

1 Ramsey's Theorem

Let $\omega = \{0, 1, 2, \dots\}$ and for any set A and $n \leq \omega$ let

$$[A]^n = \{B \subseteq A : |B| = n\}$$

where $|B|$ is the cardinality of the set B . So, for example, $[\omega]^\omega$, is the set of all infinite subsets of ω .

Theorem 1.1 (*Pigeon Hole Principal*¹) Suppose $f : \omega \rightarrow k$. Then there exists $H \in [\omega]^\omega$ such that $f \upharpoonright H$ is constant.

Theorem 1.2 *Ramsey's Theorem* ([7]) for any $m, k < \omega$ and $f : [\omega]^k \rightarrow m$ there exists $H \in [\omega]^\omega$ such that $f \upharpoonright [H]^k$ is constant.

proof:

The set H is said to be homogeneous for the function f .

We begin with the standard proof for $k = 2$. Construct $a_0 < a_1 < \dots < a_{n-1}$ and $X_n \in [\omega]^\omega$ as follows:

Let $a_n = \min\{X_{n-1}\}$ and find $X_n \in [X_{n-1} \setminus \{a_n\}]^\omega$ so that for every $a \leq a_n$ and $u, v \in X_n$

$$f(a, u) = f(a, v).$$

We can construct such a set by iteratively applying the pigeon hole principal as follows. Note given any $a \in \omega$ and $Y \in [\omega]^\omega$ there is a $Z \in [Y]^\omega$ and $i < m$ such that for every $z \in Z$ we have $f(\{a, z\}) = i$. Now we just iterate this

$$X_{n-1} = Z_0 \supseteq Z_1 \supseteq Z_2 \dots \supseteq Z_m = X_n$$

taking care of all $a \leq a_n$ (so $m = a_n + 1$).

Finally consider the set $K = \{a_n : n \in \omega\}$. It is “tail homogeneous”, i.e., for any u, v, w distinct elements of K if $u < v$ and $u < w$, then $f(u, v) =$

¹One of my colleagues told me about the pigeon head principal. If you get into an elevator and there are more buttons pressed than there are people in the elevator, then there must be a pigeon head.

$f(u, w)$. Thus we can define $g : K \rightarrow m$ by $g(u) = f(u, v)$ for any $v > u$ in K . By the pigeon hole principle there exists $H \in [K]^\omega$ such that $g \upharpoonright H$ is constant and therefore $f \upharpoonright [H]^2$ is constant.

Instead of giving the standard proof for $k > 2$ for novelty we give a proof² using model theory. Let $f : [\omega]^{k+1} \rightarrow m$ be any function. Consider the model

$$\mathfrak{A} = (\omega, <, f, \underline{n})_{n \in \omega}.$$

By applying the compactness theorem we can find a model $\mathfrak{B}$ which is a proper elementary extension of $\mathfrak{A}$. This means it contains a “hyperfinite” integer H , i.e., an element of the model $\mathfrak{B}$ satisfying $n < H$ for every $n \in \omega$. We construct a sequence $a_0 < a_1 < \dots < a_n$ in ω with the following property:

For any $u_1 < u_2 < \dots < u_k \leq a_{n-1}$ we have that

$$f(u_1, \dots, u_k, a_n) = f^{\mathfrak{B}}(u_1, \dots, u_k, H).$$

This can be done, because if we define $i_{u_1, \dots, u_k} < m$ by

$$f^{\mathfrak{B}}(u_1, \dots, u_k, H) = i_{u_1, \dots, u_k}$$

then

$$\mathfrak{B} \models \exists x > a_n \bigwedge_{u_1 < \dots < u_k \leq a_{n-1}} f(u_1, \dots, u_k, x) = i_{u_1, \dots, u_k}$$

(namely $x = H$) so by elementarity

$$\mathfrak{A} \models \exists x > a_n \bigwedge_{u_1 < \dots < u_k \leq a_{n-1}} f(u_1, \dots, u_k, x) = i_{u_1, \dots, u_k}$$

so now choose a_n to be any such $x \in \omega$.

Now our set $K = \{a_n : n \in \omega\}$ is tail-homogeneous, i.e., given any $u_1 < u_2 < \dots < u_k$ and $v, w > u_k$ in K we have

$$f(u_1, \dots, u_k, u) = f(u_1, \dots, u_k, v)$$

(since both are equal to $f^{\mathfrak{B}}(u_1, \dots, u_k, H)$). As before, we define

$$g : [K]^{k+1} \rightarrow m \text{ by } g(u_1, \dots, u_k) = f^{\mathfrak{B}}(u_1, \dots, u_k, H)$$

and apply induction to g to get $H \in [K]^\omega$ on which g is constant and then $f \upharpoonright [H]^{k+1}$ is constant. ■

²This proof was found by my fellow graduate student Charlie Gray (circa 1975) and is based on the idea of Simpson’s model theoretic proof of the Erdos-Rado Theorem.

Corollary 1.3 *Finite Ramsey Theorem.* For any $m, k, h < \omega$ there exists $N < \omega$ such that for every $f : [N]^k \rightarrow m$ there exists $H \in [N]^h$ such that $f \upharpoonright [H]^k$ is constant.

proof:

Suppose there is no such N and let $f_N : [N]^k \rightarrow m$ be a counterexample for each N . Define

$$g : [\omega]^{k+1} \rightarrow m \text{ by } g(a_1, a_2, \dots, a_k, b) = f_b(a_1, a_2, \dots, a_k)$$

where $a_1 < a_2 < \dots < a_k < b$. By Ramsey's Theorem, there exists $H \in [\omega]^\omega$ such that $g \upharpoonright [H]^{k+1}$ is constant. But if $a_1 < a_2 < \dots < a_h < b$ are any $h + 1$ elements of H then $\{a_1, \dots, a_h\}$ is a homogeneous set for f_b , a contradiction.

