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ISOLATING CARDINAL INVARIANTS

JINDŘICH ZAPLETAL
Department of Mathematics, University of Florida,
Gainesville FL 32611, USA
zapletal@math.ufl.edu

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There is an optimal way of increasing certain cardinal invariants of the continuum.

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0. Introduction

The theory of cardinal invariants of the continuum is a large subfield of set theory [2]. Its subject of study is the comparison of various cardinal numbers typically defined as "the smallest size of a set of reals with certain properties". Occasionally it is possible to prove inequalities between these cardinals, but more often than not the inequalities are independent of the usual axioms of set theory. Historically, certain forcing extensions were identified as the standard tools for proving these independence results; let me name various iterations of Sacks, Cohen, Solovay or Laver real forcings as good examples. In this paper, I prove that in a certain precise sense some of these extensions are really the optimal tools for establishing a broad syntactically defined class of independence results. I will deal with the following class of invariants.

Definition 0.1. A *tame* invariant is one defined as $\min\{|A| : A \subset \mathbb{R}, \phi(A) \wedge \psi(A)\}$ where the quantifiers of $\phi(A)$ are restricted to the set A or to the natural numbers and $\psi(A)$ is a sentence of the form $\forall x \in \mathbb{R}\ \exists y \in A\ \theta(x, y)$ where θ is a formula whose quantifiers range over natural and real numbers only, without mentioning the set A. A real parameter is allowed in both formulas ϕ and ψ.

Most cardinal invariants considered today are tame. For example:

- $\mathfrak{a} = \min\{|A| : A \subset [\omega]^\omega, \phi(A) \wedge \psi(A)\}$ where $\phi(A) =$ "A is an infinite set consisting of mutually almost disjoint sets" and $\psi(A) =$ "$\forall x \in [\omega]^\omega\ \exists y \in A\ x \cap y$ is infinite".
- $\mathtt{add}(\text{meager}) = \min\{|A| : A \subset \mathbb{R}, \psi(A)\}$ where $\psi(A) =$ "$\forall x \in \mathbb{R}\ \exists y \in A$ if x codes a countable sequence of closed nowhere dense sets then y codes a closed nowhere dense set not covered by their union". In this case the sentence ϕ is not needed, that is we set $\phi =$ true.

From these examples it is clear that in a definition of a tame invariant the sentence ϕ describes the internal structure of the set A while ψ is a statement about "large size" of the set A. It is a routine matter to write invariants like $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{s}$ as well as all the invariants in the Cichon diagram [2] in a tame form. On the other hand, $\mathfrak{g}$ and $\mathfrak{h}$ apparently cannot be so written.

Theorem 0.2. *Suppose that there is a proper class of measurable Woodin cardinals. If $\mathfrak{x}$ is a tame cardinal invariant such that $\mathfrak{x} < \mathfrak{c}$ holds in some set forcing extension then $\mathfrak{x} < \mathfrak{c}$ holds in the iterated Sacks extension.*

Here, the iterated Sacks extension is obtained as usual by a countable support iteration of length $\mathfrak{c}^+$ of Sacks forcing [2]. The theorem says that this extension is the optimal tool for proving the consistency of inequalities of the type $\mathfrak{x} < \mathfrak{c}$ where $\mathfrak{x}$ is a tame cardinal invariant. There are two immediate consequences; I will state them in a rather imprecise form to retain their flavor. First, as in $P_{\max}$ [16], we get mutual consistency: if $\mathfrak{x}_i : i \in I$ are tame invariants such that $\mathfrak{x}_i < \mathfrak{c}$ is consistent for each $i \in I$ then even the conjunction of these inequalities is consistent. Restated, $\mathfrak{c}$ cannot be written as a nontrivial maximum of several tame invariants. And second, if $\mathfrak{x}$ is a tame invariant such that $\mathfrak{x} < \mathfrak{c}$ is consistent then so is $\aleph_1 = \mathfrak{x} < \mathfrak{c} = \aleph_2$.

The proof of the theorem is flexible enough to give a host of related results.

Definition 0.3. A cardinal invariant $\mathfrak{y}$ can be *isolated* if there is a forcing $P_{\mathfrak{y}}$ such that for every tame invariant $\mathfrak{x}$, if $\mathfrak{x} < \mathfrak{y}$ holds in some set forcing extension then it holds in the $P_{\mathfrak{y}}$ extension.

