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Generating Borel measurable mappings with continuous mappings ☆

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ABSTRACT

We prove that the relative rank $\mathrm{r}(\mathcal{B}(X):\mathcal{C}(X))$ of the semigroup of Borel mappings $\mathcal{B}(X)$ from X to X (with the composition of mappings as the semigroup operation) with respect to the semigroup of continuous functions $\mathcal{C}(X)$ from X to X is equal to $\aleph_1$ if X is an uncountable Polish space which either can be retracted to a Cantor subset of X, or contains an arc, or is homeomorphic to its Cartesian square X^2.

1. Introduction

If S is a semigroup and U is a subset of S then the relative rank of S with respect to U is the minimal cardinality of a subset V of S such that U together with V generate S. The principal result in the theory of relative ranks of semigroups of mappings, where the composition of mappings is the semigroup operation, is due to Sierpiński [13]. It says that the relative rank of the semigroup of all mappings A^A from an infinite set A to A with respect to any subsemigroup is either uncountable or finite and then equal to 0, 1 or 2 (Banach in [1] gave a very nice short proof of this theorem). In [6] Galvin proved an analogous result for groups of permutations. Relative ranks of semigroups and groups of mappings have been studied in various contexts ([12] is a survey of some directions of this research). One of them is purely combinatorial and concerns the relative ranks of A^A with respect to natural subsemigroups of A^A, where A has no structure, e.g. a relative rank of A^A with respect to the group of all permutations of A [9]. When A is equipped with some structure (order, metric, etc.) one can consider relative ranks of A^A with respect to semigroups that are related to this structure, e.g. semigroups of mappings from A to A that preserve order [7,8]. In [4] the authors examined the relative rank of A^A with respect to the semigroup of mappings satisfying the Lipschitz condition for various metrics defined on A. For discrete metric spaces the semigroup A^A of all mappings from A to A is also the semigroup $\mathcal{C}(A)$ of all continuous mappings from A to A. Of course, this is not the case if the metric considered on A is not discrete. The relative rank of the semigroup of continuous mappings

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with respect to the semigroup of Lipschitz mappings for various classical metric spaces was examined in [3]. The results of [3] and [4] show that this rank depends very much on the metric structure of the space A, as it could be expected, because the Lipschitz condition is a metric notion. In this paper we consider a relative rank that depends only on topology, namely, the relative rank of the semigroup of all Borel measurable mappings with respect to the semigroup of continuous mappings for some classical topological spaces. Our main result, Theorem 3.6, states that this rank is equal to the first uncountable cardinal $\aleph_1$ for a wide family of Polishable topological spaces X, namely those which either can be retracted to a Cantor subset of X, so, in particular, all uncountable zero-dimensional spaces, or contain a topological copy of the interval $[0,1]$ (so, in particular, Euclidean spaces), or are homeomorphic to their Cartesian square X^2.

2. Notation, definitions, auxiliary results

For sets A, B, C, $f: A\times B\to C$, $a\in A$ and $b\in B$, let $f_a(b)=f(a,b)$.

Let S be a semigroup and $A\subseteq S$. By $\langle A\rangle$ we denote the semigroup of S generated by A.

A topological space X is *Polishable* if there is a metric on this space that induces the topology of X and X with this metric is a Polish space.

The topological spaces considered in this paper are assumed to be metrizable. In fact, we consider only Polish(able) spaces.

Let Q denote the Hilbert cube $[0,1]^{\mathbb{N}}$. Let $\mathbf{C}$ denote the classical ternary Cantor set (the set of those elements from $[0,1]$ which in the ternary notation can be expressed using only digits 0 and 2). Sets homeomorphic with $\mathbf{C}$ will be simply called *Cantor sets*. It is known that every uncountable Polish space contains a Cantor set [11, Chapter 3, Section 36, V, Corollary 2]. Let $\mathcal{N}$ be the Baire space $\mathbb{N}^{\mathbb{N}}$, where the topology on $\mathbb{N}$ is discrete.

