sentence $\theta$ of $L_{\omega_1,\omega}(\rho)$ such that for any countable $\rho$-structure $B$*

$$A \simeq B \textit{ iff } B \models \theta$$

A subset of $X_\rho$ is invariant iff it is closed under isomorphism. Lopez-Escobar (1965) showed that invariant Borel subsets of $X_\rho$ are the models of an $L_{\omega_1,\omega}(\rho)$-sentence. Vaught proved a hierarchy version of this:

**Theorem 10.3** *(Vaught [24]) Any $\Pi^0_\alpha$ subset of $X_\rho$ which is closed under isomorphism is the set of models of a $\Pi_\alpha$ sentence of $L_{\omega_1,\omega}(\rho)$*

proof:

Let $S_\infty \subseteq \omega^\omega$ be the Polish group of bijections of $\omega$. It's action on $X_\rho$ is isomorphism. For example, given $\pi \in S_\infty$ and $R \subseteq \omega \times \omega$ a binary relation, then $(\omega, R)$ is isomorphic to $(\omega, S)$ via $\pi$ where $S$ is defined by $R(x, y)$ iff $S(\pi(x), \pi(y))$. $S_\infty \times X_\rho \to X_\rho$ $(\pi, R) \mapsto S$ is a continuous action.

The following is the Vaught transform: For each $n \in \omega$ and one-to-one map $s : n \to \omega$ define $(A, s) \in \mathcal{B}^{*n}$ iff there are comeagerly many $\pi \in [s]$ such that $\pi^{-1}(A) \in \mathcal{B}$. For $n = 0$ then $\mathcal{B}^*$ is the set of all $A \in X_\rho$ for which there are comeagerly many $\pi \in S_\infty$ such that $\pi^{-1}(A) \in \mathcal{B}$ (or equivalent comeagerly many $\pi$ with $\pi(A) \in \mathcal{B}$).

**Lemma 10.4** *Suppose $(\mathcal{B}_n \subseteq X_\rho : n < \omega)$ and $m < \omega$, then*

$$(\bigcap_{n<\omega} \mathcal{B}_n)^{*m} = \bigcap_{n<\omega} \mathcal{B}_n^{*m}$$

proof:

The countable intersection of comeager sets is comeager.

□

**Lemma 10.5** *Suppose $\mathcal{B} \subseteq X_\rho$ is Borel and $n < \omega$, then for any $s : n \to \omega$ one-to-one,*

$$(A, s) \in \mathcal{B}^{*n} \quad \textit{iff} \quad \neg\exists t \supseteq s \;\; (A, t) \in (\sim \mathcal{B})^{*|t|}$$

*where $\sim \mathcal{B}$ is the complement of $\mathcal{B}$.*

proof:

For any $(A, s)$ the set $\{\pi \in S_\infty \;:\; \pi^{-1} \in \mathcal{B}\}$ is the continuous preimage of a Borel set and hence is Borel and so has the property of Baire. For a set with the property of Baire, it either is comeager or its complement is somewhere comeager.

□

**Lemma 10.6** *For any $n, \alpha \geq 1$ and $\mathcal{B} \subseteq X_\rho$ a $\Pi^0_\alpha$ set, there is a $\Pi_\alpha$ formula $\theta(\vec{v})$ such that for any $A \in X_\rho$ and $s : n \to \omega$ one-to-one*

$$(A, s) \in \mathcal{B}^{*n} \textit{ iff } (A, s) \models \theta(s(0), \ldots, s(n-1)).$$

proof:

First assume $\alpha = 1$ and $\mathcal{B} \subseteq X_\rho$ is clopen. Then for some quantifier-free finite formula

$$\mathcal{B} = \{A \in X_\rho \ : \ A \models \theta(0, 1, 2, \ldots, m)\}$$

Without loss, we may assume $m > n$.

Then by definition $(A, s) \in \mathcal{B}^{*n}$ iff $\{\pi \ : \pi^{-1}(A) \in \mathcal{B}\}$ is comeager in $[s]$. But since $\mathcal{B}$ is clopen, $\{\pi \ : \pi^{-1}(A) \in \mathcal{B}\}$ is comeager in $[s]$ iff $\pi^{-1}(A) \in \mathcal{B}$ for all $\pi \supseteq s$. But now $\pi^{-1}(A) \in \mathcal{B}$ iff $\pi^{-1}(A) \models \theta(0, 1, 2, \ldots, m)$. iff $A \models \theta(\pi(0), \pi(1), \ldots, \pi(m)$.