There is another proof which works by invoking the compactness theorem.

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Ramsey's theorem is not a corollary of its finite version. This follows from the fact that there is a recursive partition with no recursive homogeneous set, a result due to Specker. See, for example, Simpson [11], and Jockusch [3].

2 Galvin-Prikry Theorem

Let $\mathcal{U} \subset [\omega]^\omega$ be arbitrary but fixed. In what follows lower case $s, t, \dots$ letters will refer to finite subsets of ω and upper case letters $X, Y, \dots$ to infinite subsets of ω .

Given $s \in [\omega]^{<\omega}$ and $Y \in [\omega]^\omega$ define

- $s \subseteq_{\text{end}} Y$ iff $s \subseteq Y$ and $\max(s) < \min(Y \setminus s)$,
- $[s, Y] = \{X \in [\omega]^\omega : s \subseteq_{\text{end}} X \subseteq Y \cup s\}$,
- Y accepts s iff $[s, Y] \subseteq \mathcal{U}$,
- Y rejects s iff $\neg \exists X \in [Y]^\omega$ X accepts s .

Proposition 2.1 *If Y accepts (rejects) s and $X \in [Y]^\omega$, then X accepts (rejects) s .*

Proposition 2.2 *Given any Y and s there exists $X \in [Y]^\omega$ such that either X accepts s or X rejects s .*

Proposition 2.3 *Given any Y and $s_1, s_2, \dots, s_n$ there exists $X \in [Y]^\omega$ such that for each $i = 1, \dots, n$ either X accepts s_i or X rejects s_i .*

proof:

Iterate, i.e., construct

$$Y = Y_0 \supseteq Y_1 \supseteq Y_2 \dots \supseteq Y_n = X$$

so that Y_i either accepts s_i or rejects s_i .

■

Proposition 2.4 *Given any $Y \in [\omega]^\omega$ there exists $Z \in [Y]^\omega$ such that for every $s \in [Z]^{<\omega}$ either Z rejects s or Z accepts s .*

proof:

Construct $a_0 < a_1 < \dots < a_n = \min(Y_n)$ with

$$Y = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots$$

so that for every n and $s \subseteq \{a_0, \dots, a_n\}$ we have that Y_{n+1} either rejects or accepts s . Let $Z = \{a_n : n < \omega\}$. If $s \in [Z]^{<\omega}$, then let $a_n = \max(s)$. By construction Y_{n+1} accepts or rejects s . But $[s, Z] = [s, Y_{n+1}]$ since $a_m \in Y_{n+1}$ for all $m \geq n + 1$. It follows that if Y_{n+1} accepts s , then Z accepts s ; and if Y_{n+1} rejects s , then Z rejects s .

■

Call such a Z as in Proposition 2.4 decisive.

Proposition 2.5 *Suppose Z is decisive and Z rejects the empty set. Then there exists $Y \in [Z]^\omega$ such that Y rejects s for every $s \in [Y]^{<\omega}$.*

In order to prove this proposition we have the following claim.

Claim. Suppose Z is decisive and Z rejects t for every $t \subseteq s$. Then for all but finitely many $n \in Z$, for every $t \subseteq s \cup \{n\}$ Z rejects t .

proof:

Suppose not. Then there are infinitely many $n \in Z$ such that for some $s_n \subset s \cup \{n\}$ we have that Z accepts s_n . Since Z rejects all subsets of s it must be that $s_n = t_n \cup \{n\}$ for some $t_n \subseteq s$. Since s is a finite set there must be $Y \in [Z]^\omega$ and $t \subseteq s$ such that for every $n \in Y$ we have that Z

accepts $t \cup \{n\}$ and hence $[t \cup \{n\}, Z] \subseteq \mathcal{U}$. Without loss we may assume $\max(s) < \min(Y)$. But

$$[t, Y] = \bigcup \{[t \cup \{n\}, Y] : n \in Y\}$$

and

$$[t \cup \{n\}, Y] \subseteq [t \cup \{n\}, Z] \subseteq \mathcal{U}.$$

This means that Y accepts t . But Z rejects t . Contradiction.

■

To prove the proposition construct

$$\{a_0, a_1, \dots, a_{n-1}\} \subseteq Z$$

inductively so that Z rejects every $t \subseteq \{a_0, a_1, \dots, a_{n-1}\}$. We can get started because Z rejects the empty set. Let $Y = \{a_0, a_1, \dots\}$.

■

Define

- $\mathcal{U} \subseteq [\omega]^\omega$ is Ramsey iff for every $X \in [\omega]^\omega$ there exists $Y \in [X]^\omega$ such that either $[Y]^\omega \subseteq \mathcal{U}$ or $[Y]^\omega \cap \mathcal{U} = \emptyset$,
- $\mathcal{U} \subseteq [\omega]^\omega$ is Completely Ramsey iff for every $X \in [\omega]^\omega$ and $s \in [\omega]^{<\omega}$ there exists $Y \in [X]^\omega$ such that either $[s, Y] \subseteq \mathcal{U}$ or $[s, Y] \cap \mathcal{U} = \emptyset$.

Since $[Y]^\omega = [\emptyset, Y]$ it is clear that Completely Ramsey implies Ramsey. The usual topology on $[\omega]^\omega$ is the topology it inherits by being considered as follows:

$$[\omega]^\omega \subseteq P(\omega) \equiv 2^\omega$$

A basic open set in the usual or product topology has the form, $[s, \omega]$. The Ellentuck Topology has as its basic open sets those of the form $[s, X]$ where $s \in [\omega]^{<\omega}$ and $X \in [\omega]^\omega$. To be a basis (as opposed to subbasis) it is necessary that the intersection of any two basic open sets is a union of basic open sets.