Thus the forcing $P_{\mathfrak{y}}$ can be understood as increasing the invariant $\mathfrak{y}$ in the gentlest way, leaving all tame invariants smaller than $\mathfrak{y}$ if possible. Hence the terminology. Theorem 0.2 says that $\mathfrak{c}$ can be isolated. I also have:

Theorem 0.4. *Suppose that there is a proper class of measurable Woodin cardinals. The following invariants can be isolated*:

- $\mathfrak{c}$; *$P_{\mathfrak{c}}$ is the iterated Sacks forcing.*
- $\mathfrak{b}$; *$P_{\mathfrak{b}}$ is the iterated Laver forcing.*
- $\mathfrak{d}$; *$P_{\mathfrak{d}}$ is the iterated Miller forcing.*
- $\mathfrak{h}$; *$P_{\mathfrak{h}}$ is the iterated Mathias forcing.*
- $\mathtt{cov}(\mathit{meager})$; *one can use either a finite support or a countable support iteration of Cohen reals.*
- $\mathtt{cov}(\mathit{null})$; *use either a large measure algebra or a countable support iteration of Solovay reals.*
- $\mathtt{non}(\mathit{strong\ measure\ zero})$; *iterate forcings known as PT_g* [1].
- $\mathtt{add}(\mathit{null})$; *the forcing does not appear in published literature.*

Amusingly enough, the proofs show that the minimum of any combination of invariants considered above can be isolated too by a countable support iteration in which the relevant forcings alternate. There are invariants which cannot

be isolated. A good example is $\mathtt{cof}$(meager ideal) since it can be written as $\max(\mathfrak{d}, \mathtt{non}(\text{meager}))$. Both of the inequalities $\mathfrak{d} < \mathtt{cof}(\text{meager})$, $\mathtt{non}(\text{meager}) < \mathtt{cof}(\text{meager})$ are consistent [2, 2.2.11, 7.6.12 and 7.5.8]. An invariant that cannot be isolated for a more complicated reason is $\mathtt{non}$(meager). As shown in [2], $\mathtt{cof}(\text{meager}) = \mathtt{cov}(I_{\text{ed}})$ where I_{ed} is the σ-ideal on ω^ω generated by the sets $A_X = \{f \in \omega^\omega : \exists g \in X\ g \cap f \text{ is infinite}\}$ as X ranges over all countable subsets of ω^ω. Now clearly $\mathtt{cov}(I_{\text{ed}}) \leq \sup\{\mathfrak{d}, \mathtt{cov}(I_{\text{ed}}(h)) : h \in \omega^\omega\}$ where $I_{\text{ed}}(h)$ is the variation of the ideal I_{ed} for the space of all functions pointwise dominated by h. However, the inequalities $\mathfrak{d} < \mathtt{cov}(I_{\text{ed}})$ as well as $\mathtt{cov}(I_{\text{ed}}(h)) < \mathtt{cov}(I_{\text{ed}})$ for every fixed function $h \in \omega^\omega$ are consistent [2, 13]. Ergo, the invariant $\mathtt{non}$(meager) cannot be isolated. This example was pointed out by Bartoszynski.

A curious twist of events occurs in the case of the tower number $\mathfrak{t}$.

Theorem 0.5. *Suppose that there is a proper class of measurable Woodin cardinals. There is a forcing $P_{\mathfrak{t}}$ such that for any tame invariant $\mathfrak{x}$, if $\aleph_1 = \mathfrak{x} < \mathfrak{t}$ holds in some forcing extension then it holds in the $P_{\mathfrak{t}}$ extension.*

Thus it may be impossible to isolate $\mathfrak{t}$ from invariants like $\mathfrak{p}$ for which $\mathfrak{p} < \mathfrak{t}$ necessitates $\aleph_1 < \mathfrak{p}$. At the same time it is possible to choose the poset $P_{\mathfrak{t}}$ to make $\mathfrak{t}$ arbitrarily large. Nothing like that occurs in the cases considered before. Also the forcing $P_{\mathfrak{t}}$ is undefinable, even though it is in some sense the expected thing.

The results stated above raise a number of obvious questions. For many invariants one would like to find out whether they can be isolated or not. If yes then what is the suitable forcing? If no, is there a clear reason? Above, I stated essentially everything I know in this direction at this point. That leaves two of the invariants in the Cichon diagram without a status. Another issue is the use of large cardinal hypotheses in the above theorems. Even though the proofs contain references to determinacy of certain integer games of transfinite length and to Σ^2_1 absoluteness, I have no indication that the hypotheses used are optimal or necessary at all.

The paper is organized as follows. The first section contains the analysis of the iterations of Sacks forcing from the descriptive set theoretic point of view. The complete proof of Theorem 0.2 can be found in Sec. 2. In Sec. 3, I indicate the changes necessary to prove that $\mathfrak{b}, \mathfrak{d}, \mathfrak{h}$ and $\mathtt{non}$(strong measure zero) can be isolated. Section 4 contains the argument for Theorem 0.5.

The paper uses two important results whose proofs remain unpublished.

Fact 0.6 ($(\Sigma^2_1$ absoluteness) (Woodin)). Suppose that there is a proper class of measurable Woodin cardinals. For every boldface Σ^2_1 sentence ϕ, if ϕ holds in some generic extension then it holds in every generic extension satisfying the continuum hypothesis.

Fact 0.7 (Transfinite projective determinacy). Suppose that there is a proper class of Woodin cardinals. Then every integer valued game of every fixed transfinite countable length with projective outcome is determined. Moreover there is a winning strategy which is weakly homogeneous in every Woodin cardinal.