Though the first uncountable cardinal $\aleph_1$ and the first uncountable ordinal ω_1 are the same set-theoretical object in ordinal inequalities we shall rather use ω_1 while $\aleph_1$ will appear in quantitative statements.

An *arc* is a topological copy (i.e. a homeomorphic image) of the unit interval $[0,1]$.

The family of all functions from a set X to a set Y will be denoted by Y^X.

We will use the classical notation for the hierarchy of Borel sets. Let X be a topological space. Let $\boldsymbol{\Sigma}^0_1(X)$ denote the topology on X, i.e. the family of all open subsets of X. Let $\boldsymbol{\Pi}^0_1(X)$ denote the family of all closed subsets of X. Then we define recursively $\boldsymbol{\Sigma}^0_\alpha(X)$ and $\boldsymbol{\Pi}^0_\alpha(X)$ for all countable ordinals α. Namely, assume that all classes $\boldsymbol{\Sigma}^0_\xi(X)$ and $\boldsymbol{\Pi}^0_\xi(X)$ have been defined for $1\leqslant\xi<\alpha$. We define $\boldsymbol{\Sigma}^0_\alpha(X)$ as

$$\boldsymbol{\Sigma}^0_\alpha(X)=\left\{\bigcup_{n=1}^{\infty}A_n\colon\ A_n\in\boldsymbol{\Pi}^0_{\xi_n}(X),\ \xi_n<\alpha\right\},$$

and next we define $\boldsymbol{\Pi}^0_\alpha(X)$ as

$$\boldsymbol{\Pi}^0_\alpha(X)=\left\{A^c\colon\ A\in\boldsymbol{\Sigma}^0_\alpha(X)\right\}.$$

The family $\mathbf{B}(X)$ of all Borel measurable subsets of X is defined as the σ-algebra generated by the class $\boldsymbol{\Sigma}^0_1(X)$ of all open subsets of X. For any metric space X we have

$$\mathbf{B}(X)=\bigcup_{\alpha<\aleph_1}\boldsymbol{\Sigma}^0_\alpha(X)=\bigcup_{\alpha<\aleph_1}\boldsymbol{\Pi}^0_\alpha(X).$$

Let X, Y be topological spaces. The class of all continuous mappings from X to Y will be denoted by $\mathcal{C}(X,Y)$. The class $\mathcal{C}(X,X)$ will be denoted by $\mathcal{C}(X)$. A mapping $f: X\to Y$ is *Borel measurable* if $f^{-1}(B)\in\mathbf{B}(X)$ for all $B\in\mathbf{B}(Y)$. The class of all Borel measurable mappings from X to Y will be denoted by $\mathcal{B}(X,Y)$. The class $\mathcal{B}(X,X)$ will be denoted by $\mathcal{B}(X)$. Borel measurable mappings will be called for short *Borel mappings*.

With some abuse of notation $\mathcal{B}(X)$ ($\mathcal{C}(X)$) will denote the semigroup of mappings whose elements are all Borel (continuous, resp.) mappings from X to X with the composition of mappings as the semigroup operation.

For $\alpha<\omega_1$ let

$$B_\alpha(X,Y)=\left\{f\in Y^X\colon\ f^{-1}(U)\in\boldsymbol{\Sigma}^0_{1+\alpha}(X)\text{ for each }U\in\boldsymbol{\Sigma}^0_1(Y)\right\}$$

(note that for infinite ordinals α we have $1+\alpha=\alpha$). Thus $B_0(X,Y)=\mathcal{C}(X,Y)$. Of course, $B_\alpha(X,Y)\subseteq\mathcal{B}(X,Y)$. The class $B_\alpha(X,X)$ will be denoted by $B_\alpha(X)$.

Let $\mathcal{A}\subseteq Y^X$. Let Z be any set. We say that a mapping $F: Z\times X\to Y$ is *universal for* $\mathcal{A}$ if

$$\mathcal{A}\subseteq\left\{F(z,\cdot)\colon\ z\in Z\right\}.$$

Let

$$\underline{M}\boldsymbol{\Sigma}^0_\alpha(X)=\left\{f\in[0,1]^X\colon\ f^{-1}\big((t,1]\big)\in\boldsymbol{\Sigma}^0_\alpha(X)\text{ for each }t\in[0,1]\right\}$$

(for $\alpha=1$ this is the class of lower semi-continuous functions with values in $[0,1]$).

