Measurability of functions with approximately continuous vertical sections and measurable horizontal sections

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A function $f : \mathbf{R} \to \mathbf{R}$ is approximately continuous iff it is continuous in the density topology, i.e., for any open set $U \subseteq \mathbf{R}$ the set $E = f^{-1}(U)$ is measurable and has Lebesgue density one at each of its points. Approximate continuity was introduced by Denjoy [7] in his study of derivatives. Denjoy proved that bounded approximately continuous functions are derivatives. It follows from this that approximately continuous functions are Baire 1, i.e., pointwise limits of continuous functions. For more on these concepts, see Bruckner [3], Lukeš, Malý, Zajiček [17], Tall [22], and Goffman, Neugebauer, Nishiura [9].

For any $f : \mathbf{R}^2 \to \mathbf{R}$ define

$$f_x(y) = f^y(x) = f(x, y)$$

for any $x, y \in \mathbf{R}$. A function $f : \mathbf{R}^2 \to \mathbf{R}$ is separately continuous if f_x and f^y are continuous for every $x, y \in \mathbf{R}$. Lebesgue [16] in his first paper proved that any separately continuous function is Baire 1. He also showed that if f_x is continuous for all x and f^y Baire α for all y, then f is Baire $\alpha + 1$ (see Kuratowski [14] p. 378). For more historical comments and generalizations see Rudin [18]. Sierpiński [21] showed that there exists a nonmeasurable $f : \mathbf{R}^2 \to \mathbf{R}$ which is separately Baire 1. (The characteristic function of a nonmeasurable subset of the plane which meets every horizontal and vertical line in at most one point.)

In this paper we shall prove:

Theorem 1 *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be such that f_x is approximately continuous and f^y is Baire 1 for every $x, y \in \mathbf{R}$. Then f is Baire 2.*

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Theorem 2 *Suppose there exists a real-valued measurable cardinal. Then for any function $f : \mathbf{R}^2 \to \mathbf{R}$ and $\alpha < \omega_1$, if f_x is approximately continuous and f^y is Baire α for every $x, y \in \mathbf{R}$, then f is Baire $\alpha + 1$ as a function of two variables.*

Theorem 3 (i) *Suppose that $\mathbf{R}$ can be covered by ω_1 closed null sets. Then there exists a nonmeasurable function $f : \mathbf{R}^2 \to \mathbf{R}$ such that f_x is approximately continuous and f^y is Baire 2 for every $x, y \in \mathbf{R}$.*
(ii) *Suppose that $\mathbf{R}$ can be covered by ω_1 null sets. Then there exists a nonmeasurable function $f : \mathbf{R}^2 \to \mathbf{R}$ such that f_x is approximately continuous and f^y is Baire 3 for every $x, y \in \mathbf{R}$.*

Theorem 4 *In the random real model for any function $f : \mathbf{R}^2 \to \mathbf{R}$ if f_x is approximately continuous and f^y is measurable for every $x, y \in \mathbf{R}$, then f is measurable as a function of two variables.*

Remarks. Davies [6] showed that any function of two variables which is separately approximately continuous is Baire 2. Theorem 1 which generalizes this was announced in Laczkovich and Petruska [15], but the proof was never published. In Davies and Dravecký [5] and Grande [10] it is shown that CH implies the existence of a nonmeasurable function f such that f_x is approximately continuous for every x and f^y is measurable for every y. It is easy to check that these constructions, in fact, give Baire 2 sections. Our Theorem 3 is a refinement of this observation. Note that Bartoszynski and Shelah [1] have shown that it is relatively consistent with ZFC that $\mathbf{R}$ is the union of ω_1 meager null sets, but not the union of ω_1 closed null sets. It is well known that $\mathbf{R}$ can be the union of ω_1 closed null sets and the continuum arbitrarily large.

