Baire measures on uncountable product spaces [1]

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Abstract

We show that assuming the continuum hypothesis there exists a nontrivial countably additive measure π on the Baire subsets of the space $\mathbb{R}^{\omega_1}$ which vanishes on each element of an open cover. Contrariwise, we show that assuming MA_{ω_1}, that there is no such measure.

Let $\mathbb{R}^{\omega_1}$ be the ω_1 product of uncountably many copies of the real line. The family of Baire sets in this context is the smallest σ-algebra which contains the basic open sets, i.e., sets of the form: $\prod_{\alpha<\omega_1} U_\alpha$ where each U_α is an open subset of $\mathbb{R}$ and all but finitely many U_α are the whole real line.

Other definitions of this family are that it is the smallest σ-algebra generated by the closed G_δ-sets. It is also true that the closed G_δ-sets are the preimages of zero under a continuous real-valued mapping. Basically the Baire sets are the Borel sets which depend on only countably many coordinates. For example, singletons are not Baire.

A Baire measure is a countably additive measure into the unit interval $[0,1]$ which is defined on the Baire sets. Denis Saveliev [8] has asked

Question: Does every Baire measure on $\mathbb{R}^{\omega_1}$ which vanishes on a cover of open Baire sets have to be trivial? In particular what happens if we assume Martin's Axiom?

Here I will answer this question by proving:

Theorem 1 *(Fremlin [3]) Assume MA_{ω_1}. Then every Baire measure on $\mathbb{R}^{\omega_1}$ which vanishes on a cover of open Baire sets is trivial.*

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[1]As I was just about finished writting this up (October 1998), I discovered that everything here is already known. See the last section of

R.J.Gardner and W.F.Pfeffer, *Borel Measures*, in **Handbook of Set Theoretic Topology**, North-Holland, 1984, 961-1044.

[2]I want to thank Joel Robbin for many helpful conversations, especially for some helpful comments which fixed my first "proof" of Theorem 2.

Theorem 2 *(Moran [7], Kemperman and Maharam [5]) Assume CH. Then there exists a nontrivial Baire measure on* $\mathbb{R}^{\omega_1}$ *which vanishes on a cover of open Baire sets.*

Proof of Theorem 1

Assume MA_{ω_1} and let μ be a nontrivial Baire measure. Let $\mathbb{P}$ be the following partially ordered set. We say that $p \subseteq \mathbb{R}^{\omega_1}$ is a finite box iff p is a product $p = \prod_{\alpha<\omega_1} p(\alpha)$ where for all but finitely many coordinates $p(\alpha) = \mathbb{R}$ and for these finitely many coordinates $p(\alpha)$ is a compact interval in the reals. Define

$$\mathbb{P} = \{p \subseteq \mathbb{R}^{\omega_1} : \; p \text{ is a finite box and } \mu(p) > 0\}$$

Order $\mathbb{P}$ by inclusion. Note that it two condition are compatible iff their intersection has positive measure. Thus $\mathbb{P}$ has the countable chain condition (ccc). (Given uncountably many elements of $\mathbb{P}$, $\{p_\alpha : \alpha < \omega_1\}$, there must be some natural number n such that uncountably many of the p_α have measure greater than $\frac{1}{n}$, but there can be at most n pairwise incompatible elements of $\mathbb{P}$ of measure $\frac{1}{n}$.)

Suppose for contradiction, there were a cover of $\mathbb{R}^{\omega_1}$ by Baire open sets of measure zero. We may assume this cover has cardinality ω_1, say it is $\{U_\beta : \beta < \omega_1\}$, where the U_β are particularly simple. Namely, each is an ω_1 product of open intervals in $\mathbb{R}$ with rational end points and all but finitely many are $(-\infty, \infty)$.

For each $\alpha < \omega_1$ define

$$D_\alpha = \{p \in \mathbb{P} : p \cap U_\alpha = \emptyset\}.$$

Each D_α is dense in $\mathbb{P}$. To see this note that the complement of the set U_α can be written as a countable union of finite boxes. To see this, let F be the finitely many coordinates where U_α is a nontrivial open interval and for each $\beta \in F$ write the complement of this open interval as a union of compact intervals J_n^β for $n \in \omega$. For each $\beta \in F$ and $n \in \omega$ let

$$p_n^\beta = \{x \in \mathbb{R}^{\omega_1} : x_\beta \in J_n^\beta\}.$$

Then the complement of U_α is the countable union of finite boxes

$$\langle p_n^\beta : \beta \in F, n \in \omega\rangle.$$

Given an arbitrary $p \in \mathbb{P}$ there exists some $\beta \in F$, $n \in \omega$ such that $q = p \cap p^\beta_n$ has positive measure and so $q \in D_\alpha$. This shows that D_α is dense.

