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Borel hierarchies

Theorem 1 *(Lebesgue 1905 see [14] Thm 2.5) For every* α *with* $1 \leq \alpha < \omega_1$

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(2^\omega) \neq \underset{\sim}{\mathbf{\Pi}}^0_\alpha(2^\omega).$$

Theorem 2 *(Bing, Bledsoe, Mauldin see [14] Thm 3.2) Suppose* $F \subseteq P(2^\omega)$ *is a countable family such that* $\text{Borel}(2^\omega) \subseteq \text{Borel}(F)$*. Then* $\text{ord}(F) = \omega_1$*.*

Theorem 3 *(Recław see see [14] Thm 3.5 [13] Thm 17) If* X *is a second countable space and* X *can be mapped continuously onto the unit interval,* $[0,1]$*, then* $\text{ord}(X) = \omega_1$*.*

Define the levels of the ω_1-Borel hierarchy of subsets of 2^ω as follows:

1. $\mathbf{\Sigma}^*_0 = \mathbf{\Pi}^*_0 =$ clopen subsets of 2^ω
2. $\mathbf{\Sigma}^*_\alpha = \{\bigcup_{\beta<\omega_1} A_\beta \ : \ (A_\beta : \beta < \omega_1) \in (\mathbf{\Pi}^*_{<\alpha})^{\omega_1}\}$
3. $\mathbf{\Pi}^*_\alpha = \{2^\omega \setminus A \ : \ A \in \mathbf{\Sigma}^*_\alpha\}$
4. $\mathbf{\Pi}^*_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta \qquad \mathbf{\Sigma}^*_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Sigma}^*_\beta$

The length of this hierarchy is the smallest $\alpha \geq 1$ such that

$$\mathbf{\Pi}^*_\alpha = \mathbf{\Sigma}^*_\alpha.$$

Theorem 4 *(Miller [16]) (*MA_{ω_1}*)* $\mathbf{\Pi}^*_\alpha \neq \mathbf{\Sigma}^*_\alpha$ *for every* $\alpha < \omega_2$*.*

Theorem 5 *(Miller [9] Thm 22) It is relatively consistent with ZFC that* $ord(X) = \omega_1$ *for all uncountable* $X \subseteq 2^\omega$*.*

Theorem 6 *(Rao [20], Kunen [8]) Assume the continuum hypothesis. Then every subset of the plane is in the* σ*-algebra generated by the abstract rectangles. In fact, at level two.*

Theorem 7 *(Kunen [8]) Assume Martin's axiom, then every subset of the plane is in the σ-algebra generated by the abstract rectangles at level two. In the Cohen real model or the random real model the well-ordering on the continuum is not in the σ-algebra generated by the abstract rectangles.*

Theorem 8 *(Rothberger [21] 1952) Suppose that $2^\omega = \omega_2$ and $2^{\omega_1} = \omega_{\omega_2}$ then the σ-algebra generated by the abstract rectangles in the plane is not the power set of the plane.*

Theorem 9 *(Bing, Bledsoe, Mauldin [2]) If every subset of the plane is in the σ-algebra generated by the abstract rectangles, then for some countable α every subset of the plane is in the σ-algebra generated by the abstract rectangles by level α.*

Theorem 10 *(Miller [9]) If every subset of a separable metric space X is Borel in X, then for some countable α every subset of X is Σ^0_α in X.*

Theorem 11 *([9]) For any countable α it is consistent to have a separable metric space X in which every subsets is Borel and the order of X is α.*

Theorem 12 *([9]) For any countable $\alpha \geq 2$ it is consistent that every subset of the plane is in the σ-algebra generated by the abstract rectangles at level α but for every $\beta < \alpha$ not every subset is at level β.*

Theorem 13 *(Larson, Miller, Steprans, Weiss [18]) Suppose $2^{<\mathfrak{c}} = \mathfrak{c}$ then the following are equivalent:*

(1) There is a Borel universal function, i.e, a Borel function $F : 2^\omega \times 2^\omega \to 2^\omega$ such that for every abstract $G :: 2^\omega \times 2^\omega \to 2^\omega$ there are $h : 2^\omega \to 2^\omega$ and $k : 2^\omega \to 2^\omega$ such that for every $x, y \in 2^\omega \quad G(x, y) = F(h(x), k(y))$.

