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Analytic subspace of the reals without an analytic basis.

A Hamel basis is a basis for the reals $\mathbb{R}$ considered as a vector space over the field of rationals $\mathbb{Q}$.

Theorem 1. *(Erdos and Sierpinski [1]) There is no analytic Hamel base.*

Proof. Suppose on the contrary that B is such a basis. $\mathbb{R}$ is the countable union of sets of the form $q_1B + \ldots q_nB$, where $q_1, \ldots, q_n \in \mathbb{Q}$. These sets are all analytic, hence have the property of Baire and thus, by the Baire category theorem, for some $q_1, \ldots, q_n \in \mathbb{Q}$, the set $A = q_1B + \ldots q_nB$ is non meager. There is an interval $I \subseteq \mathbb{R}$ such that A is comeager in I. Pick any distinct $x_1, \ldots, x_n \in B$. Let $q \in \mathbb{Q}$ and $J \subseteq I$ be a subinterval such that $q(x_1 + \ldots + x_n) + A$ is comeager J. Note that q can be chosen so that $q \neq q_i$, for each i. This will ensure that there will be none of the x_i will be canceled out by elements of A. Then

$$W = (q(x_1 + \ldots + x_n) + A) \cap A \neq \emptyset$$

because both terms of the intersection above are comeager in J. Any element of W will be at the same time a linear combination of at least $n+1$ elements of B and also n elements of B. This contradicts the linear independence of B ⊠

Theorem 2. *Every proper analytic subspace of $\mathbb{R}$ is measure zero and meager.*

Proof. For the first claim, suppose that A is an analytic subspace of $\mathbb{R}$ has positive measure. Then by Steinhaus' theorem, $A - A$ (the set of differences of elements of A) contains a nontrivial interval. Hence A must be all of $\mathbb{R}$.

For the second claim, suppose that A is an analytic subspace which is non-meager. A has the property of Baire and hence there is an open interval I in which A is comeager. Fix any $\alpha \in \mathbb{R}$. Let $q \in \mathbb{Q}$ be small enough that $q\alpha$ is less than the length of I. Then $J = (q\alpha + I) \cap I$ is non empty and $q\alpha + A$ and A are both comeager in J. In other words, there exist $x, y \in A$ such that $q\alpha + x = y$. Hence $\alpha = \frac{1}{q}(y - x) \in A$. We see that $A = \mathbb{R}$.

⊠

The following answers a question raised by Ashutosh Kumar.

Theorem 3. *There exists a proper (and hence meager) analytic subspace of* $\mathbb{R}$ *with no analytic basis.*

We begin by describing the subspace in question.

Let ϵ_n be a decreasing sequence of positive rational numbers such that for every k and each $N \geq k$,

$$\sum_{n>N} k\epsilon_n \leq \frac{1}{4}\epsilon_N$$

This condition requires the ϵ_n to be a very rapidly decreasing sequence. Now let P be the set defined by

$$P = \{\sum_{n\in\omega} x_n\epsilon_n : x_n \in \{-1, 0, 1\}\}$$

P is essentially a very sparse Cantor set. We now take the subspace A to be span(P). Our first objective is to show that A is a proper subspace of $\mathbb{R}$. To this end, we make the following observations: A is the union of all sets of the form $q_1P + \ldots + q_nP$, where the q_i are rational numbers. By taking common denominators, we can write such sets as $\frac{1}{m}(p_1P + \ldots + p_nP)$, for some $p_1, \ldots, p_n \in \omega$. If we let $k = p_1 + \ldots + p_n$, then

$$(p_1P + \ldots + p_nP) \subset \underbrace{P + P + \ldots + P}_{k \text{ times}}$$

We give this latter set the name Q_k. Observe that Q_k can be described by

$$Q_k = \{\sum_{n\in\omega} x_n\epsilon_n : x_n \in \{-k, -k+1, \ldots, k-1, k\}\}$$

$p_1P + \ldots + p_nP \subset Q_k$ and hence $\frac{1}{m}(p_1P + \ldots + p_nP) \subset \frac{1}{m}Q_k$. Note that of course each $\frac{1}{m}Q_k$ is also a subset of A.

Before proceeding, note that throughout we will use the notation $\hat{\sigma}$ for the rational number $\sum_{n<|\sigma|}\sigma(n)\epsilon_n$, where σ is a finite sequence of integers.

We now show that A is a proper subspace via the following two lemmas.