The assumptions of the previous Fact are not optimal. Its proof consists of three parts. The determinacy of the games was independently established by Neeman and Woodin. By a result of Martin [8], the games must have winning strategies in a certain definability class. All sets in that definability class turn out to be weakly homogeneous as shown by Neeman and Woodin independently.

The following fairly well known fact is the only property of weakly homogeneous sets we shall need.

Fact 0.8 (Weakly homogeneous determinacy and absoluteness [15]). Suppose that δ is a supremum of Woodin cardinals with a measurable cardinal above it and $T \subset (\omega \times Ord)^{<\omega}$ is a $< \delta$-weakly homogeneous tree. Then $L(\mathbb{R})[p[T]] \models \mathrm{AD}$ and the theory of the model $L(\mathbb{R})[p[T]]$ with an arbitrary real parameter is invariant under forcing extensions of size $< \delta$.

My notation follows the set theoretic standard set forth in [4], with one exception: the concatenation of sequences $\mathbf{r}$ and $\mathbf{s}$ is denoted simply by $\mathbf{rs}$. Sequences of reals are denoted by $\mathbf{r}$, $\mathbf{s}\ldots$ For a Polish space X the expression $\mathrm{Borel}(X)$ stands for the collection of all Borel subsets of X. The spaces $\mathbb{R}^\alpha$ for a countable ordinal α are understood to come equipped with the product topology. A projective formula is one whose quantifiers range over reals and integers only, and Σ^2_1 sentences are those of the form $\exists A \subset \mathbb{R}\ \theta(A)$ where θ is projective. Projective sets are usually confused with their definitions. For a tree T the symbol $[T]$ stands for the set of all its branches and $p[T]$ for the projection of this set into a suitable Polish space. $\mathrm{AD}\mathbb{R}$ is the statement "all real games of length ω are determined". For a Woodin cardinal δ the expressions $\mathbb{P}_{<\delta}$ and $\mathbb{Q}_{<\delta}$ stand for the full nonstationary tower forcing on δ and its countably based variation respectively. The reader is referred to [2, 4, 12, 15] for all unfamiliar concepts.

1. The Sacks Forcing

The key to the proof of Theorem 0.2 is the understanding of Sacks forcing and its countable length countable support iterations in the context of determinacy. The well-known perfect set theorem can be restated to say that under $\mathrm{ZF} + \mathrm{AD}$ the Sacks forcing is (isomorphic to) a dense subset of the algebra $\mathrm{Power}(\mathbb{R})$ modulo the ideal of countable sets, ordered by inclusion. It turns out that under the stronger determinacy hypothesis of $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}\mathbb{R}$, for every countable ordinal α the countable support iteration of Sacks forcing of length α is a dense subset of the algebra $\mathrm{Power}(\mathbb{R}^\alpha)$ modulo a suitable σ-ideal I_α on $\mathbb{R}^\alpha$. This is the driving idea behind the arguments.

1.1. *The geometric reformulation of Sacks forcing iterations*

First, it is necessary to restate the definition of the countable support iteration of countable length of Sacks forcing in order to make the complexity analysis possible. A similar if not identical work was done by Kanovei in [5].

Definition 1.1. For an ordinal $\alpha \in \omega_1$ define the poset $\mathbb{S}_\alpha$ to consist of the nonempty Borel sets $p \subset \mathbb{R}^\alpha$ satisfying these three conditions:

- For every ordinal $\beta \in \alpha$ the set $p \restriction \beta = \{\mathbf{s} \in \mathbb{R}^\beta : \exists \mathbf{r} \in p\ \mathbf{s} \subset \mathbf{r}\}$ is Borel. (The projection condition; really for convenience only.)
- For every ordinal $\beta \in \alpha$ and every sequence $\mathbf{s} \in p \restriction \beta$, the set $\{t \in \mathbb{R} : \mathbf{s}\langle t\rangle \in p \restriction \beta + 1\}$ is perfect. (The Sacks condition.)
- For every increasing sequence $\beta_0 \in \beta_1 \in \cdots$ of ordinals below α and every inclusion increasing sequence of sequences $\mathbf{s}_0 \in p \restriction \beta_0$, $\mathbf{s}_1 \in p \restriction \beta_1 \ldots$, the sequence $\bigcup_n \mathbf{s}_n$ is in the set $p \restriction \bigcup_n \beta_n$. (The countable support condition.)

The sets $\mathbb{S}_\alpha$ are ordered by inclusion.

It is not hard to see that the posets $\mathbb{S}_\alpha$ are naturally isomorphic to the countable support iteration of α many Sacks reals, if $\alpha \in \omega_1$. If $G \subset \mathbb{S}_\alpha$ is a generic filter then G is given by the sequence $\mathbf{r}_{\rm gen} \in \mathbb{R}^\alpha$ with $\{\mathbf{r}_{\rm gen}\} = \bigcap\{p : p \in G\}$. This is done in the following lemma. The proof is completely unenlightening and should be skipped on the first reading of the paper. It is however useful to notice that the argument depends only on the definability and properness of Sacks forcing.