Proposition 2.6 *Suppose $[s, X]$ and $[t, Y]$ are two basic open sets. Then either they are disjoint or*

$$[s, X] \cap [t, Y] = [s \cup t, X \cap Y].$$

proof:

If neither $s \subseteq_{end} t$ or $t \subseteq_{end} s$, then they are disjoint, since then there are no Z with $s \subseteq_{end} Z$ and $t \subseteq_{end} Z$. So suppose $s \subseteq_{end} t$. Now if it is not true that $(t \setminus s) \subseteq X$, then again they are disjoint, since $t \subseteq Z$ for every $Z \in [t, Y]$. Finally if $X \cap Y$ is finite, then they are disjoint. If none of these things happen, then $s \cup t = t$, $t \setminus s \subseteq X$, and both sides can be described as the set of infinite Z such that $t \subseteq_{end} Z$ and $(Z \setminus t) \subseteq (X \cap Y)$.
 ■

Lemma 2.7 *If $\mathcal{U}$ is open in the Ellentuck Topology, then $\mathcal{U}$ is Ramsey.*

proof:

Given $X \in [\omega]^\omega$ apply the Proposition 2.4 and find $Z \in [X]^\omega$ such Z is decisive. Now if Z accepts the empty set, then $[Z]^\omega = [\emptyset, Z] \subseteq \mathcal{U}$ and we are done. If Z rejects the empty set, then apply Proposition 2.5 and obtain $Y \in [Z]^\omega$ such that Y rejects all of its finite subsets. To finish the proof it is enough to prove the following

Claim. $[Y]^\omega \cap \mathcal{U} = \emptyset$.

proof:

If not, there is some $Z \in [Y]^\omega \cap \mathcal{U}$. In the Ellentuck topology the sets of the form $[Z \cap n, Z]$ form a neighborhood basis for Z . Since $\mathcal{U}$ is open, it must be that for some n

$$Z \in [Z \cap n, Z] \subseteq \mathcal{U}.$$

But this means that Z accepts $Z \cap n$ contradicting the fact that Y rejects $Z \cap n$.
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Lemma 2.8 *If $\mathcal{U}$ is open in the Ellentuck Topology, then $\mathcal{U}$ is Completely Ramsey.*

proof:

We can either say - just do the whole proof over again but start with $[s, Y]$ instead of $[\emptyset, Y]$ or we can use the following argument.

Fix s and Y with $\max(s) < \min(Y)$. Let $h : [\omega]^\omega \rightarrow [s, Y]$ be defined as follows. Let $Y = \{y_n : n < \omega\}$ be written in increasing order. For each $X \in [\omega]^\omega$ let $h(X) = s \cup \{y_n : n \in X\}$. It is easy to check that h is a

homeomorphism in the Ellentuck topology, in fact, it takes basic open sets to basic open sets. Let $\mathcal{V} = h^{-1}(\mathcal{U})$. Then $\mathcal{V}$ is open and hence by Lemma 2.7 it is Ramsey, and so there exists $H \in [\omega]^\omega$ such that either $[H]^\omega \subseteq \mathcal{V}$ or $[H]^\omega \cap \mathcal{V} = \emptyset$. Let $Z = \{y_n : n \in H\}$. Then either $[s, Z] \subseteq \mathcal{U}$ or $[x, Z] \cap \mathcal{U} = \emptyset$.

■

Lemma 2.9 *The Completely Ramsey sets form a σ -algebra, i.e., a family of sets closed under taking compliments and taking countable unions.*

proof:

It is easy to see that the compliment of a Completely Ramsey set is Completely Ramsey. It also easy to see that the union of two Completely Ramsey sets is Completely Ramsey. So it suffices to prove that the countable union of an increasing union of Completely Ramsey sets is Completely Ramsey. Let $\mathcal{U} = \cup_{n < \omega} \mathcal{U}_n$ be an increasing union of Completely Ramsey sets. We begin by showing that $\mathcal{U}$ is Ramsey. The acceptance-rejection terminology is with respect to a fixed background set $\mathcal{V}$ so we write “modulo $\mathcal{V}$ ” to indicate which one. For any $X \in [\omega]^\omega$ there exists $Y \in [X]^\omega$ such that either

- Y accepts the empty set modulo $\mathcal{U}$, and so $[Y]^\omega \subseteq \mathcal{U}$, or
- Y rejects all of its finite subsets modulo $\mathcal{U}$.

In the first case, we are done, so we must analyze the second case. If Y rejects s modulo $\mathcal{U}$, then since $\mathcal{U}_n$ is smaller it must also reject s modulo $\mathcal{U}_n$. But $\mathcal{U}_n$ is Completely Ramsey, so there must be $Z \in [Y]^\omega$ such that

$$[s, Z] \cap \mathcal{U}_n = \emptyset.$$

Lemma 2.10 *Suppose for every finite $s \in [Y]^\omega$, and $Z \in [Y]^\omega$, there exists $W \in [Z]^\omega$ such that $[s, Z] \cap \mathcal{U}_n = \emptyset$. Then there exists $Z \in [Y]^\omega$ such that*

$$[Z]^\omega \cap (\cup_{n < \omega} \mathcal{U}_n) = \emptyset.$$

proof:

Construct $a_0 < a_1 < \dots < a_{n-1} < a_n = \min Y_n$ with

$$Y = Y_0 \supseteq Y_1 \supseteq \dots$$

Apply the hypothesis to obtain $Y_{n+1} \in [Y_n]^\omega$ with the property that $a_n < \min(Y_{n+1})$ and so that for every $s \subseteq \{a_0, a_1, \dots, a_n\}$

$$[s, Y_{n+1}] \cap \mathcal{U}_n = \emptyset.$$

Now let $Z = \{a_n : n \in \omega\}$. We claim that: $[Z]^\omega \cap (\cup_{n < \omega} \mathcal{U}_n) = \emptyset$. If not, there exists $W \in [Z]^\omega \cap \mathcal{U}_n$ for some n . Since the $\mathcal{U}_n$ are increasing, we may (by choosing n larger, $\mathcal{U}_n$ is an increasing sequence) assume that $a_n \in W$. But then letting $s = W \cap (a_n + 1)$

$$W \in [s, W] \subseteq [s, Y_{n+1}]$$

contradicting the fact that

$$[s, Y_{n+1}] \cap \mathcal{U}_n = \emptyset.$$

■

This proves that the countable union of Completely Ramsey sets is Ramsey. The proof that it is Completely Ramsey is the same but done by relativizing the entire argument to a fixed $[s, Y]$.