In Theorem 2 we only use that for any family of continuum many subsets of the real line there exists a measure extending Lebesgue measure and making the family measurable. This is slightly weaker than a real-valued measurable and has the consistency strength of a weakly compact cardinal (see Carlson [4]).

It follows from Lebesgue's argument that any function $f : \mathbf{R}^2 \to \mathbf{R}$ such that f_x is continuous and f^y is measurable for all $x, y \in \mathbf{R}$ must be measurable as a function of two variables. Theorems 3 and 4 show that this fact is independent of set theory if we replace continuous by approximately continuous.

Proof of Theorem 1. This is an immediate consequence of the following theorem due to Bourgain, Fremlin and Talagrand [2].

Theorem 5 (Bourgain, Fremlin, Talagrand) *Let (X, Σ, μ) be a probability space and let $f : X \times \mathbf{R} \to \mathbf{R}$ be bounded. If f_x is Baire 1 for every $x \in X$ and f^y is measurable for every $y \in \mathbf{R}$, then the function*

$$y \mapsto \int_X f^y d\mu(x) \ (y \in \mathbf{R})$$

is Baire 1.

Suppose that f_x is approximately continuous and f^y is Baire 1 for every $x, y \in \mathbf{R}$. Without loss of generality we may assume that f is bounded. (Otherwise, let $h : \mathbf{R} \to (0,1)$ be a homeomorphism. Then $h \circ f$ is approximately continuous when x is fixed and measurable when y is fixed. Hence $h \circ f$ is Baire 2 and therefore $h^{-1} \circ h \circ f = f$ is Baire 2.)

It follows from Theorem 5, that for every fixed y, the function

$$x \mapsto \int_0^y f_x dt \ (x \in \mathbf{R})$$

is Baire 1.

This implies that the function

$$F(x,y) = \int_0^y f_x dt$$

is Baire 1, since F^y is Baire 1 and the family $\{F_x : x \in \mathbf{R}\}$ is uniformly continuous (in fact, uniformly Lipschitz). The proof is this. Let $F_n : \mathbf{R}^2 \to \mathbf{R}$ be the function such that $F_n(x, i/n) = F(x, i/n)$ for every $x \in \mathbf{R}$ and every integer i, and let $F_n(x_0, y)$ be linear in $y \in [(i-1)/n, i/n]$ for every integer i and every fixed x_0. Then F_n is Baire 1. Indeed, let $F(x, i/n) = \lim_{j \to \infty} g_{i,j}(x)$, where $g_{i,j} : \mathbf{R} \to \mathbf{R}$ continuous. Let $G_j(x, i/n) = g_{i,j}(x)$, let G_j be continuous in y and linear for $y \in [(i-1)/n, i/n]$ for every fixed x. Then G_j is continuous and $G_j \to F_n$, so that F_n is Baire 1. Finally, $F_n \to F$ uniformly, so that F is Baire 1 (see Kuratowski [14] p. 386).

Finally, since

$$f(x,y) = \lim_{n \to \infty} \frac{F(x, y + (1/n)) - F(x,y)}{1/n},$$

it follows that f is Baire 2.

■

Proof of Theorem 2. This is the same as the proof of Theorem 1 except we use the following generalization of the Bourgain-Fremlin-Talagrand Theorem 5:

Lemma 6 *Let (X, Σ, μ) be a probability space such that every subset of X is in Σ and let $f : X \times \mathbf{R} \to \mathbf{R}$ be bounded. For $\alpha < \omega_1$ if f_x is Baire α for every $x \in X$, then the function*

$$F(y) = \int_X f^y d\mu(x) \text{ for } y \in \mathbf{R}$$

is Baire α.