Lemma 1.2. *Suppose α is a countable ordinal.*

(1) *For every $\beta \in \alpha$ the poset $\mathbb{S}_\beta$ is naturally completely embedded in $\mathbb{S}_\alpha$ and the factor $\mathbb{S}_\alpha/\mathbb{S}_\beta$ is naturally isomorphic to $\mathbb{S}_{\alpha-\beta}$.*
(2) *$\mathbb{S}_\alpha \Vdash$ for some unique sequence $\mathbf{r}_{\rm gen} \in \mathbb{R}^\alpha$ the generic filter is just the set $\{p \in \check{\mathbb{S}}_\alpha : \mathbf{r}_{\rm gen} \in p\}$.*
(3) *If α is a limit ordinal then $\mathbb{S}_\alpha$ is the inverse limit of the posets $\{\mathbb{S}_\beta : \beta \in \alpha\}$.*
(4) *$\mathbb{S}_\alpha$ is proper.*
(5) *For every countable elementary submodel M of sufficiently large structure containing all the relevant information, for every forcing $P \in M$ adding a real $\dot{s} \in M$ and every P-name $\dot{p} \in M$ for a condition in $\mathbb{S}_\alpha$ there is a Borel relation $B \subset \mathbb{R} \times \mathbb{R}^\alpha$ so that*
   (a) *whenever $\beta \in \alpha$ then the relation $B \restriction \beta = \{(s, \mathbf{r}) \in \mathbb{R} \times \mathbb{R}^\beta : \exists \mathbf{t}\ (s, \mathbf{rt}) \in B\}$ is Borel,*
   (b) *whenever $(s, \mathbf{r}) \in B$ then s is M-generic for P, $\mathbf{r}$ is $M[s]$-generic for $\mathbb{S}_\alpha$ and $\mathbf{r} \in \dot{p}/s$,*
   (c) *for every M-generic real s for P the set $\{\mathbf{r} \in \mathbb{R}^\alpha : (s, \mathbf{r}) \in B\}$ is a condition in $\mathbb{S}_\alpha$. It follows from* (b) *that this condition strengthens $\dot{p}/s$.*

Here (5) really amounts to saying that there is a constructive method for obtaining master conditions. Note that by (2) and an absoluteness argument the condition obtained in 5(c) must be master for the model $M[s]$.

Proof. This is a completely standard simultaneous transfinite induction argument. I will show how (5) is obtained at limit stages and why (1) holds at successor stages.

The reader should refer to [2, 12] for many similar arguments. The following simple computation will be used throughout.

Claim 1.3. *Suppose M is a countable transitive model of ZFC, $P \in M$ is a partial order adding a single real $\dot s_{\rm gen}$ and $\tau \in M$ is a P-name for a real. Then*

(1) *the set $A = \{s \in \mathbb{R} :$ the equation $s = \dot s_{\rm gen}$ defines an M-generic filter$\}$ is Borel,*
(2) *the function $\tau/s : A \to \mathbb{R}$ is a Borel function.*

Proof. I will prove (1); (2) is similar. The set B of all M-generic filters on $RO(P)^M$ is Borel in the product topology on $\mathrm{Power}(RO(P)^M)$ since its elements x are subject to the Borel conditions "x is closed upwards", "x is a filter" and "x meets every open dense set in M". The function $F : \mathrm{Power}(RO(P)^M) \to \mathbb{R}$ defined by $F(x)(n) = 0$ if and only if $\|\dot s_{\rm gen}(\check n) = 0\|^M \in x$ is continuous, and one to one on the set B. Thus the set A is a one to one continuous image of a Borel set, therefore Borel. □

To see how Lemma 1.2(1) is obtained at a successor stage $\alpha = \beta + 1$, we will prove that $\mathbb{S}_\alpha \Vdash \mathbf{r}_{\rm gen}(\beta)$ is $V[\mathbf{r}_{\rm gen} \restriction \beta]$-generic Sacks real. In order to do that, suppose $p_0 \in \mathbb{S}_\alpha$ is an arbitrary condition and $(q_0, \tau) \in \mathbb{S}_\beta * \mathrm{Sacks}$ is a condition such that $q \subset p \restriction \beta$ and $q_0 \Vdash \tau$ is a perfect subset of the set $\{t \in \mathbb{R} : \mathbf{r}_{\rm gen}\langle t\rangle \in p\}$. It will be enough to produce a condition $p_1 \subset p_0$ in $\mathbb{S}_\alpha$ such that $p_1 \Vdash \mathbf{r}_{\rm gen} \restriction \beta \in q_0$ and $\mathbf{r}_{\rm gen}(\beta) \in \tau/\mathbf{r}_{\rm gen} \restriction \beta$. And indeed, if M is a countable elementary submodel of some large structure containing all relevant objects and $q_1 \subset q_0$ is a condition in $\mathbb{S}_\beta$ consisting of sequences M-generic for this poset — and such a condition exists by the induction hypothesis (5) —, then we can put $p_1 = \{\mathbf{r} \in p_0 : \mathbf{r} \restriction \beta \in q_1$ and $\mathbf{r}(\beta) \in \tau/\mathbf{r} \restriction \beta\}$ and the condition $p_1 \in \mathbb{S}_\alpha$ will be as required.

