

Namba forcing - A. Miller Oct 2012 (1)

Prity 1968

We can have $M \subseteq M[G]$

(1) $(\text{Card})^M = (\text{Card})^{M[G]}$ (" $\leq$ " always true)

(2) $\exists \kappa \in M$
 $M \models \kappa \text{ reg. limit card}$ & $M[G] \models \text{cf}(\kappa) = \omega$

Note $M \models \kappa \text{ reg. limit card}$

Ley 1965? collapse ω_2 to ω_1

$$IP = (\omega_2^{<\omega_1})^M$$

$$\kappa^{<\lambda} = \bigcup_{\alpha < \lambda} \kappa^\alpha$$

$$\kappa^\alpha = \{p : p \cdot \alpha \rightarrow \kappa\}$$

$$G \text{ IP-generic / } M \Rightarrow g = \bigcup G$$

$$g: \omega_1 \rightarrow \omega_2$$

$$p \leq q$$

$$\text{iff}$$

$$p \supseteq q$$

$$\text{or } s \text{ ---}$$

$$IP \text{ ctky-closed} \Rightarrow M^\omega \cap M[G] \subseteq M$$

$$\text{so } M[G] \models \text{cf}(\omega_2^M) = \omega_1^M$$

$$\& \omega_1^M = \omega_1$$

(2)

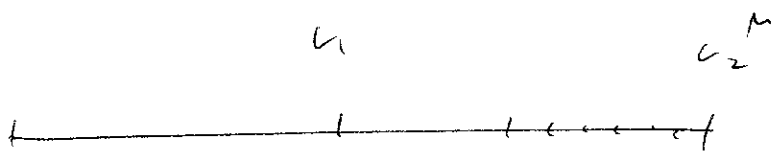
Namba (1971)

If $M \models \text{"} P \text{ is Namba forcing"}$ or $C \models \text{"} P \text{ is } \mathbb{P} \text{-gen}/M \text{ then}$

$$\textcircled{1} M[G] \models " \omega_1 = \omega_1^M + \text{cof}(\omega_2^M) = \omega "$$

$$(\text{if } M \models \text{CH}) \textcircled{2} (\omega_1^M)^M = (\omega_1^M)^{M[G]}$$

$$\text{or equivalently } (P(\omega_1))^M = (P(\omega_1))^{M[G]}$$



Def $p \leq \omega_2^{<\omega}$ is a Namba tree
iff

$$\textcircled{1} p \neq \emptyset \quad \& \quad \forall s, t \ (s \leq t \in p \rightarrow s \in p)$$

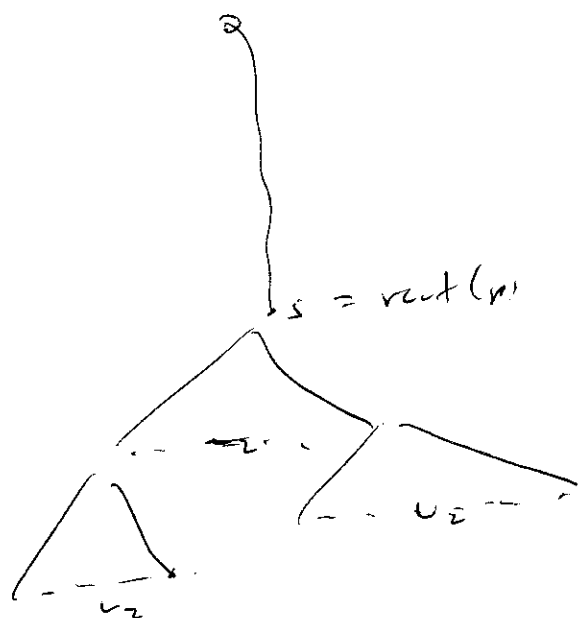
$$\textcircled{2} \exists s \in p \ (s = \text{root}(p))$$

$$p_s = \text{the } \{ t \in p : t \leq s \text{ or } s \leq t \} = \text{the } \{ t \in p : s \leq t \}$$

$$p_s = p \quad \& \quad \forall t \in A(p) = \{ t \in p : s \leq t \}$$

$$|\{ \alpha < \omega_2 : t^\alpha \in p \}| = \omega_2$$

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$$p \leq s \text{ iff } p \leq s$$

$$P = (N_{\text{new}} - k - 1)^M$$

If a P -sub $/M$ the MCLT "col(u_1^*) = u ".

exercice Let $g = U \{ \text{root}(p) : p \in L \}$.