Lemma 4. *Suppose that $\sigma, \tau \in \{-k, \ldots, k\}^{<\omega}$ such that $|\sigma| = |\tau| > k$. If $\sigma <_{lex} \tau$, then every point in Q_k^σ is less than every point in Q_k^τ.*

Proof. It suffices to prove this lemma for the case in which there exists γ such that $\sigma = \gamma\hat{\ }i$ and $\tau = \gamma\hat{\ }(i+1)$, for some $i \in \{-k, \ldots, k-1\}$. Let $M = |\gamma|$. The greatest element of Q_k^σ is

$$\alpha = \hat{\sigma} + \sum_{n>M} k\epsilon_n = \hat{\gamma} + i\epsilon_M + \sum_{n>M} k\epsilon_n$$

and the least element of Q_k^τ is

$$\beta = \hat{\tau} - \sum_{n>M} k\epsilon_n = \hat{\gamma} + (i\epsilon_M - \sum_{n>M} k\epsilon_n)$$

Therefore,

$$\beta - \alpha = \epsilon_M - \sum_{n>M} 2k\epsilon_n \geq \frac{1}{2}\epsilon_M > 0$$

⊠

Lemma 5. *Each Q_k^σ is nowhere dense, for $|\sigma| > k$.*

Proof. Fix any interval I such that $I \cap Q_k^\sigma \neq \emptyset$. Choose $\tau \supseteq \sigma$ such that $\hat{\tau}\hat{\ }i, \tau\hat{\ }(i\hat{+}1) \in I$, for some $i \in \{-k, \ldots, k-1\}$. Then every element of $Q_k^{\tau\hat{\ }i}$ is less than every element of $Q_k^{\tau\hat{\ }(i+1)}$ by the previous lemma. Therefore, between all Q_k^σ are closed sets, we may take an interval J between $Q_k^{\tau\hat{\ }i}$ and $Q_k^{\tau\hat{\ }(i+1)}$. J is disjoint from Q_k^σ, because Q_k^σ is the disjoint union of Q_k^γ for $|\gamma| = |\tau| + 1$ and by the previous lemma, no such Q_k^γ intersects J.

⊠

This shows that each Q_k^σ is nowhere dense. Hence Q_k is as well, being a finite union of such Q_k^σ. It follows that each $\frac{1}{m}Q_k$ is nowhere dense and hence $A = \bigcup_{m,k\in\omega} \frac{1}{m}Q_k$ is meager. A is therefore proper.

Now we get to our main claim.

Lemma 6. *A has no analytic basis as a vector space over $\mathbb{Q}$.*

We begin with some remarks about the set P. As in the above lemma, for $\sigma \in \{-1,0,1\}^{<\omega}$, we define

$$N_\sigma = \{\sum_{n\in\omega} x_n\epsilon_n : x_n \in \{-1,0,1\} \ \&\ \sigma(n) = x_n \text{ for } n < |\sigma|\}$$

Note that although the N_σ are closed sets in $\mathbb{R}$ (and hence in P), they are also relatively open in P. In fact, they form a base for the relative topology on P.

Proof of Lemma 6. Suppose towards a contradiction that B is an analytic basis for A. We may assume, without loss of generality, that $1 \in B$. Otherwise, suppose that $x_1,\ldots,x_n \in B$ and $q_1,\ldots q_n \in \mathbb{Q}$ are such that $1 = q_1x_1 + \ldots + q_nx_n$. Then $[(q_1-1)x_1 + q_2x_2 + \ldots q_nx_n] + x_1 = 1$. Hence $[(q_1-1)x_1 + q_2x_2 + \ldots q_nx_n] + B$ is an analytic basis for A which contains 1.

Since B is a basis, the generating set P of A must be covered by a union of set of the form

$$q_1(B \cap I_1) + \ldots + q_n(B \cap I_n)$$

where $q_1,\ldots,q_n \in \mathbb{Q}$ and $I_1,\ldots,I_n$ are pairwise disjoint intervals with rational endpoints. To avoid confusion later on, we assume here that all q_j are nonzero and that each $B \cap I_j$ is nonempty.

P is an uncountable closed set and hence a Baire space when regarded as a topological subspace of $\mathbb{R}$. Because the union described above is countable, the Baire category theorem yields that there are $q_1,\ldots,q_n, I_1,\ldots,I_n$ as above such that

$$W = q_1(B \cap I_1) + \ldots + q_n(B \cap I_n)$$

is non-meager in P. W is analytic, hence has the Baire property. We therefore obtain $\sigma \in \{-1,0,1\}^\omega$ such that W is comeager in N_σ. (Because the N_σ are a base for the relative topology on P.)

We now define a homeomorphism π of N_σ as follows: If $z \in N_\sigma$, then $z = \hat{\sigma} + \sum_{n\geq|\sigma|} x_n\epsilon_n$, for some sequence $\langle x_n : n \in \omega\rangle \in \{-1,0,1\}^\omega$. We define $\pi(z) = \hat{\sigma} - \sum_{n\geq|\sigma|} x_n\epsilon_n$. It is clear that π is an autohomeomorphism of N_σ.