For each α and natural number n, let

$$E^n_\alpha = \{p \in \mathbb{P} : \text{ the diameter of } p(\alpha) \text{ is less than } 1/n \ \}.$$

To see that E^n_α is a dense subset of $\mathbb{P}$, let J_k for $k < \omega$ be a cover of $\mathbb{R}$ by closed intervals of length less than $1/n$. Given an arbitrary $p \in \mathbb{P}$ let

$$p_k = \{x \in p : x_\alpha \in J_k\}.$$

Each p_k is a closed Baire set and since p has positive measure at least on p_k must have positive measure. Hence, there exists k such that $p_k \leq p$ and $p_k \in E^n_\alpha$.

By MA_{ω_1} there exists a $\mathbb{P}$-filter G such that for all $\alpha < \omega$ and n, both

$$G \cap E^n_\alpha \neq \emptyset \text{ and } G \cap D_\alpha \neq \emptyset.$$

First we claim that $\bigcap G$ is nonempty, in fact, a singleton $\{x\}$. To see that it is nonempty, note that we cannot directly use compactness since our space $\mathbb{R}^{\omega_1}$ is not compact. However, for each fixed α the intersection $\bigcap\{p(\alpha) : p \in G\}$ is a nonempty singleton since G meets E^n_α for all n. Let $x \in \mathbb{R}^{\omega_1}$ be defined by

$$\{x(\beta)\} = \bigcap\{p(\alpha) : p \in G\}$$

for every $\beta < \omega_1$. By its definition, $x \in p$ for every $p \in G$.

Since G meets each D_α it must be that $x \notin U_\alpha$ and this contradicts the fact that U_α were supposed to be a cover of $\mathbb{R}^{\omega_1}$.

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Proof of Theorem 2

We will begin by showing that there exists such a measure π on the Baire subsets of $2^\omega \times \omega^{\omega_1}$.

Assuming CH let $2^\omega = \{x_\alpha : \ \alpha < \omega_1\}$.

For any $\alpha < \omega_1$ let $\langle \sigma^\alpha_n \in 2^{n+1} : n \in \omega\rangle$ be defined by

$$\sigma^\alpha_n(i) = \begin{cases} x_\alpha(i) & \text{if } i < n \\ 1 - x_\alpha(n) & \text{if } i = n \end{cases}$$

Note that $\{[\sigma_n^\alpha] : n \in \omega\}$ is a partition of $2^\omega \setminus \{x_\alpha\}$.

For the convenience of the reader we first descibe what π would be on $2^\omega \times \omega$.

Let λ be the usual product measure on 2^ω. For any Borel $B \subseteq 2^\omega \times \omega$ and $n \in \omega$ define

$$B_n = \{x \in 2^\omega : (x, n) \in B \text{ and } x \in [\sigma_n^0]\}.$$

Then define

$$\pi(B) = \sum_{n=0}^{\infty} \lambda(B_n).$$

Note that since B_n is a subset of $[\sigma_n^0]$ and since the $[\sigma_n^0]$ partition $2^\omega \setminus \{x_0\}$, we get that π is a probability measure on $2^\omega \times \omega$. Also note that for any Borel $C \subseteq 2^\omega$ we will have that $\pi(C \times \omega) = \lambda(C')$ where $C' = C \setminus \{x_0\}$. Finally note that for any $n \in \omega$ the point $(x_0, n) \in 2^\omega \times \omega$ has an open neighborhood which has π-measure zero, namely $\pi([x_0 \restriction m] \times \{n\}) = 0$ for any $m > n$.

Now we indicate what π is on all of $2^\omega \times \omega^{\omega_1}$.

Let $\mathcal{A}$ be the algebra of Baire sets $B \subseteq 2^\omega \times \omega^{\omega_1}$ whose support in ω_1 is finite. This means, that there exists a finite $F \subseteq \omega_1$ such that:

for every $(x, y), (x, y') \in 2^\omega \times \omega^{\omega_1}$ if $y \restriction F = y' \restriction F$, then $(x, y) \in B$ iff $(x, y') \in B$.

For each $s : F \to \omega$ define

$$C_s = \cap\{[\sigma_{s(\alpha)}^\alpha] : \alpha \in F\} \subseteq 2^\omega.$$

A lot of the C_s are empty, however they partition $2^\omega \setminus F$. (For each $x \in 2^\omega \setminus F$ define $s : F \to \omega$ by $s(\alpha)$ is the largest n such that $x(n) = x_\alpha(n)$. Then $x \in C_s$.)