Hence we get that $(A, s) \in \mathcal{B}^{*n}$ iff
$(A, s) \models \forall v_n, \ldots, v_m \ \ (D(s, \vec{v}) \rightarrow \theta(s(0), \ldots, s(n-1), v(n), \ldots, v(m)))$

where $D$ is the first-order formula saying that the $v_i$ are distinct and different from all the $s(j)$.

To finish the case of $\alpha = 1$ just use that if $\mathcal{B}$ is $\Pi^0_1$ then $\mathcal{B} = \bigcap_{m<\omega} \mathcal{B}_m$ where each $\mathcal{B}_m$ is clopen and $\mathcal{B}^{*n} = \bigcap_{m<\omega} \mathcal{B}^{*n}_m$ by Lemma 10.4.

Now assume $\alpha > 1$ and $\mathcal{B} \subseteq X_\rho$ is $\Sigma^0_\beta$ for some $\beta < \alpha$. Then $\sim \mathcal{B}$ is $\Pi^0_\beta$ and so by Lemma 10.5: $(s, A) \in \mathcal{B}^{*n}$ iff $\neg\exists t \supseteq s \ (A, t) \in (\sim \mathcal{B})^{*|t|}$ By induction hypothesis $(A, t) \in (\sim \mathcal{B})^{*|t|}$ iff $(A, t) \models \theta_m(t(0), \ldots, t(m))$ for some $\Pi_\beta$ formula $\theta_m$. And so, $(A, s) \in \mathcal{B}^{*n}$ iff
$(A, s) \models \forall v_n, \ldots, v_m \ \ (D(s, \vec{v}) \rightarrow \neg\theta(s(0), \ldots, s(n-1), v(n), \ldots, v(m)))$

Note that this is a $\Pi_{\beta+1}$ formula and so a $\Pi_\alpha$ formula. To finish this case if $\mathcal{B}$ is $\Pi^0_\alpha$ then $\mathcal{B} = \bigcap_{m<\omega} \mathcal{B}_m$ where $\mathcal{B}_m$ is $\Sigma^0_{\beta_m}$ for some $\beta_m < \alpha$. Hence applying Lemma 10.4 us the result since the countable conjunction of $\Pi_\alpha$ formulas is a $\Pi_\alpha$ formula.

□

Vaught's Theorem 10.3 follows immediately from the Lemma for the case $n = 0$ and $\mathcal{B}$ invariant, i.e., $\mathcal{B}^* = \mathcal{B}$. It also the case that invariant $\Sigma^0_\alpha$ sets are the models of a $\Sigma_\alpha$ sentence by considering complements.

**Theorem 10.7** *(Hausdorff Difference Hierarchy)* $\mathcal{B} \in \Delta^0_{\alpha+1}$ *iff there exists a countable sequence of decreasing* $\Pi^0_\alpha$ *sets* $\mathcal{B}_\beta$ *for* $\beta < \gamma$ *such that*

$$\mathcal{B} = \bigcup_{\beta \ \mathit{even} \ <\gamma} \mathcal{B}_\beta \backslash \mathcal{B}_{\beta+1}$$

For a proof see Kuratowski [8] page-section ???.

**Theorem 10.8** *(Douglas E. Miller [19]) If $\mathcal{B}$ is also invariant, then*

$$\mathcal{B} = \bigcup_{\beta \ even \ <\gamma} \mathcal{B}^*_\beta \backslash \mathcal{B}^*_{\beta+1}$$

proof:

The dual of the Vaught *-transform is $\mathcal{B}^\triangle = \sim (\sim \mathcal{B}^*)$. Equivalently $A \in \mathcal{B}^\triangle$ iff $\pi(A) \in \mathcal{B}$ for non-meagerly many $\pi \in S_\infty$. Two other transforms are

$$\mathcal{B}^+ = \{A \ : \ \exists \pi \in S_\infty \ \ \pi(A) \in \mathcal{B}\} \quad \text{and} \quad \mathcal{B}^- = \{A \ : \ \forall \pi \in S_\infty \ \ \pi(A) \in \mathcal{B}\}.$$