Now suppose that α is a countable limit ordinal and Lemma 1.2(1)–(5) have been verified for all ordinals $\beta \in \alpha$. To prove (5) at α fix a countable elementary submodel M of some large structure H_λ containing α and choose a forcing $P \in M$ adding a real $\dot s$ and a P-name $\dot p \in M$ for a condition in $\mathbb{S}_\alpha$. Choose an increasing sequence $\langle \alpha_n : n \in \omega\rangle$ of ordinals converging to α starting with $\alpha_0 = 0$, and an enumeration $\langle \dot D_n : n \in \omega\rangle$ of all P-names for open dense subsets of $\mathbb{S}_\alpha$ in M. By induction on $n \in \omega$ choose $P * \mathbb{S}_{\alpha_n}$-names $\dot p_n \in M$ so that $\dot p = \dot p_0$ and

- $P * \mathbb{S}_{\alpha_n} \Vdash \dot p_n \in (\mathbb{S}_\alpha)^{V^P}$, $\dot p_n \in \dot D_{n-1}$, $\mathbf{r}_{\rm gen} \in \dot p_n \restriction \check\alpha_n$,
- $P * \mathbb{S}_{\alpha_{n+1}} \Vdash$ if $\mathbf{r}_{\rm gen} \in \dot p_n \restriction \alpha_{n+1}$ then $\dot p_{n+1} \subset \dot p_n$.

So for each $n \in \omega$ $\dot q_n = \{\mathbf{t} \in \mathbb{R}^{\alpha - \alpha_n} : \mathbf{r}_{\rm gen}\mathbf{t} \in \dot p_n\}$ is an $P * \mathbb{S}_{\alpha_n}$-name for a condition in $\mathbb{S}_{\alpha - \alpha_n}$, $\dot q_n \in M$.

By the induction hypothesis there are Borel relations $B_n \subset \mathbb{R} \times \mathbb{R}^{\alpha_n} \times \mathbb{R}^{\alpha_{n+1} - \alpha_n}$ so that

- for every natural number n and all $(s, \mathbf{r}, \mathbf{t}) \in B_n$ we have that s is an M-generic real for the poset P, $\mathbf{r}$ is an $M[s]$-generic sequence for $\mathbb{S}_{\alpha_n}$ and $\mathbf{t}$ is an $M[s][\mathbf{r}]$-generic sequence for $\mathbb{S}_{\alpha_{n+1} - \alpha_n}$ such that $\mathbf{t} \in (\dot q_n/s, \mathbf{r}) \restriction [\alpha_n, \alpha_{n+1})$,

- whenever s is an M-generic real for the poset P and $\mathbf{r}$ is an $M[s]$-generic sequence for $\mathbb{S}_{\alpha_n}$ then the set $\{\mathbf{t} \in \mathbb{R}^{\alpha_{n+1}-\alpha_n} : (s, \mathbf{r}, \mathbf{t}) \in B_n\}$ is a condition in $\mathbb{S}_{\alpha_{n+1}-\alpha_n}$.

Let $B \subset \mathbb{R} \times \mathbb{R}^\alpha$ be the relation given by $(s, \mathbf{r}) \in B \leftrightarrow \forall n \in \omega\ (s, \mathbf{r} \restriction \alpha_n, \mathbf{r} \restriction [\alpha_n, \alpha_{n+1}) \in B_n$. This is obviously a Borel relation, 5(a) holds for it and 5(b) and 5(c) can be easily verified:

- If $(s, \mathbf{r}) \in B$ then for all natural numbers $n < m$ we have $\mathbf{r} \restriction \alpha_m \in (\dot{p}_n/s, \mathbf{r} \restriction \alpha_n) \restriction \alpha_m$ by the choice of the names $\dot{p}_n$, $\dot{q}_n$ and the relations B_n. By the countable support condition applied to the sets $\dot{p}_n/s$ it must be the case that $\mathbf{r} \in \dot{p}_n/s, \mathbf{r} \restriction \alpha_n$ for all $n \in \omega$, in particular $\mathbf{r} \in \dot{p}/s$ and $\mathbf{r}$ is an $M[s]$-generic sequence for $\mathbb{S}_\alpha$.
- Whenever s is an M-generic real for the poset P, we have $\{\mathbf{r} \in \mathbb{R}^\alpha : (s, \mathbf{r}) \in B\} = \{\mathbf{r} \in \mathbb{R}^\alpha : \forall n \in \omega\ (s, \mathbf{r} \restriction \alpha_n, \mathbf{r} \restriction [\alpha_n, \alpha_{n+1}) \in B_n\}$ and the latter set is easily verified to be a condition in $\mathbb{S}_\alpha$.