Show $g : \omega \rightarrow \omega_2^M$ map colid in ω_2^M .

Def $p \leq_0 s$ iff $p \leq s$ at $\text{root}(p) = \text{root}(s)$

Lemma: Suppose p lt $\tau < \check{u}_i$ then $\exists q \leq_0 p$

$\exists \alpha < u_i$ q lt $\tau = \check{\alpha}$.

pf

Def s good if $\exists q \leq_0 p$ $\exists \alpha < u_i$

q lt $\tau = \check{\alpha}$.

bad = not good

Assume for contradiction $\text{root}(p)$ bad.

Claim If $s \in A(p)$ but then

$$|\{ \alpha < \omega_2 : s^{\wedge} \alpha \text{ good} \}| \leq \omega_1$$

pf

Suppose not $G = \{ \alpha < \omega_2 : s^{\wedge} \alpha \in p \text{ and } s^{\wedge} \alpha \text{ good} \}$

$$|G| = \omega_2$$

$$\forall \alpha \in G \quad \exists q_\alpha \leq p_{s^\wedge \alpha} \quad \exists \beta_\alpha < \omega_1$$

$$q_\alpha \Vdash \tau = \check{p}_\alpha$$

$$\exists G_0 \subseteq G \quad \exists p \quad \forall \alpha \in G_0 \quad p_\alpha = p$$

$$(G_0) = \omega_2$$

$$\text{Let } q = \bigcup_{\alpha \in G_0} q_\alpha$$

$$q \leq p$$

$$q \Vdash \tau = \check{p}$$

contradict

s good.



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Let $Q = \{s \in p \mid s \subseteq \text{root}(p) \text{ or}$

$s \in A(p) \forall t$

$\text{root}(p) \leq t \leq s$
 $t \text{ bad } \}.$



$q \leq p$

$\forall s \in A(q) \quad p, b \text{ bad.}$

let $r \leq q \quad r \Vdash \tau = \check{\alpha}$

Conclude $r \leq q_s \leq p_s \quad s = \text{root}(r)$
 $\& r \leq p_s \rightarrow s \text{ good.}$

(*)

Lemma Suppose $p \Vdash \tau: \check{\omega} \rightarrow \check{\omega}_1$ then

$\exists q \leq p \quad \exists \pi: q \rightarrow \omega_1^{<\omega} \quad \forall s \in q \quad |s| = n$

$q_s \Vdash \pi(n) = \pi(s).$

$\omega_1 \approx \omega_1^*$

so $\forall s \in p \quad \exists q_s \leq p_s \quad \exists \pi(s)$

$q_s \Vdash \pi(n) = \pi(s)$

To get on q
use Fodor

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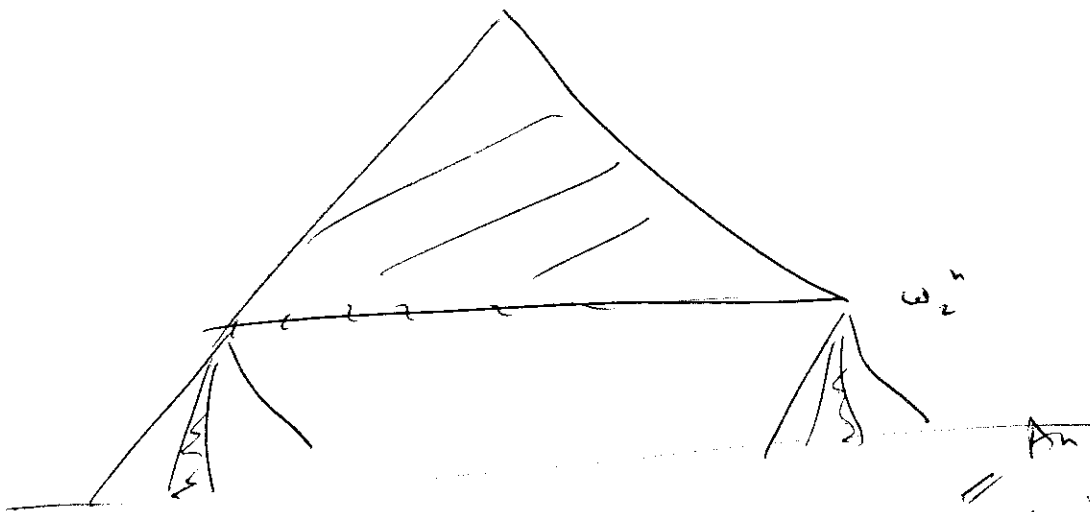
To simplify notation assume $\text{rank}(p) = \infty$