Note that

1. $\mathcal{B}^- \subseteq \mathcal{B}^* \subseteq \mathcal{B}^\triangle \subseteq \mathcal{B}^+$
2. $\mathcal{B}$ is invariant iff $\mathcal{B}^- = \mathcal{B}^+$
3. $(\bigcup_{n<\omega} \mathcal{B}_n)^\triangle = \bigcup_{n<\omega} \mathcal{B}_n^\triangle$
4. $\mathcal{B}_1^* \cap \mathcal{B}_2^\triangle \subseteq (\mathcal{B}_1 \cap \mathcal{B}_2)^\triangle$
5. $(\sim \mathcal{B})^* = \sim \mathcal{B}^\triangle$
6. $\mathcal{B}_1^* \backslash \mathcal{B}_2^* = \mathcal{B}_1^* \cap \ \sim (\mathcal{B}_2^*) = \mathcal{B}_1^* \cap (\sim \mathcal{B}_2)^\triangle \subseteq (\mathcal{B}_1 \backslash \mathcal{B}_2)^\triangle$

To prove the theorem note that

$$\bigcup_{\text{even } \beta<\gamma} \mathcal{B}^*_\beta \backslash \mathcal{B}^*_{\beta+1} \subseteq \bigcup_\beta (\mathcal{B}_\beta \backslash \mathcal{B}_{\beta+1})^\triangle = (\bigcup_\beta \mathcal{B}_\beta \backslash \mathcal{B}_{\beta+1})^\triangle = \mathcal{B}^\triangle = \mathcal{B}$$

where these unions are all taken over even $\beta < \gamma$. Hence if we let

$$\text{Diffeven}(\mathcal{B}_\alpha : \alpha < \gamma) =^{def} \bigcup \{\mathcal{B}_\alpha \backslash \mathcal{B}_{\alpha+1} \ : \ \text{ even } \alpha < \gamma\}$$

Then we conclude that

$$\text{Diffeven}(\mathcal{B}^*_\alpha : \alpha < \gamma) \subseteq \mathcal{B}.$$

To get the reverse inclusion note that the complement of a difference set is also a difference set. To see this we may without loss of generality assume that $\gamma$ is a limit ordinal by padding with the empty set if necessary. Then $\sim \mathcal{B} = \sim (\bigcup \{\mathcal{B}_\beta \backslash \mathcal{B}_{\beta+1} \ : \ \beta \text{ even } < \gamma\}$ is the union of the following sets:

- $X_\rho \backslash \mathcal{B}_0$
- $\mathcal{B}_\beta \backslash \mathcal{B}_{\beta+1}$ for odd $\beta < \gamma$
- $(\cap_{\alpha<\lambda} \mathcal{B}_\alpha) \backslash \mathcal{B}_\lambda$ for limit $\lambda < \gamma$
- $\cap_{\alpha<\gamma} \mathcal{B}_\alpha$

Let us denote this union as Diffodd$(\mathcal{B}_\alpha : \alpha < \gamma)$. Then by the argument above we get that

$$\text{Diffodd}(\mathcal{B}^*_\alpha : \alpha < \gamma) \subseteq\sim \mathcal{B}$$

But since Diffeven$(\mathcal{B}^*_\alpha : \alpha < \gamma)$ and Diffodd$(\mathcal{B}^*_\alpha : \alpha < \gamma)$ are complements, it follows that $\mathcal{B} = \text{Diffeven}(\mathcal{B}^*_\alpha : \alpha < \gamma)$.
□

**Corollary 10.9** *If the isomorphism class of a countable structure is* $\Delta^0_{\alpha+1}$ *then it must be either* $\Pi^0_\alpha$, $\Sigma^0_\alpha$, *or the difference of two invariant* $\Pi^0_\alpha$ *sets.*

**Theorem 10.10** *(Miller [11]) The isomorphism class of a countable model cannot be properly* $\Sigma^0_1$ *or properly* $\Sigma^0_2$. *For* $\lambda$ *a countable limit ordinal, it cannot be properly* $\Sigma^0_\lambda$ *or properly the difference of two* $\Pi^0_\lambda$ *sets.*

**Theorem 10.11** *(Miller [11], Hjorth [5]) In all other cases of there are examples of countable structures whose isomorphism class is properly of that Borel class.*

# References

[1] Barwise, Jon; Back and forth through infinitary logic. Studies in model theory, pp. 5-34. MAA Studies in Math., Vol. 8, Math. Assoc. Amer., Buffalo, N.Y., 1973.