■

Corollary 2.11 (*Galvin-Prikry [2]*) *The Borel subsets of $[\omega]^\omega$ are Ramsey, i.e., for any Borel set $B \subseteq [\omega]^\omega$ there exists an $H \in [\omega]^\omega$ such that either $[H]^\omega \subseteq B$ or $[H]^\omega \cap B = \emptyset$. In fact, the Borel subsets of $[\omega]^\omega$ in the Ellentuck topology are Completely Ramsey.*

Ramsey's Theorem is also a corollary of the Galvin-Prikry Theorem. Given $f : [\omega]^n \rightarrow 2$ define $B \subseteq [\omega]^\omega$ by

$$B = \{X \in [\omega]^\omega : f(\{x_1, \dots, x_n\}) = 0\}$$

where $\{x_1, \dots, x_n\}$ is the first n elements X . Then B is Borel (in fact clopen) and a homogeneous set for it, is homogeneous for f .

3 Rosenthal's Theorem

A sequence of subsets of a set X $\langle A_n : n \in \omega \rangle$ converges iff for any $x \in X$ we have that $x \in A_n$ for all but finitely many n or $x \notin A_n$ for all but finitely

many n . This is the same as saying that the characteristic functions are pointwise convergent.

A sequence of sets, $\langle A_n : n \in \omega \rangle$, is independent iff given any two disjoint finite subsets of ω , s and t the set

$$\left(\bigcap_{n \in s} A_n\right) \cap \left(\bigcap_{n \in t} \sim A_n\right)$$

is nonempty. ($\sim A_n$ is the compliment of A_n in the set X .)

Theorem 3.1 (Rosenthal [9]) *Given $\langle A_n : n \in \omega \rangle$, sequence of subsets of a set X , there exists a $C \in [\omega]^\omega$ such that either $\langle A_n : n \in C \rangle$ is convergent or $\langle A_n : n \in C \rangle$ is independent.*

proof:

This proof was found by Farahat (see also Lindenstrauss and Tzafriri [5] page 100.) Define $\mathcal{Q} \subseteq [\omega]^\omega$ by $Y \in \mathcal{Q}$ iff $Y = \{y_0 < y_1 < \dots\}$ and for every $n \in \omega$

$$A_0 \cap \sim A_1 \cap A_2 \cap \sim A_3 \cap \dots \cap \sim A_{2n-1} \cap A_{2n} \neq \emptyset.$$

That is, we take the compliment of every other one. Then $\mathcal{Q}$ is a closed set. Therefore, by the Galvin-Prikry Theorem, there exists an infinite $H \subseteq \omega$ such that either

1. $[H]^\omega \subseteq \mathcal{Q}$ or
2. $[H]^\omega \cap \mathcal{Q} = \emptyset$.

In the second case take $C = H$. Then it must be that $\langle A_n : n \in H \rangle$ is convergent, otherwise we could find $x \in X$ and an infinite subsequence of H , say $K = \{k_0 < k_1 < \dots\}$, with $x \in A_{k_{2n}}$ and $x \notin A_{k_{2n+1}}$ for each n , but then $K \in \mathcal{Q}$, contradiction.

In the first case, $[H]^\omega \subseteq \mathcal{Q}$, let $H = \{h_n : n < \omega\}$ and take

$$C = \{h_{2n+1} : n < \omega\}.$$

Then $\langle A_n : n \in C \rangle$ is independent. This is because, given any disjoint s and t in $[C]^{<\omega}$, we can find $K \in [H]^\omega$ (by filling in on the even ones as needed) so that

$$s \subseteq \{h_{2n} : n < \omega\} \text{ and } t \subseteq \{h_{2n+1} : n < \omega\}.$$

But since $K \in \mathcal{Q}$ this implies

$$\left(\bigcap_{n \in s} A_n\right) \cap \left(\bigcap_{n \in t} \sim A_n\right) \neq \emptyset.$$

■

4 Ellentuck's Theorem

Silver [10] generalized the Galvin-Prikry Theorem (Thm 2.11) by showing the Σ_1^1 sets are Ramsey (in fact Completely Ramsey). His proof was meta-mathematical. Ellentuck [1] gave a more general and simpler proof of Silver's Theorem.

In order to state Ellentuck's result we begin by reviewing the notion of Property of Baire or Baire Property. Let X be any topological space. Define

- $N \subseteq X$ is nowhere dense iff its closure has no interior. Or equivalently for any nonempty basic open set U there exists a nonempty basic open set $V \subseteq U$ such that $V \cap N = \emptyset$.
- $M \subseteq X$ is meager iff M is the countable union of nowhere dense subsets of X . Meager is also refer to as “first category” in X .
- $G \subseteq X$ is comeager iff $X \setminus G$ is meager.
- $B \subseteq X$ has the Property of Baire iff there exists an open set U and a meager set M such that

$$B = U \Delta M$$

where $U \Delta M = (U \setminus M) \cup (M \setminus U)$ is the symmetric difference. It equivalent to say $B \Delta U = M$.