Thus (5) has been proved for α. □

1.2. *The dichotomy*

The following is the key dichotomy and the only new result in this section.

Lemma 1.4.

(1) ($ZF + DC + AD\mathbb{R}$) *Suppose that $\alpha \in \omega_1$ and $A \subset \mathbb{R}^\alpha$. Then either there is a condition $p \in \mathbb{S}_\alpha$ with $p \subset A$ or there is a function $g : \mathbb{R}^{<\alpha} \to [\mathbb{R}]^{\aleph_0}$ such that $\forall \mathbf{r} \in A\ \exists \beta \in \alpha\ \mathbf{r}(\beta) \in g(\mathbf{r} \restriction \beta)$.*

(2) (ZFC + *there is a proper class of Woodin cardinals*) *Suppose that $\alpha \in \omega_1$ and $A \subset \mathbb{R}^\alpha$ is a projective set. Then the same dichotomy as in* (1) *holds for the set A.*

The first item can be reworded thus: under ZF + DC + AD$\mathbb{R}$, the poset $\mathbb{S}_\alpha$ is a dense subset of the algebra Power($\mathbb{R}^\alpha$) modulo the σ-ideal I_α generated by the sets $B_g = \{\mathbf{r} \in \mathbb{R}^\alpha : \exists \beta \in \alpha\ \mathbf{r}(\beta) \in g(\mathbf{r} \restriction \beta)\}$ as g varies through all functions from $\mathbb{R}^{<\alpha}$ to $[\mathbb{R}]^{\aleph_0}$. This is a handsome way of putting things. However, the proof of (1) uses some hard unpublished theorems of Martin and Woodin and works in a choiceless environment unfamiliar to some prospective readers. Since I will need the dichotomy for projective sets only, I choose to include just the proof of (2). The assumption of (2) can be reduced to the existence of ω_1 Woodin cardinals.

Proof of Theorem 1.4(2). Let $\alpha \in \omega_1$ and $A \subset \mathbb{R}^\alpha$ be a projective set. Consider a real game of length α where players Adam and Eve play reals s_β and r_β respectively for $\beta \in \alpha$ so that the real s_β codes in some fixed way a countable set of reals and r_β is not one of them. Eve wins if the α-sequence of her answers belongs to the set A. Since real games of length α are easily simulated by integer games of length $\omega \cdot \alpha$, by the Transfinite Determinacy Fact 0.7 the game is determined and moreover

there is a weakly homogeneous winning strategy. It is therefore enough to prove the following two claims: □

Claim 1.5. *Adam has a winning strategy iff there is a function* $g : \mathbb{R}^{<\alpha} \to [\mathbb{R}]^{\aleph_0}$ *such that* $\forall \mathbf{r} \in A\ \exists \beta \in \alpha\ \mathbf{r}(\beta) \in g(\mathbf{r} \restriction \beta)$.

Claim 1.6. *Eve has a weakly homogeneous winning strategy iff there is a condition* $p \in \mathbb{S}_\alpha$ *with* $p \subset A$.

Now the first claim is a virtual triviality. The right-to-left direction of the second claim is not hard either. If $p \subset A$ for some condition $p \in \mathbb{S}_\alpha$ then Eve can defeat Adam merely making sure that at each stage $\beta \in \alpha$ the sequence $\mathbf{r}_\beta$ of answers she produced so far is in the set $p \restriction \beta$ and choosing her next answer from the perfect set $\{t \in \mathbb{R} : \mathbf{r}_\beta \langle t \rangle \in p \restriction \beta + 1\}$ minus the countable set coded by Adam's challenge s_β. With a little care the choice can be made uniformly so that the winning strategy is not only weakly homogeneous but Borel.

That leaves us with the left-to-right direction of Claim 1.6. Let σ be a weakly homogeneous winning strategy for Eve. Call a pair $\langle \mathbf{s}, \mathbf{r} \rangle$ of real sequences of length $\leq \alpha$ *good* if it represents a (partial) play of the game in which Eve follows the strategy σ. Thus there is a suitably weakly homogeneous tree T whose projection is the set of all good pairs of sequences of length $\leq \alpha$.

By transfinite induction on $\beta \leq \alpha$ prove that for every ordinal $\gamma \in \beta$ and every good pair $\langle \mathbf{s}_0, \mathbf{r}_0 \rangle \in \mathbb{R}^\gamma \times \mathbb{R}^\gamma$ there is a condition $p \in \mathbb{S}_{\beta-\gamma}$ such that for every sequence $\mathbf{r}_1 \in p$ there is $\mathbf{s}_1 \in \mathbb{R}^{\beta-\gamma}$ such that the pair $\langle \mathbf{s}_0 \mathbf{s}_1, \mathbf{r}_0 \mathbf{r}_1 \rangle \in \mathbb{R}^\beta \times \mathbb{R}^\beta$ is good. This will clearly suffice considering the case $\beta = \alpha$, $\gamma = 0$ and $\mathbf{s}_0 = \mathbf{r}_0 = 0$ and the fact that σ is a winning strategy for Eve.