Proof. This is proved by induction on α. If $\alpha = 0$; that is, if f_x is continuous for every x, then the continuity of F follows from the dominated convergence theorem. For $\alpha > 0$, let β_n be a nondecreasing sequence of ordinals such that $\sup_{n \in \omega}(\beta_n + 1) = \alpha$. Let $\langle f_n : n \in \omega \rangle$ be a sequence of uniformly bounded functions such that $(f_n)_x$ is Baire β_n for each n and

$$\lim_{n \to \infty} f_n(x, y) = f(x, y).$$

Then by induction the function

$$F_n(y) = \int_X f_n^y d\mu(x)$$

is Baire β_n. By the dominated convergence theorem

$$\lim_{n \to \infty} F_n(y) = F(y)$$

is Baire α.

■

Since there is a real-valued measurable cardinal we can find an extension μ of Lebesgue measure λ which makes every set of reals measurable. The rest of the proof is the same as Theorem 1.

■

Proof of Theorem 3. Let $\mathbf{R} = \cup_{\alpha < \omega_1} C_\alpha$, where C_α is a closed set of measure zero for every $\alpha < \omega_1$.

By a Lemma of Zahorski [23] (see also Bruckner [3] p. 28) for any G_δ measure zero set $G \subseteq \mathbf{R}$ there exists an approximately continuous $g : \mathbf{R} \to [0,1]$ such that $g^{-1}\{0\} = G$. So for each α let $g_\alpha : \mathbf{R} \to [0,1]$ be an approximately continuous function such that $g_\alpha^{-1}\{0\}$ is a measure zero set covering $\bigcup_{\beta<\alpha} C_\beta$. We define $f(x,y) = g_\alpha(y)$, where α is the smallest ordinal such that $x \in C_\alpha$.

Obviously, f_x is approximately continuous for every x. For any fixed y, let α be such that $y \in C_\alpha$. If $x \notin \bigcup_{\beta<\alpha} C_\alpha$, then $f(x,y) = 0$. It is also clear that f^y is constant on each of the G_δ sets $C_\beta \setminus \cup_{\gamma<\beta} C_\gamma$. It follows that f^y is Baire 2, since the range of f^y is countable and the preimage of any set is a countable union of G_δ-sets. Finally, f is not measurable, since

$$\int_{\mathbf{R}}\left(\int_{\mathbf{R}} f_x dy\right) dx > 0 = \int_{\mathbf{R}}\left(\int_{\mathbf{R}} f^y dx\right) dy.$$

For the second part, let $\mathbf{R} = \cup_{\alpha<\omega_1} C_\alpha$, where $\lambda(C_\alpha) = 0$ for every $\alpha < \omega_1$. We may assume that each C_α is a G_δ set. Following the proof of (i), we obtain a nonmeasurable function f such that f_x is approximately continuous for every x. Also, for every y, the preimage of any set by f^y is a countable union of $F_{\sigma\delta}$ sets, and thus f^y is Baire 3.

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Proof of Theorem 4. We will use the following lemmas. For a set in the plane $H \subseteq \mathbf{R} \times \mathbf{R}$ and $x, y \in \mathbf{R}$ let

$$H_x = \{y \in \mathbf{R} : (x,y) \in H\} \text{ and } H^y = \{x \in \mathbf{R} : (x,y) \in H\}.$$

Lemma 7 *The following statements are equivalent.*

(i) *There exists a nonmeasurable function $f : \mathbf{R}^2 \to \mathbf{R}$ such that f_x is approximately continuous and f^y is measurable for every $x, y \in \mathbf{R}$.*

(ii) *There exists a set $H \subseteq \mathbf{R}^2$ such that $\lambda(H^y) = 0$ for every $y \in \mathbf{R}$, but the set $\{x : \lambda(\mathbf{R} \setminus H_x) = 0\}$ has positive outer measure.*

Proof. (ii)$\Longrightarrow$(i): Suppose (ii) and let $A = \{x : \lambda(\mathbf{R} \setminus H_x) = 0\}$. For every $x \in A$ there is a G_δ null set $B_x \subseteq \mathbf{R}$ such that $\mathbf{R} \setminus H_x \subseteq B_x$. This implies by Zahorski's Lemma that for every $x \in A$ there exists an approximately continuous function $g_x : \mathbf{R} \to \mathbf{R}$ such that $g_x(y) = 0$ if $y \in B_x$ and $0 < g_x(y) \leq 1$ if $y \notin B_x$.