Define

$$\begin{aligned} B_s &= \{x \in 2^\omega : \exists y \in \omega^{\omega_1} \;\; s \subseteq y, \;\; (x, y) \in B, \text{ and } x \in C_s\} \\ B^* &= \bigcup\{B_s : \;\; s : F \to \omega\} \\ \pi(B) &= \lambda(B^*) \end{aligned}$$

To verify that this is well-defined we should check that if $F \subseteq F' \subseteq \omega_1$ for any F' finite that

$$\sum_{s:F\to\omega} \lambda(B_s) = \sum_{t:F'\to\omega} \lambda(B_t).$$

But for any fixed $s : F \to \omega$

$$C_s = \cup\{C_t : \ \ t : F' \to \omega, \ \ s \subseteq t\} \cup (\{x_\alpha \in C_s : \alpha \in F'\}.$$

Now since F is the support of B for each $s : F \to \omega$, B_s will differ from

$$\bigcup\{B_t : \ \ t : F' \to \omega, s \subseteq t\}$$

by at most a finite set, hence

$$\lambda(B_s) = \sum\{\lambda(B_t) : \ \ t : F' \to \omega, \ \ s \subseteq t\}.$$

Thus the set B^* (which depends on F) is the same up to a finite set.

To verify that π extends to the σ-algebra generated by $\mathcal{A}$, according to Halmos [6], we need to show that whenever B_n are disjoint elements of $\mathcal{A}$ such that their union $B = \cup\{B_n : n \in \omega\}$ is in $\mathcal{A}$, we have

$$\pi(B) = \sum_{n=0}^{\infty} \pi(B_n).$$

For each B_n suppose that it is supported by the finite set F_n and B is supported by the finite set F. Without loss of generality, we may assume $F \subseteq F_n$ for all n. To verify our equation it is enough to see that for any x except possibly the x_α for $\alpha \in \bigcup\{F_n : n \in \omega\}$ that

$$x \in B^* \text{ iff } x \in \bigcup\{B_n^* : n \in \omega\}.$$

Let us verify this. Suppose $x \in B^*$. Then for some $s : F \to \omega$ and $y \supseteq s$ we have that $(x, y) \in B$ and $x \in C_s$. Since the support of B is F we can redefine y so that for every

$$\alpha \in \bigcup\{F_n : n \in \omega\} \setminus F$$

$y(\alpha)$ is the first n such that $x(n) \neq x_\alpha(n)$ (i.e. so that $x \in [\sigma^\alpha_{y(\alpha)}]$). Since $(x, y) \in B$ there exists n such that $(x, y) \in B_n$. But since we are assuming $x \neq x_\alpha$ for any $\alpha \in F_n$, there exists $t : F_n \to \omega$ such that $t \supseteq s$ and if $t = y \restriction F_n$, then we have that $x \in C_t$ and hence $x \in B_n^*$.

Now let us check the converse. Suppose $x \in B_n^*$ for some n. Then for some $t : F_n \to \omega$ and some $y \supseteq t$ we have $(x, y) \in B_n$ and $x \in C_t$. But then since $F \subseteq F_n$ we have that $(x,y) \in B$ and $x \in C_s$ where $s = t \upharpoonright F$ and so $x \in B^*$.

It follows that π extends to the σ-algebra generated by $\mathcal{A}$, i.e., the Baire subsets of $2^\omega \times \omega^{\omega_1}$. Finally, we note that for every element $(x,y) \in 2^\omega \times \omega^{\omega_1}$ there exists a clopen set C with $\pi(C) = 0$ and $(x,y) \in C$. Let $x = x_\alpha$ and suppose that $y(\alpha) = n$. Then let $s : \{\alpha\} \to \omega$ be defined by $s(\alpha) = n$ and let $t = x \upharpoonright (n+1)$. Then $(x,y) \in C = [t] \times [s]$ and $\pi([t] \times [s]) = 0$.

To obtain the same result for $\mathbb{R}^{\omega_1}$, first note that we have such a measure on ω^{ω_1}, since $2^\omega \times \omega^{\omega_1}$ is a closed Baire set in $\omega^\omega \times \omega^{\omega_1}$ and we can identify $\omega^\omega \times \omega^{\omega_1}$ with $\omega^{\omega+\omega_1} = \omega^{\omega_1}$. Now to "lift" π to $\hat{\pi}$ on the Baire subsets of $\mathbb{R}^{\omega_1}$, note that for every Baire $B \subseteq \mathbb{R}^{\omega_1}$ the set $B \cap \omega^{\omega_1}$ is Baire in ω^{ω_1}. So we can let $\hat{\pi}(B) = \pi(B \cap \omega^{\omega_1})$. The measure $\hat{\pi}$ vanishes on an open neighborhood of every point. To check this, suppose $x \in \mathbb{R}^{\omega_1}$. Then if there exists some α such that $x(\alpha) \notin \omega$, then letting $\epsilon > 0$ be a quarter of the distance from $x(\alpha)$ and the nearest element of ω, we have that