Note that a set is nowhere dense iff its closure is nowhere dense. Thus any meager set can be covered by a meager F_σ , i.e., a countable union of closed nowhere dense sets. Also any subset of a nowhere dense set is nowhere dense, and hence the meager subsets of X form a σ -ideal. One terminology for “sets with the Baire Property” is to refer to them as “sets which are almost open”. Another equivalent definition is the following: a set $B \subseteq X$ has the Baire property iff there exists a comeager set G and an open set U such that $B \cap G = U \cap G$. Thus when we restrict B to a comeager subset, it is open.

Theorem 4.1 (*Baire*) *The sets with Property of Baire in X form a σ -algebra, i.e., if B has the Property of Baire then so does $X \setminus B$; and if $\langle B_n : n \in \omega \rangle$ each have the Baire property, then so does $\cup_{n < \omega} B_n$.*

proof:

If U is any open set let V be the interior of $X \setminus U$ and note that $X \setminus (U \cup V)$ is nowhere dense. This is true because for any nonempty open W either W meets U (and so $W \cap U \subseteq W$ is a nonempty open set missing $X \setminus (U \cup V)$) or W is disjoint from U and hence contained in V , the interior of the closed set $X \setminus U$, (and so W already misses $X \setminus (U \cup V)$).

Suppose $B \cap G = U \cap G$ where U is open and G is comeager. Then

$$(X \setminus B) \cap G' = V \cap G'$$

where V is the interior of $X \setminus U$ and $G' = G \cap (U \cup V)$ is comeager.

If $B_n \cap G_n = U_n \cap G_n$ where each G_n is comeager and U_n open, then

$$(\cup_{n < \omega} B_n) \cap (\cap_{n < \omega} G_n) = (\cup_{n < \omega} U_n) \cap (\cap_{n < \omega} G_n).$$

■

Theorem 4.2 (*Ellentuck [1]*) *A set $B \subseteq [\omega]^\omega$ is Completely Ramsey iff it has the Baire Property in the Ellentuck topology.*

proof:

This will follow easily from the following lemma:

Lemma 4.3 *Any meager set in the Ellentuck topology is nowhere dense.*

Before proving the Lemma let us deduce from it, Theorem 4.2. Suppose that $\mathcal{B} \subseteq [\omega]^\omega$ is Completely Ramsey. Let

$$\mathcal{U} = \cup\{[s, X] : [s, X] \subseteq \mathcal{B}\},$$

i.e., $\mathcal{U}$ is the interior of $\mathcal{B}$ in the Ellentuck topology. To see, that $\mathcal{B}$ has the property of Baire, it is enough to show that $\mathcal{B} \setminus \mathcal{U}$ is nowhere dense. Since $\mathcal{U}$ is open it is Completely Ramsey, hence $\mathcal{B} \setminus \mathcal{U}$ is Completely Ramsey. Given any $[s, A]$ there exists $B \in [A]^\omega$ such that either

$$[s, B] \subseteq (\mathcal{B} \setminus \mathcal{U}) \text{ or } [s, B] \cap (\mathcal{B} \setminus \mathcal{U}) = \emptyset.$$

But the first cannot happen by the definition of $\mathcal{U}$, so the second, $[s, B] \cap (\mathcal{B} \setminus \mathcal{U}) = \emptyset$, must happen. And since $[s, A]$ is an arbitrary basic open set, it follows that $\mathcal{B} \setminus \mathcal{U}$ is nowhere dense.

Conversely, suppose $\mathcal{B}$ has the property of Baire in the Ellentuck topology. Then there exists an open set $\mathcal{U}$ and meager set $\mathcal{M}$ such that $\mathcal{B} \Delta \mathcal{U} \subseteq \mathcal{M}$. By Lemma 4.3 There is a closed nowhere dense $\mathcal{C}$ with $\mathcal{M} \subseteq \mathcal{C}$. Now since open sets are Completely Ramsey, for any $[s, X]$ there exists $Y \in [X]^\omega$ such that $[s, Y] \subseteq \mathcal{U}$ or $[s, Y] \cap \mathcal{U} = \emptyset$. Since closed sets are Completely Ramsey, there exists $Z \in [Y]^\omega$ such that either $[s, Z] \subseteq \mathcal{C}$ or $[s, Z] \cap \mathcal{C} = \emptyset$. But the first doesn't happen, because $\mathcal{C}$ is nowhere dense. It follows that $[s, Z] \subseteq \mathcal{B}$ or $[s, Z] \cap \mathcal{B} = \emptyset$ (according to what was true for $\mathcal{U}$). And so $\mathcal{B}$ is Completely Ramsey.

Proof of Lemma 4.3. This proof is somewhat analogous to the proof that the countable union of Completely Ramsey sets is Completely Ramsey. Suppose that $\mathcal{Q}_n$ for $n \in \omega$ are nowhere dense in the Ellentuck topology. Since their closures are also nowhere dense, and the finite union of nowhere dense set is nowhere dense, we may assume without loss that they are closed and increasing. Since they are closed each $\mathcal{Q}_n$ is completely Ramsey. Now given any $A \in [\omega]^\omega$ we can construct a sequence

$$a_0 < a_1 < \dots < a_{n-1} < \min(A_n)$$

as follows. Let $A_0 = A$. Given

$$a_0 < a_1 < \dots < a_n = \min(A_n)$$

find $A_{n+1} \subseteq A_n$ with the property that for every $s \subseteq \{a_0, \dots, a_n\}$

$$[s, A_{n+1}] \cap \mathcal{Q}_n = \emptyset.$$

Now consider $B = \{a_n : n < \omega\}$. We claim that

$$[B]^\omega \cap \bigcup_{n < \omega} \mathcal{Q}_n = \emptyset.$$

If not, there exists C and n with $a_n \in C$ and $C \in \mathcal{Q}_n$. But if

$$s = C \cap \{a_0, \dots, a_n\}$$

then $C \in [s, A_{n+1}]$ contradicting the fact that

$$[s, A_{n+1}] \cap \mathcal{Q}_n = \emptyset.$$

By relativizing this argument to any $[s, A]$ it follows that $\bigcup_{n < \omega} \mathcal{Q}_n$ is nowhere dense and the lemma is proved.