For every $y \in \mathbf{R}$ we define $f(x,y) = g_x(y)$ if $x \in A$, and $f(x,y) = 0$ if $x \notin A$. Then f_x is approximately continuous for every x. Also, f^y is measurable for every y, since $f^y(x) = 0$ for a.e. x. Indeed,

$$f^y(x) \neq 0 \Longrightarrow x \in A,\ y \notin B_x, \Longrightarrow y \in H_x \Longrightarrow x \in H^y$$

and hence

$$\lambda(\{x : f^y(x) \neq 0\}) \leq \lambda(H^y) = 0.$$

This implies that

$$\int_{\mathbf{R}} \left(\int_{\mathbf{R}} f^y dx \right) dy = 0.$$

On the other hand,

$$\int_{\mathbf{R}} \left(\int_{\mathbf{R}} f_x dy \right) dx > 0,$$

since $\int_{\mathbf{R}} f_x dy > 0$ for every $x \in A$ and A has positive outer measure. Therefore f cannot be measurable.

(i)$\Longrightarrow$(ii): Suppose (i); we may also assume that f is bounded.

Since every approximately continuous function is Baire 1, it follows as in the proof of Theorem 1, that the function

$$F(x,y) = \int_0^x f^y dt$$

is Baire 1. Let

$$g(x,y) = \begin{cases} \lim_{n\to\infty} n \cdot (F(x + (1/n), y) - F(x,y)) & \text{if this limit exists} \\ 0 & \text{if it does not.} \end{cases}$$

Then g is Borel measurable, and for every fixed y, we have $g(x,y) = f(x,y)$ for a.e. x by Lebesgue's classical theorem.

Claim. For any $g : \mathbf{R}^2 \to \mathbf{R}$ measurable, there exists a Borel set $B \subseteq \mathbf{R}^2$ such that $\lambda_2(B) = 0$ and for every $(x,y) \notin B$ the function g_x is approximately continuous at y.

Proof. This easily follows from the fact that if $E \subseteq \mathbf{R}^2$ is measurable then there is a Borel set $B \subseteq \mathbf{R}^2$ such that $\lambda_2(B) = 0$ and y is a density point of E_x for every $(x,y) \in E \setminus B$; see the argument on pp. 130-131 of Saks

[20]. For the convenience of the reader we sketch the proof here. Without loss of generality, we may assume E is compact. Fix $\epsilon > 0$ and define

$$A_n^\epsilon = \{(x,y) \in E : \lambda(E_x \cap I) \geq (1-\epsilon)\lambda(I) \text{ whenever } y \in I \text{ and } |I| < 1/n\}.$$

(We use I to range over nondegenerate closed intervals.) Then it can be shown that A_n^ϵ is closed since E is. Therefore

$$N_\epsilon = E \setminus \bigcup_{n \in \omega} A_n^\epsilon$$

is measurable. By the Lebesgue density theorem, $(N_\epsilon)_x$ has measure zero for every x and hence by Fubini's Theorem N_ϵ has planar measure zero. Let

$$B = \bigcup_{\epsilon > 0} N_\epsilon.$$

Then $\lambda_2(B) = 0$ and y is a density point of E_x for every $(x,y) \in E \setminus B$. To obtain the result for g let B be a measure zero subset of the plane such that for every U in some countable basis for $\mathbf{R}$ if $(x,y) \in g^{-1}(U) \setminus B$, then y is a density point of $(g^{-1}(U))_x = g_x^{-1}(U)$. It follows that g_x is approximately continuous at y for every $(x,y) \in \mathbf{R} \setminus B$. This proves the Claim.