Construct $p \geq p_1 \geq p_2 \geq \dots$

with strong p.p.i. $p_n \cap \omega_2^n = p_n \cap \omega_2^n$

$\forall s \in \omega_2^n \cap p_n \exists \pi(s)$

$(p_n)_s \Vdash \pi \upharpoonright n = \pi(s)$



to get p_n

$q^s \leq (p_n)_s$

$\forall s \in \omega_2^n$ chain

$q^s \Vdash \pi \upharpoonright n = \pi(s)$

let $p_n = \bigcup_{s \in \omega_2^n \cap p_n} q^s$

Note $\bigcap_n p_n = q$ (i) $\in \text{Nak}$ for $q \leq p$

Recall

Topology on ω_2^ω

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basic clops $[s] = \{x \in \omega_2^\omega : s \leq x\}$
 $s \in \omega_2^{<\omega}$

$[p] = \{x \in \omega_2^\omega : \forall n \ x \upharpoonright n \in p\}$
 closed set.

π induces a continuous map.

$\tilde{\pi} : [q] \rightarrow \omega_1^\omega$

$\tilde{\pi}(x) = \bigcup_{n < \omega} \pi(x \upharpoonright n)$

$\left(\begin{array}{l} s \leq t \in q \rightarrow q_t \leq q_s \\ q_t \Vdash x \upharpoonright_n = \pi(t) \\ \pi(t) \geq \pi(s) \end{array} \right) \rightarrow \pi(t) \geq \pi(s)$

Recall $\mathcal{F}$ filter on X $\mathcal{F} \subseteq \mathcal{P}(X)$

① $A, B \in \mathcal{F} \rightarrow A \cap B \in \mathcal{F}$

② $A \subseteq B, A \in \mathcal{F} \rightarrow B \in \mathcal{F}$

③ $\emptyset \notin \mathcal{F}$

$\mathcal{I}$ dual ideal $\mathcal{I} = \{X \setminus A : A \in \mathcal{F}\}$

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$$f^+ = \{ A \subseteq X : A \neq \emptyset \} = \{ A \subseteq X : \forall B \in f, A \cap B \neq \emptyset \}$$



ex. (1) $f = \{ X \subseteq [0,1] : \lambda(X) = 1 \}$ $\mathcal{I} = \text{measures}$ $f^+ = \text{prob. measures}$

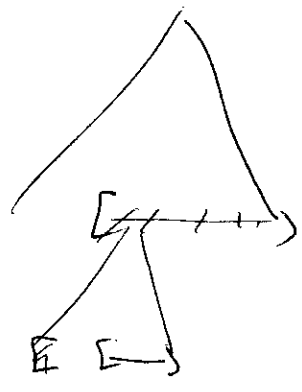
(2) $f = \text{continuous } \omega$ $\mathcal{I} = \text{filters}$ $f^+ = \text{int-filters}$

(3) $f = \text{club filter on } \omega_1$ $f^+ = \text{stationary sets}$

(4) $\mathcal{I} = \text{measures}$ $f = \text{co-measures}$ $f^+ = \text{non-measures}$

Def. T Fat-Number iff T Number

$$\forall s \in A(T) \exists \alpha_0 < \omega_1 \forall \alpha_0 \leq \alpha < \omega_1 s^{\wedge \alpha} \in T$$



Lemma: If T_α $\alpha < \omega_1$ are Fat-Numbers with root s then $P = \bigcap_{\alpha < \omega_1} T_\alpha$ is Fat Number with root s .