(2) Every subset of the plane is in the σ-algebra generated by the abstract rectangles.

Furthermore the universal function has level α iff every subset of the plane is in the σ-algebra generated by the abstract rectangles at level α.

Theorem 14 *([18]) If $2^{<\kappa} = \kappa$, then there is an abstract universal function $F : \kappa \times \kappa \to \kappa$.*

Theorem 15 *([18]) It is relatively consistent with ZFC, that there is no abstract universal function $F : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$.*

Theorem 16 *([18]) There does not exist a Borel function $F : 2^\omega \times 2^\omega \to 2^\omega$ such that for every Borel $G : 2^\omega \times 2^\omega \to 2^\omega$ there are $h, k : 2^\omega \to 2^\omega$ such that k is Borel and for every $x, y \in 2^\omega$*

$$G(x, y) = F(h(x), k(y))$$

Theorem 17 *(Miller [19]) The answer to question 4.6 [18] is no. If we drop the condition that k above is Borel in Theorem 16 it is consistent that there be no such F Borel.*

Theorem 18 *([18]) Universal functions F of higher dimensions reduce to countably many cases where the only thing that matters is the arity of the parameter functions, e.g.*

$$G(x, y, z) = F(h(x), k(y), l(z))$$

$$G(x, y, z) = F(h(x, y), k(y, z), l(x, z))$$

$$G(x_1, x_2, x_3, x_4) = F(h(x_2, x_3, x_4), k(x_1, x_3, x_4), l(x_1, x_2, x_4), i(x_1, x_2, x_3))$$

Theorem 19 *([18]) In the Cohen real model for every $n \geq 1$ there is a universal function on ω_n where the parameter functions have arity $n + 1$ but no universal function where the parameters functions have arity n.*

Theorem 20 *(Miller [19]) The answer to question 7.13 [18] is yes. If there is a Borel Sierpinski universal and $2^{<\mathfrak{c}} = \mathfrak{c}$, then there is a Borel map H such that for every cardinal $\kappa < \mathfrak{c}$ for every $G : \kappa \times \kappa \to \kappa$ there are $x_\alpha \in 2^\omega$ for $\alpha < \kappa$ such that for $\alpha, \beta, \gamma < \kappa$*

$$G(\alpha, \beta) = \gamma \text{ iff } H(x_\alpha, x_\beta) = x_\gamma$$

Theorem 21 *(Galvin, Mycielski, Solovay [5]) (a) X is countable iff Black has a winning strategy in G(X). (b) X does not have strong measure zero iff White has a winning strategy in the game $G(X)$.*

Theorem 22 *([5]) X has strong measure zero iff for any dense G_δ set D there exists z such that $z + X \subseteq D$.*

Theorem 23 *(Cancino, Guzmán, Miller [3]) $\mathfrak{d} \leq \mathfrak{s}_{mm}$. Any irredundant family $\mathcal{A} \subseteq [\omega]^\omega$ with $|\mathcal{A}| < \mathfrak{d}$ is not maximal. Irredundant means no element of $\mathcal{A}$ is covered mod finite by a finite union of other elements of $\mathcal{A}$.*

Theorem 24 *([3]) It is consistent with the continuum arbitrarily large to have maximal irredundant families of size* ω_1*.*

Theorem 25 *(Marczewski see Miller[15]) If* I *is a ccc* σ*-ideal in the Borel sets then the family of* I*-measurable sets is closed under the Souslin operation.*