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Let

$$K = \{(x,y) : g(x,y) \neq f(x,y)\};$$

then $\lambda(K^y) = 0$ for every y. Let x be fixed. Then, for $y \notin B_x$, the functions f_x and g_x are both approximately continuous at y. Therefore, if $(x,y) \in K$ then the set

$$K_x = \{y : f_x(y) \neq g_x(y)\}$$

is measurable and of positive measure. (This is because if two functions are approximately continuous at a point x and take on different values there, then there exists a measurable set with density one at x where they differ.)

Hence for any x, K_x is measurable, and either $K_x \subseteq B_x$ or K_x has positive measure. Let $A = \{x : \lambda(K_x) > 0\}$, then $K \subseteq B \cup (A \times \mathbf{R})$. If $\lambda(A) = 0$ then $\lambda_2(K) = 0$ and $f = g$ almost everywhere, contradicting our assumption that f is not measurable. Thus A has positive outer measure.

Now, putting $H = \{(x, y + r) : (x, y) \in K,\ r \in \mathbf{Q}\}$, we obtain a set such that $\lambda(H^y) = 0$ for every y and $\lambda(\mathbf{R} \setminus H_x) = 0$ for $x \in A$; and hence (ii) holds.

■

By the random real model we refer to any model of set theory which is a generic extension of a countable transitive ground model of CH by adding ω_2 random reals, i.e., forcing with the measure algebra on 2^{ω_2}.

Lemma 8 *In the random real model the following two facts hold:*

1. $\mathbf{R}$ *is not the union of* ω_1 *measure zero sets.*
2. *Any* $Y \subseteq \mathbf{R}$ *with positive outer measure contains a subset* $Z \subseteq Y$ *of cardinality* ω_1 *with positive outer measure.*

Proof. Lemma 8.1 is due to Solovay [19] and is also proved in Kunen [13] 3.18 and probably Jech [11]. Lemma 8.2 is probably due to Kunen (see remark in Tall [22] p. 283), but we don't know of a published proof, so we include one here. The category version of Lemma 8.2 appears in Komjáth [12].

Since 2^ω and $[0, 1]$ are measure isomorphic, we may work in 2^ω. For any set Σ let 2^Σ be product space of the two point set $2 = \{0, 1\}$ with the usual product measure and topology. Let $\mathbb{B}(\Sigma)$ denote the measure algebra, i.e., the Borel subsets of 2^Σ modulo the measure zero sets. This is a complete boolean algebra which satisfies the countable chain condition.

Let M be a countable standard model of ZFC+CH. For any set Σ in M let $\mathbb{B}(\Sigma)^M$ denote the measure algebra in M. A generic filter may be regarded as a map $G : \Sigma \to 2$.

We use the following facts which are probably all due to Solovay:

1. (see Kunen [13] 3.13) For any two disjoint sets Σ and Γ in a countable standard model M,
   (a) G is $\mathbb{B}(\Sigma \cup \Gamma)$-generic over M iff
   (b) $G \restriction \Sigma$ is $\mathbb{B}(\Sigma)^M$-generic over M and $G \restriction \Gamma$ is $\mathbb{B}(\Gamma)^{M[G\restriction\Sigma]}$-generic over $M[G \restriction \Sigma]$.

2. (Kunen [13] 3.22) Suppose $G : \Sigma \to 2$ is $\mathbb{B}(\Sigma)$-generic over M and $Y \in M$ is such that

$$M \models Y \subseteq 2^\omega \text{ has positive outer measure.}$$

Then

$$M[G] \models Y \subseteq 2^\omega \text{ has positive outer measure.}$$

3. (Well-known) Suppose $G : \omega_2 \to 2$ is $\mathbb{B}(\omega_2)$-generic over M and

$$M[G] \models Y \text{ has positive outer measure.}$$

Then there exists a set $\Sigma \subseteq \omega_2$ in M of cardinality ω_1 in M such that if $Z = M[G \restriction \Sigma] \cap Y$, then

$$M[G \restriction \Sigma] \models Z \text{ has positive outer measure.}$$

Fact 3 is proved with a Lowenheim-Skolem argument as follows. Let $f : 2^\omega \to 2 \times 2^\omega$ be a map with the following property: If $f(x) = (i, z)$, then

a $i = 1$ iff $x \in Y$ and

b if x is a code for a Borel set of measure zero set $Z(x)$, then $z \in Y \setminus Z(x)$.