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The following theorem was proved in [5].

Theorem 2.1. *Let* $1 \leqslant \alpha < \omega_1$. *If* X *is a Polish space, then there exists* $F \in \underline{M}\boldsymbol{\Sigma}^0_\alpha(\mathbf{C} \times X)$ *universal for* $\underline{M}\boldsymbol{\Sigma}^0_\alpha(X)$.

The following theorem is due to Kuratowski [10].

Theorem 2.2. *If* X, Y *are Polish spaces of the same cardinality then there exists a bijection* $\psi \in B_1(X, Y)$ *from* X *onto* Y *such that* $\psi^{-1} \in B_1(Y, X)$.

A bijection $\psi \in B_1(X, Y)$ such that $\psi^{-1} \in B_1(Y, X)$ is called a $(1, 1)$-*homeomorphism* between X and Y [10].

Lemma 2.3. *Let* X *be an uncountable Polish space and* $C \subseteq X$ *be a Cantor set. For every* $\alpha < \omega_1$, *there exists a Borel measurable mapping* $F : C \times X \to Q$ *universal for* $B_\alpha(X, Q)$.

Proof. By Theorem 2.1 there exists a function $f \in \underline{M}\boldsymbol{\Sigma}^0_{1+\alpha}(C \times X)$ universal for the class $\underline{M}\boldsymbol{\Sigma}^0_{1+\alpha}(X)$. The spaces C and $C^{\mathbb{N}}$ are homeomorphic. Let $\phi : C \to C^{\mathbb{N}}$ be any homeomorphism between them. Let $x \in C$, $y \in X$. Let $G : X \to Q$ be defined by

$$G(x, y) = \big(f\big(\phi_i(x), y\big)\big)_i.$$

The mapping G is universal for $B_\alpha(X, Q)$. Indeed, let $g = (g_i)_i \in B_\alpha(X, Q)$. Then $g_i \in B_\alpha(X, [0, 1])$. Thus $g_i(\cdot) = f(z_i, \cdot)$ for some $z_i \in C$, because f is universal for $\underline{M}\boldsymbol{\Sigma}^0_{1+\alpha}(X)$ and $B_\alpha(X, [0, 1]) \subseteq \underline{M}\boldsymbol{\Sigma}^0_{1+\alpha}(X)$. Let $x = \phi^{-1}((z_i)_i)$. It is easy to see that $G(x, y) = g(y)$ for each $y \in X$. $\square$

Lemma 2.4. *Let* X *be an uncountable Polish space. For every* $\alpha < \omega_1$, *there exists a Borel measurable mapping* $f : C \times X \to X$ *universal for* $B_\alpha(X)$.

Proof. From Lemma 2.3 we know that there exists a mapping $\Gamma \in \mathcal{B}(C \times X, Q)$ universal for the class $B_\alpha(X, Q)$. As every Polish space has a homeomorphic copy in Q we can assume that $X \subseteq Q$. Because Γ is universal for $B_\alpha(X, Q)$ for each $f \in B_\alpha(X)$ there exists $c \in C$ such that $f(\cdot) = \Gamma(c, \cdot)$. The set $\Gamma^{-1}(X)$ is Borel. The desired mapping is given by the formula

$$f(x, y) = \begin{cases} \Gamma(x, y), & \text{if } (x, y) \in \Gamma^{-1}(X), \\ x_0, & \text{otherwise}, \end{cases}$$

where $x_0 \in X$ is fixed. $\square$

3. Results

We will now establish the following general result concerning all uncountable Polish spaces.