■

5 Souslin Operation

The Souslin operation is the following. Given $\{A_s : s \in \omega^{<\omega}\}$ define

$$S\langle A_s : s \in \omega^{<\omega} \rangle = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}.$$

This is the Souslin operation also called operation A. Given a family of sets $\mathcal{A}$ define

$$\mathcal{S}(\mathcal{A}) = \{S\langle A_s : s \in \omega^{<\omega} \rangle : \{A_s : s \in \omega^{<\omega}\} \subseteq \mathcal{A}\}.$$

Typically $\mathcal{A}$ is the family of all closed subsets of a topological space X and then $\mathcal{S}(\mathcal{A})$ is known as the family of Souslin sets.

Theorem 5.1 *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an arbitrary family of sets with $X \in \mathcal{A}$. Then:*

1. $\mathcal{A} \subseteq \mathcal{S}(\mathcal{A})$,
2. $\mathcal{S}(\mathcal{A})$ is closed under countable unions, i.e., $X_n \in \mathcal{S}(\mathcal{A})$ for each n implies $\bigcup_{n < \omega} X_n \in \mathcal{S}(\mathcal{A})$,
3. $\mathcal{S}(\mathcal{A})$ is closed under countable intersections, and
4. $\mathcal{S}(\mathcal{S}(\mathcal{A})) = \mathcal{S}(\mathcal{A})$.

proof:

See [8] for example.

The first item is obvious, since if $A_s = A$ for all s then

$$A = S\langle A_s : s \in \omega^{<\omega} \rangle.$$

For countable unions, note that if $X_k = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}^k$, then

$$\bigcup_{k < \omega} X_k = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} B_{f \upharpoonright n}$$

where $B_s = A_{\langle s(1), \dots, s(n) \rangle}^{s(0)}$ for each $s \in \omega^{n+1}$. Put $B_{\langle \rangle} = X$.

For countable intersections, a coding argument is required. Let $\langle n, m \rangle \in \omega$ be a bijective map between ω^2 and ω satisfying, $i < j$ implies $\langle k, i \rangle < \langle k, j \rangle$. For example, $\langle k, i \rangle = 2^k(2i + 1) - 1$. Given $X_k = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}^k$ then

$$\bigcap_{k < \omega} X_k = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} B_{f \upharpoonright n}$$

where

$$B_s = A_t^k$$

where t and k are determined by s as follows. Let $s = (s(0), \dots, s(n))$, then $n = \langle k, m \rangle$ and $t = (t(0), \dots, t(m))$ where $t(i) = s(\langle k, i \rangle)$ for each $i \leq m$. This encoding is based on:

$$\bigcap_{k < \omega} \bigcup_{f \in \omega^\omega} \bigcap_{m < \omega} A_{f \upharpoonright n}^k = \bigcup_{\langle f_k \in \omega^\omega : k \in \omega \rangle} \bigcap_{m, k < \omega} A_{f \upharpoonright m}^k = \bigcup_{h \in \omega^\omega} \bigcap_{n < \omega} B_{h \upharpoonright n}$$

where $f_k(m) = h(\langle k, m \rangle)$.

To see that $\mathcal{S}(\mathcal{S}(\mathcal{A})) = \mathcal{S}(\mathcal{A})$ suppose:

$$A_s = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}^s$$

and for each $s, t \in \omega^{<\omega}$ there is $A_t^s \in \mathcal{A}$ with

$$A_s = \bigcup_{g \in \omega^\omega} \bigcap_{k < \omega} A_{g \upharpoonright k}^s$$

then

$$\bigcup_f \bigcap_n \bigcup_g \bigcap_k A_{g \upharpoonright k}^{f \upharpoonright n} = \bigcup_{f, \langle g_n : n < \omega \rangle} \bigcap_{n, k} A_{g_n \upharpoonright k}^{f \upharpoonright n}$$

To define B_s suppose $s \in \omega^N$, $N = \langle n, k \rangle$, $s_0, s_1 \in \omega^N$ are defined by $s(i) = \langle s_0(i), s_1(i) \rangle$, and $r = s_0 \upharpoonright n$ and $t \in \omega^k$ is defined by $t(i) = s_1(\langle n, i \rangle)$. Put $B_s = A_t^r$. Then

$$\bigcup_{f, \langle g_n : n < \omega \rangle} \bigcap_{n, k} A_{g_n \upharpoonright k}^{f \upharpoonright n} = \bigcup_h \bigcap_N B_{h \upharpoonright N}.$$

■

Another way to define the Souslin sets is in terms of the projection operation.

For $B \subseteq U \times X$ let

$$\text{proj}_X(B) = \{x \in X : \exists u \in U \ (u, x) \in B\}$$

For $\mathcal{B}$ a family of subsets of $U \times X$ let

$$\text{proj}_X(\mathcal{B}) = \{\text{proj}_X(B) : B \in \mathcal{B}\}.$$

For X a topological space let $\text{cl}(X)$ be the family of closed subsets of X .

Theorem 5.2 *For any topological space X*

$$\text{proj}_X(\text{cl}(\omega^\omega \times X)) = \mathcal{S}(\text{cl}(X))$$

proof:

If $B = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} C_{f \upharpoonright n}$ where each C_s is closed in X , then let

$$C = \{(f, x) : \forall n < \omega \ x \in C_{f \upharpoonright n}\} \subseteq \omega^\omega \times X.$$

Then C is closed and $B = \text{proj}_X(C)$.