[2] Bing, R. H.; Bledsoe, W. W.; Mauldin, R. D.; Sets generated by rectangles. Pacific J. Math. 51 (1974), 27-36.

[3] Tim Carlson, On $\omega_1$-Borel sets, unpublished 1982.

[4] Fleissner, William G.; Miller, Arnold W.; On Q Sets, Proceedings of the American Mathematical Society, 78(1980), 280-284.

`http://www.math.wisc.edu/~miller/res/qsets.pdf`

[5] Hjorth, G.; An orbit that is exactly $\Sigma_{\lambda+1}$, handwritten note Feb 1995.

`http://www.math.wisc.edu/~miller/res/unpub/hjorth.pdf`

[6] Kechris, Alexander S.;**Classical descriptive set theory.** Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995. xviii+402 pp. ISBN: 0-387-94374-9

[7] Kunen, Kenneth; INACCESSIBILITY PROPERTIES OF CARDINALS. Thesis (Ph.D.) Stanford University. 1968.

[8] Kuratowski, topology

[9] Miller, Arnold W.; On the length of Borel hierarchies, Annals of Math Logic, 16(1979), 233-267.

`http://www.math.wisc.edu/~miller/res/hier.pdf`

[10] Miller, Arnold W.; Generic Souslin sets, Pacific Journal of Mathematics, 97(1981), 171-181.

`http://www.math.wisc.edu/~miller/res/gensous.pdf`

[11] Miller, Arnold W.; The Borel classification of the isomorphism class of a countable model, Notre Dame Journal of Formal Logic, 24(1983), 22-34.

`http://www.math.wisc.edu/~miller/res/ctblmod.pdf`

[12] Miller, Arnold W.; Mapping a Set of Reals Onto the Reals, Journal of Symbolic Logic, 48(1983), 575-584.

`http://www.math.wisc.edu/~miller/res/map.pdf`

[13] Miller, Arnold W.; Special subsets of the real line, in **Handbook of Set Theoretic Topology**, North Holland, (1984), 201-233.

`http://www.math.wisc.edu/~miller/res/special.pdf`

[14] Miller, Arnold W.; Projective subsets of separable metric spaces Annals of Pure and Applied Logic, 50(1990), 53-69.

`http://www.math.wisc.edu/~miller/res/proj.pdf`

[15] Miller, Arnold W.; Special sets of reals, in **Set Theory of the Reals**, ed Haim Judah, Israel Mathematical Conference Proceedings, 6(1993), 415-432, American Math Society.

`http://www.math.wisc.edu/~miller/res/survey.pdf`

[16] Miller, Arnold W.; Descriptive Set Theory and Forcing: how to prove theorems about Borel sets the hard way, Lecture Notes in Logic 4(1995), Springer-Verlag. New edition 4-2001 now published by Association for Symbolic Logic.

`http://www.math.wisc.edu/~miller/res/dstfor.pdf`

[17] Miller, Arnold W.; Ramsey Theory

`http://www.math.wisc.edu/~miller/old/m873-96/ramsey.pdf`

[18] Miller, Arnold W.; The hierarchy of $\omega_1$-Borel sets, eprint July 2011

`http://www.math.wisc.edu/~miller/res/omega1.pdf`

[19] Miller, Douglas E.; The invariant $\Pi^0_\alpha$ separation principle. Trans. Amer. Math. Soc. 242 (1978), 185-204.

[20] Larson, Paul B.; Miller, Arnold W.; Steprans, Juris; Weiss, William A.R.; **Universal Functions** eprint Apr 2012

`http://www.math.wisc.edu/~miller/res/univ.pdf`

[21] Rao, B.V.; On discrete Borel spaces and projective sets. Bull. Amer. Math. Soc. 75 1969 614-617.

[22] Rothberger, Fritz; A remark on the existence of a denumerable base for a family of functions. Canadian J. Math. 4, (1952). 117-119.

[23] Steprans, Juris; Cardinal arithmetic and $\aleph_1$-Borel sets. Proc. Amer. Math. Soc. 84 (1982), no. 1, 121–126.

[24] Vaught, Robert; Invariant sets in topology and logic. Collection of articles dedicated to Andrzej Mostowski on his sixtieth birthday, VII. Fund. Math. 82 (1974/75), 269-294.

[25] Willard, Stephen Some examples in the theory of Borel sets. Fund. Math. 71 (1971), no. 3, 187-191.