Theorem 3.1. *If* X *is an uncountable Polish space and* α *is a countable ordinal, then*

$$\mathrm{r}\big(\mathcal{B}(X) : B_\alpha(X)\big) \geqslant \aleph_1.$$

Proof. Aiming at a contradiction, assume that

$$\mathcal{B}(X) = \big\langle B_\alpha(X) \cup \{\psi_n\colon n \in \mathbb{N}\}\big\rangle,$$

where $\{\psi_n\colon n \in \mathbb{N}\} \subseteq \mathcal{B}(X)$. Then $\psi_n \in B_{\alpha_n}(X)$ for some countable ordinal α_n. Let

$$\gamma = \max\big(\sup\{\alpha_n\colon n \in \mathbb{N}\}, \alpha\big).$$

Of course, γ is also a countable ordinal. Let now $f \in \langle B_\alpha(X) \cup \{\psi_n\colon n \in \mathbb{N}\}\rangle$. Then

$$f = f_{k+1} \circ \psi_{n_k} \circ f_k \circ \cdots \circ \psi_{n_1} \circ f_1$$

for some $f_1, \ldots, f_{k+1} \in B_\alpha(X)$, $n_1, \ldots, n_k \in \mathbb{N}$, $k \in \mathbb{N}$. But then

$$f \in B_{\gamma\omega}(X).$$

Hence $\langle B_\alpha(X) \cup \{\psi_n\colon n \in \mathbb{N}\}\rangle \subseteq B_{\gamma\omega}(X) \subsetneq \mathcal{B}(X)$. This is a contradiction. $\square$

Setting $\alpha = 0$ we obtain the following corollary.

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Corollary 3.2. *If X is an uncountable Polish space, then*

$$\mathrm{r}\big(\mathcal{B}(X):\mathcal{C}(X)\big) \geqslant \aleph_1.$$

We will prove the inverse inequality for the family of Polish spaces from the hypothesis of the next lemma.

Lemma 3.3. *If X is an uncountable Polish space which either* (i) *can be retracted to a Cantor subset of X, or* (ii) *contains an arc* (*i.e. a topological copy of the unit interval* $[0,1]$), *or* (iii) *is homeomorphic to its Cartesian square X^2 then X has the following property*:

($\star$) *there are Cantor sets $C, C' \subseteq X$ and a homeomorphism $\varphi : C^2 \to C'$ such that for every $c \in C$ there exists $f \in \mathcal{C}(X)$ extending φ_c.*

Proof. (i) Let $r : X \to C$ be a retraction of X onto some Cantor subset of X. Let $\varphi : C^2 \to C$ be any homeomorphism from C^2 onto C. Let $c \in C$. Then $f(x) = \varphi_c(r(x))$ is the extension desired.

(ii) Let X contain a topological copy of $[0,1]$. Losing no generality we can simply assume $[0,1] \subseteq X$. Let $C, C' = \mathbf{C} \subseteq [0,1]$. Of course, there exists a homeomorphism $\varphi : \mathbf{C}^2 \to \mathbf{C}$. By Tietze's extension theorem φ_c can be extended to $f : X \to [0,1] \subseteq X$ for all $c \in \mathbf{C}$.

(iii) Let $h : X^2 \to X$ be any homeomorphism from X^2 onto X. Let $C \subseteq X$ be a Cantor set. Let $\varphi : C^2 \to h(C^2)$ be defined as $\varphi = h|C^2$. Let $c \in C$. We have $\varphi_c = h_c|C$, and h_c is the extension desired. □

Lemma 3.4. *Suppose that X is an uncountable Polish space satisfying the conclusion* ($\star$) *of Lemma 3.3. Let C be the set from* ($\star$). *Then for any Borel mapping $\Phi : C \times X \to X$ there exist Borel mappings $\Xi, \Psi \in \mathcal{B}(X)$ such that for every $c \in C$ there exists $f \in \mathcal{C}(X)$ with $\Phi_c(x) = \Xi(f(\Psi(x)))$ for every $x \in X$.*