Suppose $B = \text{proj}_X(C)$ where $C \subseteq \omega^\omega \times X$ is closed. For each $s \in \omega^{<\omega}$ define

$$A_s = \text{closure}(\{x \in X : \exists f \supseteq s \ (f, x) \in C\}).$$

We claim that

$$B = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}.$$

If $x \in B = \text{proj}_X(C)$, then for some $f \in \omega^\omega$ we have that $(f, x) \in C$ and therefore $x \in A_{f \upharpoonright n}$ for each $n \in \omega$. Contrariwise, if $x \in \bigcap_{n < \omega} A_{f \upharpoonright n}$ then for some $g \in \omega^\omega$ for each $m < \omega$

$$x \in \text{closure}(\{x \in X : \exists f \supseteq g \upharpoonright m \ (f, x) \in C\}).$$

But this implies that $(g, x) \in C$ since C is closed and therefore $(g, x) \notin C$ would imply that there exists an m and open $U \subseteq X$ with

$$(g, x) \in [g \upharpoonright m] \times U$$

and $[g \upharpoonright m] \times U$ disjoint from C , contradiction.

■

The Borel subsets of X , $\text{Borel}(X)$, is the smallest σ -algebra containing the open subsets of X .

Theorem 5.3 *Suppose X is a topological space and every closed subset of X is a countable intersection of open sets. (More generally, assume every open set is in $\mathcal{S}(\text{cl}(X))$). Then the following classes are all the same:*

1. $\mathcal{S}(\text{cl}(X))$
2. $\mathcal{S}(\text{Borel}(X))$

$$3. \text{proj}_X(\text{cl}(\omega^\omega \times X))$$

$$4. \text{proj}_X(\text{Borel}(\omega^\omega \times X))$$

proof:

Obviously (1) implies (2), (3) implies (4), and we already know (1) and (3) are equivalent. The fact that (2) implies (4) is proved similarly to Theorem 5.2.

So we only need to see that (4) implies (3): Let $Y = \omega^\omega \times X$. Then $\text{Borel}(Y) \subseteq \mathcal{S}(\text{cl}(Y))$. This is true because $\mathcal{S}(\text{cl}(Y))$ is closed under countable union and countable intersections and contains the closed sets. So it is enough to see that every open subset of Y is in $\mathcal{S}(\text{cl}(Y))$. But (because ω^ω has countable base) every open subset of Y is a countable union of open rectangles, i.e., set of the form $U \times V$ where $U \subseteq \omega^\omega$ is open and $V \subseteq X$ is open. It follows that

$$\begin{aligned} \text{proj}_X(\text{Borel}(\omega^\omega \times X)) &\subseteq \text{proj}_X(\text{proj}_{\omega^\omega \times X}(\text{cl}(\omega^\omega \times \omega^\omega \times X))) \\ &= \text{proj}_X(\text{cl}(\omega^\omega \times \omega^\omega \times X)) = \text{proj}_X(\text{cl}(\omega^\omega \times X)). \end{aligned}$$

■

This class of sets also includes $\text{proj}_X(\text{Borel}(Z \times X))$ for all sufficiently nice Z . For more on projective sets see Miller [6].

Marczewski proved a general theorem which gives a sufficient condition under which a σ -field is closed under the Souslin operation. A σ -field $\mathcal{B}$ of subsets of a set X is a family of sets closed under complementation and countable unions

A σ -ideal $\mathcal{I}$ in $\mathcal{B}$ is a subfamily of $\mathcal{B}$ which is closed under countable unions and taking subsets, i.e., if $Z \in \mathcal{I}$ and $W \subseteq Z$ then $W \in \mathcal{I}$.

Theorem 5.4 (Marczewski, See Kuratowski [4]) *Suppose the σ -field $\mathcal{B}$ on the set X and a σ -ideal $\mathcal{I}$ in $\mathcal{B}$ satisfy the following minimal covering property:*

For every $Y \subseteq X$ there exists $B \in \mathcal{B}$ such that $Y \subseteq B$ and for every $C \in \mathcal{B}$ if $Y \subseteq C \subseteq B$, then $B \setminus C \in \mathcal{I}$.

Then $\mathcal{S}(\mathcal{B}) = \mathcal{B}$, i.e., $\mathcal{B}$ is closed under the Souslin operation.

proof:

Suppose $A_s \in \mathcal{B}$ and

$$A = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}.$$

For each $s \in \omega^{<\omega}$ define

$$A^s = \bigcup_{s \subseteq f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}.$$

Note

1. $A = A^\langle \rangle$ where $\langle \rangle$ is the empty sequence,
2. $A^s \subseteq A_s$,
3. $A^s = \bigcup_{n < \omega} A^{s \hat{\ } n}$

Apply the minimal cover assumption to all A^s , to get $B_s \supseteq A^s$ with $B_s \in \mathcal{B}$ a minimal cover. Using (2) we may as well assume $B_s \subseteq A_s$. By using (3) we may replace $B_{s \hat{\ } n}$ with $B_{s \hat{\ } n} \cap B_s$, so without loss, we may assume $B_{s \hat{\ } n} \subseteq B_s$ for each s and n . Now note that since

$$A^s = \bigcup_{n < \omega} A^{s \hat{\ } n} \subseteq \bigcup_{n < \omega} B_{s \hat{\ } n} \subseteq B_s$$

we have that by the minimal cover property that

$$(B_s \setminus \bigcup_{n < \omega} B_{s \hat{\ } n}) \in \mathcal{I}.$$

To finish the proof, since $A = A^\langle \rangle$, $A^\langle \rangle \subseteq B_\langle \rangle$ and $B_\langle \rangle \in \mathcal{B}$, it is enough to show $B_\langle \rangle \setminus A^\langle \rangle \in \mathcal{I}$.