It follows that $\pi^{-1}(W)$ is also comeager in N_σ and hence $W \cap \pi^{-1}(W) \neq \emptyset$. Let $z \in W \cap \pi^{-1}(W)$. Then $z, \pi(z) \in W$. Note that we may assume that z

(and hence $\pi(z)$) are irrational. This follows from the fact that the rationals are meager in N_σ.

We may now take $x_j, y_j \in B \cap I_j$ such that

$$z = q_1 x_1 + \ldots + q_n x_n$$
$$\pi(z) = q_1 y_1 + \ldots + q_n y_n$$

Thus

$$z + \pi(z) = q_1(x_1 + y_1) + \ldots + q_n(x_n + y_n)$$

By the definition of π, $z + \pi(z) = \hat{\sigma} \in \mathbb{Q}$. Note that since the I_j are disjoint, for each j and $i \neq j$ $x_j \neq x_i, y_i$. Further, because $z \notin \mathbb{Q}$, $z \neq \pi(z)$, we have that for at least one j, $x_j \neq y_j$. We have therefore expressed a rational number (namely $\hat{\sigma}$) as a sum of $n+1$ distinct elements of the basis B. On the other hand, $1 \in B$ and any rational can be expressed as a rational scalar multiple of 1, i.e. a linear combination of length 1. By independence, such linear combinations are unique and so the above leads to a contradiction. ⊠

We conclude with some further notes.

Theorem 7. *For all $\alpha > 2$ there exists a $\mathbb{Q}$–subspace A of $\mathbb{R}$ which is $\mathbf{\Sigma}^0_\alpha$, but not $\mathbf{\Pi}^0_\alpha$.*

Proof. Let $C \subseteq \mathbb{R}$ be a perfect, linearly independent set. Choose $B \subseteq C$ which is $\mathbf{\Sigma}^0_\alpha$, but not $\mathbf{\Pi}^0_\alpha$. Take A to be the linear span of B.

First of all, A is not $\mathbf{\Pi}^0_\alpha$. To see this, observe that, by the independence of C, $A \cap C = B$. If A were $\mathbf{\Pi}^0_\alpha$, then B would be as well.

Secondly, A is $\mathbf{\Sigma}^0_\alpha$. Observe that A is the union of sets of the form.

$$q_1(B \cap I_1) + \ldots + q_n(B \cap I_n)$$

Where the q_i are nonzero rational numbers and the I_i are disjoint intervals with rational endpoints. We can define a homeomorphism

$$\prod_{i=1}^{n}(C \cap I_i) \to q_i(C \cap I_i) + \ldots + (C \cap I_n)$$

by $\langle x_1, \ldots, x_n\rangle \mapsto q_1x_1 + \ldots + q_nx_n$. Under this map, $\prod_{i=1}^n (B \cap I_i)$ maps onto $q_1(B \cap I_1) + \ldots + q_n(B \cap I_n)$. Hence this latter set of is of the same Borel class as B, namely $\mathbf{\Sigma}^0_\alpha$. Since the union above is countable, A is also $\mathbf{\Sigma}^0_\alpha$.

$\boxtimes$

Theorem 7 is also a consequence of Theorem 2.5 of Farah and Solecki [2] but has a shorter proof.

Theorem 8. *For all $\alpha > 3$, there exists a $\mathbb{Q}$-subspace W of $\mathbb{R}$ which is $\mathbf{\Pi}^0_\alpha$ and not $\mathbf{\Sigma}^0_\alpha$.*

Proof. Let $C \subset \mathbb{R}$ be a perfect, independent set over $\mathbb{Q}$. Let $A_0 \supset A_1 \supset \ldots$ be subsets of C which are $\mathbf{\Sigma}^0_{<\alpha}$ and such that $A = \bigcap_{n\in\omega} A_n$ is $\mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$. Let $W_n = \mathrm{span}_{\mathbb{Q}}(A_n)$ and $W = \bigcap_{n\in\omega} W_n$. Then each W_n is $\mathbf{\Sigma}^0_{<\alpha}$ as in the proof of Theorem 7. Thus W is $\mathbf{\Pi}^0_\alpha$, but not $\mathbf{\Sigma}^0_\alpha$. If W were $\mathbf{\Sigma}^0_\alpha$, then $A = W \cap C$ would be as well. $\boxtimes$

References

[1] Erdös, P.; Some remarks on set theory. Proc. Amer. Math. Soc. 1, (1950). 127-141.

[2] Farah, Ilijas; Solecki, Sławomir; Borel subgroups of Polish groups. Adv. Math. 199 (2006), no. 2, 499-541.