Proof. Let C' be the set from ($\star$). Let $\varphi : C^2 \to C'$ be the homeomorphism from ($\star$) and let $\Psi : X \to C$ be a Borel bijection (Theorem 2.2). Let $\Xi : X \to X$ be a Borel mapping whose restriction to C' satisfies for all $c, d \in C$:

$$\Xi\big(\varphi(c,d)\big) = \Phi\big(c, \Psi^{-1}(d)\big).$$

Let $c \in C$. By ($\star$) there exists $f \in \mathcal{C}(X)$ such that $\varphi_c = f|C$ (recall that $\varphi_c(\cdot) = \varphi(c,\cdot)$). Then for any $x \in X$

$$\Xi\big(f\big(\Psi(x)\big)\big) = \Xi\big(\varphi_c\big(\Psi(x)\big)\big) = \Xi\big(\varphi\big(c, \Psi(x)\big)\big) = \Phi\big(c, \Psi^{-1}\big(\Psi(x)\big)\big) = \Phi(c,x). \qquad \square$$

Theorem 3.5. *If X is an uncountable Polish space which either can be retracted to a Cantor subset of X, or contains an arc, or is homeomorphic to its Cartesian square X^2, then* $\mathrm{r}(\mathcal{B}(X):\mathcal{C}(X)) \leqslant \aleph_1$.

Proof. By Lemma 2.4 for $\alpha < \omega_1$ there are Borel mappings $\Phi^{(\alpha)} : C \times X \to X$ such that

$$\mathcal{B}(X) = \bigcup\big\{\Phi_c^{(\alpha)}:\ c \in C \text{ and } \alpha < \omega_1\big\}$$

and the conclusion follows from Lemma 3.4. □

Theorems 3.1 and 3.5 imply now the following main result of this article.

Theorem 3.6. *If X is an uncountable Polish space which either can be retracted to a Cantor subset of X, contains an arc, or is homeomorphic to its Cartesian square X^2, then* $\mathrm{r}(\mathcal{B}(X):\mathcal{C}(X)) = \aleph_1$.

Hence we have $\mathrm{r}(\mathcal{B}(X):\mathcal{C}(X)) = \aleph_1$ if X is an uncountable zero-dimensional Polish space (because it is retractable on a Cantor subset [11, Chapter 2, Section 26, II, Corollary 2]), in particular if X is a Cantor set $X = \mathbf{C}$ or X is Baire space $X = \mathcal{N}$, if X is a Euclidean space $X = \mathbb{R}^n$ (because it contains an arc), or if X is a countable product of any non-singleton Polish space Y, $X = Y^{\mathbb{N}}$ (because it is homeomorphic to its Cartesian square X^2).

Remark. As we noticed above for most of the classical Polish spaces X we have $\mathrm{r}(\mathcal{B}(X):\mathcal{C}(X)) = \aleph_1$. On the other hand there are bizarre uncountable Polish spaces for which $\mathcal{C}(X)$ consists only of the identity mapping and the constants, see [2]. In this case, as it is easy to see, $\mathrm{r}(\mathcal{B}(X):\mathcal{C}(X)) = \mathfrak{c}$.

Using Theorem 2.2 and repeating the proof of Lemma 3.4 replacing C and C' by X and Ψ by the identity mapping on X we obtain the following counterpart of Lemma 3.4 for mappings from $B_1(X)$.

Lemma 3.7. *Let X be an uncountable Polish space. Then for any Borel mapping $\Phi : X^2 \to X$ there exists a Borel mapping $\Xi \in \mathcal{B}(X)$ such that for each $t \in X$ there exists $f \in B_1(X)$ such that $\Phi_t(x) = \Xi(f(x))$ for all $x \in X$.*

The following theorem follows from Lemma 3.7 in an analogous way as Theorem 3.5 follows from Lemma 3.4.

Theorem 3.8. *If X is an uncountable Polish space, then* $\mathrm{r}(\mathcal{B}(X) : B_1(X)) \leqslant \aleph_1$.

Theorems 3.1 and 3.8 imply the following theorem.

Theorem 3.9. *If X is an uncountable Polish space, then*

$$\mathrm{r}\big(\mathcal{B}(X) : B_\alpha(X)\big) = \aleph_1$$

for each $1 \leqslant \alpha < \omega_1$.

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