Note: $[T] = \bigcap_{\alpha < \omega_1} [T_\alpha]$

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Def: $\mathcal{F}_s$ = filter gen. by Fct-Nak with $\text{root}(s)$.

$$= \{ X \subseteq \omega_2 : \exists T \text{ F-N tree } [T] \subseteq X \}.$$

Note: $\mathcal{F}_s$ closed under ω_1 -intersections

$$[p] \in \mathcal{F}_s^+ \text{ when } s = \text{root}(p).$$

If $q = p \cap T$ then $q \leq p$.

Lemma Suppose $A \in \mathcal{F}_s^+$ then

$$Q = \{ \alpha < \omega_2 : A \in \mathcal{F}_{s \restriction \alpha}^+ \} \quad |Q| = \omega_2.$$

pf

$$\text{Suppose } |Q| \leq \omega_1 \quad \forall \alpha \in Q^c$$

$$A \in \mathcal{I}_{s \restriction \alpha} \quad \exists T_\alpha \text{ Fct-Nak tree } s \restriction \alpha$$

$$[T_\alpha] \cap A = \emptyset.$$

$$\text{Let } P = \bigcup_{\alpha \in Q^c} T_\alpha \quad T \text{ Fct-Nak tree } s$$

$$[P] \cap A = \emptyset.$$

$$[P] = \bigcup_{\alpha \in Q^c} [T_\alpha] \quad \text{no } A \in \mathcal{I}_s \quad \square$$

Lemme Suppos $A \subseteq L(p)$

$A \in \mathcal{F}_s^+$ $s = \text{root}(p)$ et A clos.

then $\exists q \subseteq_0 p$ $[q] \subseteq A$.

pt

$q = \{s \in p \mid \text{If } s \in A(p) \text{ then}$

$\forall t \text{ root}(p) \leq t \leq s \rightarrow A \in \mathcal{F}_t^+\}$.

Then $q \subseteq_0 p$ $A \subseteq L(p)$

~ if $A \in \mathcal{F}_s^+$ to sep.

Claim $[q] \subseteq A$

pt $\forall x \in [q] \quad \forall n > |\text{root}(p)| \quad A \in \mathcal{F}_{x \upharpoonright n}^+$

Hence $A \cap [x \upharpoonright n] \neq \emptyset$.

A clos so $x \in A$. $\square$.

Recall $p, \tilde{\pi}$

$$\forall s \in p \quad p_s \Vdash \check{\tau} \upharpoonright_{|s|} = \check{\pi}(s)$$

$$\tilde{\pi}: [p] \rightarrow \omega_1^\omega \quad \text{contin-}$$

Assume CH $\omega_1^\omega = \{x_\alpha : \alpha < \omega_1\}$.

$$A_\alpha = \tilde{\pi}^{-1}(x_\alpha) \quad \text{closed.}$$

$$[p] = \bigcup_{\alpha < \omega_1} A_\alpha$$

$$[p] \in \mathcal{F}_s^+ \quad s = \text{root}(p)$$

$$\text{so } \exists \alpha_0 \quad A_{\alpha_0} \in \mathcal{F}_s^+$$

$$\text{By Lemma } \exists q \leq_0 p \quad [q] \subseteq A_{\alpha_0}.$$

$$\text{But } q \Vdash \check{\tau} = \check{x}_{\alpha_0}^\vee.$$

To summarize: $\forall p \quad \forall \tau \quad \text{if } p \Vdash \tau \in \omega_1^\omega$
 then $\exists x \in \omega_1^\omega \quad \exists q \leq p \quad q \Vdash \tau = \check{x}$

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$$A = \{ x \in C : \forall n \frac{f_n(x)}{n} \leq 1 \}$$

$$A \subseteq C \subseteq LIT$$

Bukowski (1975)

If $M \models 2^u = u_1 + 2^v = u_2$

then $M \models \text{"} \text{cot}(w_1^M) = u_1 \text{"}$

1. Jensen's coding Lemma

2. Woodin-Shelah

$M \models CH \quad \exists x \in 2^\omega$ s.t.

$M[x] \models \neg CH + u_1 = u_1^M$

or

Alt:

$$C = \{ x : x(0) = 0 \}$$

$$C = \{ u : \exists v : u \leq v \}$$

$$D = \{ u : \exists v : v \leq u \}$$