$$x \in U = \{y \in \mathbb{R}^{\omega_1} : |y(\alpha) - x(\alpha)| < \epsilon\}$$

and $\hat{\pi}(U) = 0$. On the otherhand if $x \in \omega^{\omega_1}$ then for some finite $F \subseteq \omega_1$ we have that $\pi([x \upharpoonright F]) = 0$, and so if we let

$$U = \{y \in \mathbb{R}^{\omega_1} : |y(\alpha) - x(\alpha)| < 1/4 \text{ for all } \alpha \in F\}$$

then $x \in U$ and $\hat{\pi}(U) = 0$.
■

Theorem 3 *(Bockstein [1] [2]) If $C \subseteq \omega^{\omega_1}$ is a closed G_δ set, then C is Baire.*

Proof:

Suppose for contradiction that C is not countably supported. This means that for each $\alpha < \omega_1$ there exists $f^i_\alpha : \omega_1 \to \omega$ for $i = 0, 1$ such that

$$f^0_\alpha \upharpoonright \alpha = f^1_\alpha \upharpoonright \alpha \text{ and } f^0_\alpha \in C \text{ and } f^1_\alpha \notin C.$$

Suppose $C = \bigcap\{U_n : n \in \omega\}$ where U_n are open. For each α choose finite F^α_n for $n \in \omega$ so that

$$[f^1_\alpha \upharpoonright F^\alpha_0] \cap C = \emptyset$$

$$[f^0_\alpha \restriction F^\alpha_n] \subseteq U_n$$

By the push down lemma there exists H and a stationary set of α such that $F^0_\alpha \cap \alpha = H$. Hence, it easy to find $\alpha < \beta$ such that

$$F^n_\alpha \cap F^0_\beta \subseteq \alpha$$

for all n. Now if we define $g : \omega_1 \to \omega$ so that

$$g \restriction F^n_\alpha = f^0_\alpha \restriction F^n_\alpha$$

and

$$g \restriction F^0_\beta = f^1_\beta \restriction F^0_\beta$$

then we get the contradiction that $g \notin C$ but $g \in \bigcap\{U_n : n \in \omega\}$.
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Theorem 4 *(A.H.Stone [9]) The space ω^{ω_1} is not normal.*

Proof:
Let

$$C_i = \{f \in \omega^{\omega_1} : \forall n \neq i \ \ |f^{-1}\{i\}| \leq 1\}$$

i.e., one-to-one except when equal to i. Then C_0 and C_1 are disjoint closed sets which cannot be separated by disjoint open sets. To see this suppose that U_0 and U_1 are disjoint open sets which separate C_0 and C_1. For each $\alpha < \omega_1$ choose $f^i_\alpha \in C_i$ for $i = 0, 1$ such that

$$f^0_\alpha \restriction \alpha = f^1_\alpha \restriction \alpha.$$

For each α there exists a finite F_α such that

$$[f^0_\alpha \restriction F_\alpha] \subseteq U_0 \text{ and } [f^1_\alpha \restriction F_\alpha] \subseteq U_1.$$

But by the Δ-systems lemma it easy to find $\alpha < \beta$ such that $F_\alpha \cap F_\beta \subseteq \alpha$. But this means we can define $g : \omega_1 \to \omega$ so that

$$g \restriction F_\alpha = f^0_\alpha \restriction F_\alpha \text{ and } g \restriction F_\beta = f^1_\beta \restriction F_\beta$$

which is a contradiction since then $g \in U_0 \cap U_1$.
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Theorem 5 *The usual product measure μ on 2^{ω_1} makes all Borel subsets measurable (not just the Baire sets).*

Proof:
Let $U \subseteq 2^{\omega_1}$ be any open set. Suppose

$$U = \bigcup_{\alpha<\omega_1} [s_\alpha].$$

Then we may choose α_0 so that for every β

$$\mu(\bigcup_{\alpha<\alpha_0} [s_\alpha]) = \mu([s_\beta] \cup \bigcup_{\alpha<\alpha_0} [s_\alpha]).$$

Let $\Gamma = \bigcup_{\alpha<\alpha_0} \text{domain}(s_\alpha)$. Because this is the product measure it must be that for every β

$$\mu([s_\beta \upharpoonright \Gamma] \setminus \bigcup_{\alpha<\alpha_0} [s_\alpha]) = 0.$$

So let

$$U_0 = \bigcup_{\alpha<\alpha_0} [s_\alpha])$$

and let

$$U_1 = U_0 \cup \bigcup_{\alpha_0 \leq \beta < \omega_1} [s_\beta \upharpoonright \Gamma]).$$

Then U_0 and U_1 are Baire open sets, $U_0 \subseteq U \subseteq U_1$, and $\mu(U_1 \setminus U_0) = 0$. Hence, U is measurable.
■

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