$$B_\langle \rangle \setminus A^\langle \rangle = B_\langle \rangle \setminus \left(\bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n} \right) \subseteq B_\langle \rangle \setminus \left(\bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} B_{f \upharpoonright n} \right)$$

This follows from the assumption that $B_s \subseteq A_s$ so we are subtracting off a smaller set.

$$B_\langle \rangle \setminus \left(\bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} B_{f \upharpoonright n} \right) \subseteq \bigcup_{s \in \omega^{<\omega}} (B_s \setminus \bigcap_{n < \omega} B_{s \hat{\ } n}).$$

Since the last set is in $\mathcal{I}$ we are done. This inclusion is true because if x not in $\bigcup_{s \in \omega^{<\omega}} (B_s \setminus \bigcap_{n < \omega} B_{s \smallfrown n})$ then whenever $x \in B_s$ there is an $n < \omega$ such that $x \in B_{s \smallfrown n}$. Hence if $x \in B_{\langle \rangle}$ but not in $\bigcup_{s \in \omega^{<\omega}} (B_s \setminus \bigcap_{n < \omega} B_{s \smallfrown n})$ we can construct $f \in \omega^\omega$ such that x is in $\bigcap_{n < \omega} B_{f \upharpoonright n}$.

■

The two main examples for which Marczewski's result holds are

1. $\mathcal{B}$ is the family of sets with the property of Baire in some topological space X and $\mathcal{I}$ is the ideal of meager sets, and
2. $\mathcal{I}$ is a ccc ideal in $\mathcal{B}$, i.e., there does not an uncountable disjoint family of sets in $\mathcal{B} \setminus \mathcal{I}$.

The second property holds, for example, when $\mathcal{B}$ is the family of measurable sets and $\mathcal{I}$ is the σ -ideal of measure zero sets where μ is some finite countably additive measure on X .

Theorem 5.5 *If $\mathcal{I}$ is a ccc ideal in $\mathcal{B}$, then they satisfy minimal cover property (see hypothesis of Theorem 5.4).*

proof:

Given $Y \subseteq X$ arbitrary, try to construct disjoint B_α as follows. Given $B_\alpha : \alpha < \beta$, consider if there exists $B \in (\mathcal{B} \setminus \mathcal{I})$ such that

1. $B \cap A = \emptyset$ and
2. $B \cap B_\alpha = \emptyset$ for all $\alpha < \beta$.

If there is such a B let B_β be any such. If there is no such B stop the construction. Since $\mathcal{I}$ is ccc the construction must eventually stop at some countable stage $\beta_0 < \omega_1$. Let

$$B = X \setminus (\bigcup_{\alpha < \beta_0} B_\alpha).$$

Then B is a minimal cover of A , because if $A \subseteq C \subseteq B$ and $C \in \mathcal{B}$, then if $B \setminus C \notin \mathcal{I}$ it would be a candidate for B_{β_0} which however never got defined.

■

The obvious generalization of the above proof is to κ -fields and κ -ideals and the κ^+ chain condition.

Theorem 5.6 *If $\mathcal{B}$ is the family of sets with the property of Baire in some topological space X and $\mathcal{I}$ the meager ideal, then they satisfy minimal cover property (as in Theorem 5.4).*

proof:

Claim. Suppose $\mathcal{U}$ is family of open sets and for every $U \in \mathcal{U}$ we have that $Y \cap U$ is meager. Then $(\cup \mathcal{U}) \cap Y$ is meager.

proof:

First assume the family $\mathcal{U}$ is pairwise disjoint. Then for each $U \in \mathcal{U}$ there would exists N_n^U nowhere dense so that

$$Y \cap U = \cup_{n < \omega} N_n^U.$$

But because the $U \in \mathcal{U}$ are disjoint it is easy to check that each

$$N_n = \cup_{U \in \mathcal{U}} N_n^U$$

is nowhere dense.

Given an arbitrary family of open sets $\mathcal{U}$ let $\mathcal{V}$ be a maximal family of pairwise disjoint open sets which refines $\mathcal{U}$, i.e., for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ with $V \subseteq U$. So we know by the first case that $(\cup \mathcal{V}) \cap Y$ is meager. But the maximality assumption implies that $\cup \mathcal{U} \setminus \cup \mathcal{V}$ is nowhere dense, hence

$$(\cup \mathcal{U}) \cap Y \subseteq (\cup \mathcal{U} \setminus \cup \mathcal{V}) \cup ((\cup \mathcal{V}) \cap Y)$$

is meager. This proves the Claim.

■

Now to prove the theorem, let $Y \subseteq X$ be arbitrary and put

$$\mathcal{U} = \{U : U \text{ open and } U \cap Y \text{ meager} \}.$$

Let

$$B = (X \setminus (\cup \mathcal{U})) \cup ((\cup \mathcal{U}) \cap Y).$$

Then B has the property of Baire since it is the union of a closed set and a meager set. Also $Y \subseteq B$. If $Y \subseteq C \subseteq B$ and C has the Baire property, then so does $B \setminus C$. So let

$$B \setminus C = (U \setminus M_1) \cup M_2$$

where U is open and M_1, M_2 are meager. Since $B \setminus C$ is disjoint from Y , $U \cap Y \subseteq M_1$. Hence $U \in \mathcal{U}$. It follows that

$$U \cap B \subseteq (\mathcal{U}) \cap Y$$

and therefore that U is meager and so $B \setminus C$ is meager.

■

This result would follow trivially from the ccc case if X had a countable base, but in the case we are interested in, the Ellentuck topology, the space is not second countable.

Corollary 5.7 (*Silver [10]*) *Every Σ_1^1 set is Ramsey.*

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