Since there is a recursive pairing function taking $2 \times 2^\omega$ to 2^ω it suffices to show that for any function $f : 2^\omega \to 2^\omega$ in $M[G]$ there exists a set $\Sigma \subseteq \omega_2$ in M of size ω_1 in M such that $2^\omega \cap M[G \restriction \Sigma]$ is closed under f and

$$f \restriction (M[G \restriction \Sigma]) \in M[G \restriction \Sigma].$$

For any $x \in 2^\omega \cap M[G]$ there exists a sequence $(B_n : n \in \omega)$ of Borel sets in M with countable support such that for any $n \in \omega$ we have $x(n) = 1$ iff $G \in B_n$ (the equivalence class of B_n is the boolean value of the statement "$x(n) = 1$"). Any such sequence $(B_n : n \in \omega)$ is called a canonical name for an element of 2^ω (see Kunen [13] 3.17). Working in the ground model M with a name for the function f, we can define a map F from canonical names to canonical names such that for any canonical name τ, $F(\tau)$ will be a canonical name for $f(\tau^G)$. Since canonical names have countable support and M satisfies the GCH there exists a set $\Sigma \subseteq \omega_2$ of cardinality ω_1 in M

such that for any canonical name τ with support from Σ, the support of $F(\tau)$ is a subset of Σ. This proves Fact 3.

To prove Lemma 8.2, suppose $Y \subseteq \mathbf{R}$ has positive outer measure. By Fact 3 above there exists a set $\Sigma \subseteq \omega_2$ in M of cardinality ω_1 in M such that if $Z = M[G] \cap Y$, then

$$M[G \restriction \Sigma] \models Z \text{ has positive outer measure.}$$

Now since M is a model of CH we have that $M[G \restriction \Sigma]$ is a model of CH (see Kunen [13] 3.14). Hence Z has cardinality ω_1. By Facts 1 and 2, it follows that Z has positive outer measure in $M[G]$.

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Finally we prove Theorem 4. By Lemma 7 if there were such a nonmeasurable function, then there would be a set $H \subseteq \mathbf{R}^2$ such that $\lambda(H^y) = 0$ for every $y \in \mathbf{R}$, and $Y = \{x : \lambda(\mathbf{R} \setminus H_x) = 0\}$ has positive outer measure. By applying Lemma 8.2 we get $Z \subseteq Y$ with positive outer measure and cardinality ω_1. By Lemma 8.1 we know that the reals are not covered by the ω_1 measure zero sets $\{\mathbf{R} \setminus H_x : x \in Z\}$. Suppose $y \notin \bigcup\{\mathbf{R} \setminus H_x : x \in Z\}$. Then $y \in \bigcap\{H_x : x \in Z\}$ which implies $Z \subseteq H^y$ contradicting the fact that H^y has zero measure.

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Remarks. The next statement is implicit in Freiling [8] (see the proof of the Theorem on p. 198). The following are equivalent:

(i) there is a function $f : [0,1]\times[0,1] \to [0,1]$ such that f_x, f^y are measurable for every x and y, and $\int(\int f_x dy)dx \neq \int(\int f^y dx)dy$;

(ii) there exists a set $H \subset [0,1] \times [0,1]$ such that H^y is a null set for every y and $[0,1] \setminus H_x$ is a null set for every x.

This is similar to our Lemma 7; also, it implies that if Fubini's theorem is not true for arbitrary bounded functions, then there is a nonmeasurable function f such that f_x is approximately continuous and f^y is measurable for every x, y.

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