Suppose first that $\beta = \beta' + 1$ is a successor ordinal and the induction hypothesis has been verified for β'. Let $\gamma \in \beta$ and let $\langle \mathbf{s}_0, \mathbf{r}_0 \rangle \in \mathbb{R}^\gamma \times \mathbb{R}^\gamma$ be a good pair. By the induction hypothesis there is a condition $p' \in \mathbb{S}_{\beta'-\gamma}$ such that for every $\mathbf{r} \in p'$ there is $\mathbf{s}$ such that the pair $\langle \mathbf{s}_0 \mathbf{s}, \mathbf{r}_0 \mathbf{r} \rangle$ is good. Now, whenever we have such a good pair then the set $X_{\mathbf{s},\mathbf{r}} = \{t \in \mathbb{R} :$ for some $u \in \mathbb{R}$ the pair $\langle \mathbf{s}_0 \mathbf{s} \langle u \rangle, \mathbf{r}_0 \mathbf{r} \langle t \rangle \rangle$ is good$\}$ must be uncountable; in the opposite case Adam would defeat the strategy σ by playing $\mathbf{s}_0, \mathbf{s}$ and then a code for the set $X_{\mathbf{s},\mathbf{r}}$. As $X_{\mathbf{s},\mathbf{r}} \in L(\mathbb{R})[p[T]]$ and $L(\mathbb{R})[p[T]] \models \mathrm{AD}$ the set $X_{\mathbf{s},\mathbf{t}}$ must have a perfect subset. By the Weakly Homogeneous Absoluteness Fact 0.8, $p' \Vdash$ there is a sequence $\mathbf{s}$ such that $\langle \mathbf{s}_0 \mathbf{s}, \mathbf{r}_0 \mathbf{r}_{\mathrm{gen}} \rangle \in p[\check{T}]$ and there is a perfect set c such that $\forall t \in c \exists u \in \mathbb{R}\ \langle \mathbf{s}_0 \mathbf{s} \langle u \rangle, \mathbf{r}_0 \mathbf{r}_{\mathrm{gen}} \langle t \rangle \rangle \in p[\check{T}]$. Pick $\mathbb{S}_{\beta'-\gamma}$ names $\mathbf{s}, \dot{c}$ for these two objects, let M be a countable elementary submodel of a large enough structure containing all the relevant information and using Lemma 1.2(5) find a condition $q \subset p'$ in $\mathbb{S}_{\beta'-\gamma}$ consisting of M-generic sequences only. Then $p = \{\mathbf{r} \in \mathbb{S}_{\beta-\gamma} : \mathbf{r} \restriction \beta' \in q \wedge \mathbf{r}(\beta') \in \dot{c}/\mathbf{r} \restriction \beta\}$ is the sought condition in the poset $\mathbb{S}_{\beta-\gamma}$.

Now suppose $\beta \leq \alpha$ is a limit ordinal and the induction hypothesis has been verified up to β. Suppose that $\gamma \in \beta$ and $\langle \mathbf{s}_0, \mathbf{r}_0 \rangle \in \mathbb{R}^\gamma \times \mathbb{R}^\gamma$ is a good pair. Let

$\gamma = \beta_{-1} \in \beta_0 \in \beta_1 \in \dots$ be an increasing ω-sequence of ordinals converging to β. By induction on $n \in \omega$ perform the following three tasks:

- Let $\mathbf{r}_{n+1}$ be the $\mathbb{S}_{\beta_n - \gamma}$-name for the part of the generic sequence of reals between β_{n-1} and β_n.
- Choose an $\mathbb{S}_{\beta_n - \gamma}$-name $\mathbf{s}_{n+1}$ for a $\beta_n - \beta_{n-1}$-sequence of reals so that $\mathbb{S}_{\beta_n - \gamma} \Vdash$ if there is a sequence $\mathbf{s}$ such that $\langle \mathbf{s}_0 \mathbf{s}_1 \dots \mathbf{s}_n \mathbf{s}, \mathbf{r}_0 \mathbf{r}_1 \dots \mathbf{r}_{n+1} \rangle \in p[\check{T}]$ then $\mathbf{s}_{n+1}$ is such a sequence.
- (Even for $n = -1$) Choose an $\mathbb{S}_{\beta_n - \gamma}$-name $\dot{p}_{n+1}$ for a condition in the forcing $\mathbb{S}_{\beta_{n+1} - \beta_n}$ such that $\mathbb{S}_{\beta_n - \gamma} \Vdash$ if $\langle \mathbf{s}_0 \mathbf{s}_1 \dots \mathbf{s}_n \mathbf{s}_{n+1}, \mathbf{r}_0 \mathbf{r}_1 \dots \mathbf{r}_{n+1} \rangle \in p[\check{T}]$ then $\dot{p}_{n+1} \subset \{\mathbf{r} \in \mathbb{R}^{\beta_{n+1} - \beta_n} : \exists \mathbf{s} \in \mathbb{R}^{\beta_{n+1} - \beta_n} \ \langle \mathbf{s}_0 \mathbf{s}_1 \dots \mathbf{s}_n \mathbf{s}_{n+1} \mathbf{s}, \mathbf{r}_0 \mathbf{r}_1 \dots \mathbf{r}_{n+1} \mathbf{r} \rangle \in p[\check{T}]\}$.