Theorem 26 *([14]) (CH) For any* α *with* $2 \leq \alpha \leq \omega_1$ *there is exists an uncountable* $X \subseteq 2^\omega$ *such that* $\mathrm{ord}(X) = \alpha$ *and every Souslin set in* X *is Borel in* X*.*

Theorem 27 *(Miller [10]) It is consistent to have* $X \subseteq 2^\omega$ *such that every subset of* X *is Souslin in* X *and the Borel order of* X *is* ω_1*.*

Theorem 28 *([10]) It is relatively consistent with ZFC that for every subset* $A \subseteq 2^\omega \times 2^\omega$ *there are abstract rectangles* $B_s \times C_s$ *with*

$$A = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} (B_{f\upharpoonright n} \times C_{f\upharpoonright n})$$

but not every subset of $2^\omega \times 2^\omega$ *is in the* σ*-algebra generated by the abstract rectangles.*

Theorem 29 *(Sierpinski 1935) Assume CH. There Luzin sets and Sierpinski sets whose square can be continuously mapped onto* 2^ω*.*

Corollary 30 *(CH) For any* α *with* $2 \leq \alpha < \omega_1$ *there is* $X \subseteq 2^\omega$ *such that*

$$\mathrm{ord}(X) = \alpha \text{ and } \mathrm{ord}(X^2) = \omega_1$$

Theorem 31 *(Miller [12]) (CH) There is an uncountable* σ*-set* $X \subseteq 2^\omega$ *which is concentrated on a countable set. (*σ*-set means* $\mathrm{ord}(X) = 2$*.)*

Theorem 32 *(Fleissner, Miller [4]) It is relatively consistent with ZFC to have an uncountable Q-set which is concentrated on a countable set.*

Theorem 33 *(Miller [19]) (CH) For any* α_0 *with* $3 \leq \alpha_0 < \omega_1$ *there are* $X_0, X_1 \subseteq 2^\omega$ *with* $\mathrm{ord}(X_0) = \alpha_0 = \mathrm{ord}(X_1)$ *and* $\mathrm{ord}(X_0 \cup X_1) = \alpha_0 + 1$*.*

Theorem 34 *(Mostowski) If* θ *is a* Σ^0_α *formula of* $L_{\omega_1,\omega}(\rho)$*, then the set of models of* θ *is a* Σ^0_α *Borel subset of* X_ρ*. Where* X_ρ *is the Polish space of* ρ*-structures with universe* ω*.*

Theorem 35 *(Scott 1964 see Barwise [1]) For any countable structure A in a countable similarity type ρ, there is a sentence θ of $L_{\omega_1,\omega}(\rho)$ such that for any countable ρ-structure B*

$$A \simeq B \text{ iff } B \models \theta$$

Theorem 36 *(Vaught [22]) Any Σ^0_α subset of X_ρ which is closed under isomorphism is the set of models of a Σ^0_α sentence of $L_{\omega_1,\omega}(\rho)$*

Theorem 37 *(Hausdorff Difference Hierarchy) $B \in \Delta^0_{\alpha+1}$ iff there exists a countable sequence of decreasing Π^0_α sets C_β such that*

$$B = \bigcup_{\gamma \text{ even}} C_\beta \setminus C_{\beta+1}$$

Theorem 38 *(Douglas E. Miller [17]) If B is also invariant, then*

$$B = \bigcup_{\gamma \text{ even}} C^*_\beta \setminus C^*_{\beta+1}$$

Corollary 39 *If the isomorphism class of a countable structure is $\Delta^0_{\alpha+1}$ then it must be either Π^0_α, Σ^0_α, or the difference of two invariant Π^0_α sets.*

Theorem 40 *(Miller [11]) The isomorphism class of a countable model cannot be properly Σ^0_1 or properly Σ^0_2. For λ a countable limit ordinal, it cannot be properly Σ^0_λ or properly the difference of two Π^0_λ sets.*

Theorem 41 *(Miller [11], Hjorth [6]) In all other cases of there are examples of countable structures whose isomorphism class is properly of that Borel class.*

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