For the third item note that the induction hypothesis has been proved up to β and that by the Weakly Homogeneous Absoluteness Fact 0.8 it holds up to β even in the $\mathbb{S}_{\beta_n - \gamma}$ extension.

Now choose a countable elementary submodel M of a large enough structure containing all the relevant information and use Lemma 1.2(5) to find Borel relations $B_n \subset \mathbb{R}^{\beta_n - \gamma} \times \mathbb{R}^{\beta_{n+1} - \beta_n}$ for $n = -1, 0, 1, 2 \dots$ such that for all pairs $\langle \mathbf{t}_n, \mathbf{t}_{n+1} \rangle \in B_n$ the sequence $\mathbf{t}_n$ is M-generic for $\mathbb{S}_{\beta_n - \gamma}$ and the sequence $\mathbf{t}_{n+1} \in \dot{p}_{n+1}/\mathbf{t}_n$ is $M[\mathbf{t}_n]$-generic for $\mathbb{S}_{\beta_{n+1} - \beta_n}$, and moreover for every M-generic sequence $\mathbf{t}_n \in \mathbb{R}^{\beta_n - \gamma}$ the set of all sequences $\mathbf{t}_{n+1}$ with $\langle \mathbf{t}_n, \mathbf{t}_{n+1} \rangle \in B_n$ is a condition in the poset $\mathbb{S}_{\beta_{n+1} - \beta_n}$. Let $p = \{\mathbf{r} \in \mathbb{R}^{\beta - \gamma} :$ for every $n = -1, 0, 1, \dots$ the pair $\langle \mathbf{r} \restriction [\gamma, \beta_n), \mathbf{r} \restriction [\beta_n, \beta_{n+1}) \rangle$ is in the relation $B_n\}$. It is not difficult to verify that $p \in \mathbb{S}_{\beta - \gamma}$ is the desired condition. □

2. The Absoluteness Argument

Towards the proof of Theorem 0.2, suppose that there is a proper class of measurable Woodin cardinals, $\mathfrak{x}$ is a tame cardinal invariant, $\mathfrak{x} = \min\{|A| : A \subset \mathbb{R}, \phi(A) \wedge \psi(A)\}$ where $\phi(A)$ is a statement quantifying over the natural numbers and elements of A, and $\psi(A)$ is a sentence of the form $\forall x \in \mathbb{R} \ \exists y \in A \ \theta(x, y)$ where θ is a formula whose quantifiers range over natural and real numbers only, and suppose that $\mathfrak{x} < \mathfrak{c}$ holds in some set generic extension $V[G]$.

Move into the model $V[G]$. There must be a set $A \subset \mathbb{R}$ such that $\phi(A) \wedge \psi(A)$ holds and $|A| < \mathfrak{c}$. I will prove that the Sacks forcing and its countable support iterations preserve the properties ϕ and ψ of the set A. Certainly $\phi(A)$ is preserved because of its simple syntactical form. However the preservation of $\psi(A)$ could pose problems since some iteration $\mathbb{S}_\alpha$ could add a real $\dot{x}$ such that $\mathbb{S}_\alpha \Vdash \forall y \in \check{A} \ \neg\theta(\dot{x}, y)$.

2.1. *The countable case*

First consider the case of an arbitrary countable ordinal $\alpha \in \omega_1$. Fix a condition $p \in \mathbb{S}_\alpha$ and an $\mathbb{S}_\alpha$-name $\dot{x}$ for a real. Strengthening the condition p if necessary we may identify $\dot{x}$ with a Borel function $\dot{x} : p \to \mathbb{R}$ with the understanding

that the new real is the value of this function on the generic α-sequence of reals. I will show

$$\exists q \leq p\ \exists y \in A\ \forall \mathbf{r} \in q\ \theta(\dot{x}(\mathbf{r}), y)\,. \tag{2.1}$$

Of course, then by projective absoluteness $q \Vdash \theta(\dot{x}, \check{y})$ and as p, $\dot{x}$ were arbitrary, $\mathbb{S}_\alpha \Vdash \forall x \in \mathbb{R}\ \exists y \in \check{A}\ \theta(x, y) = \psi(\check{A})$ as desired.

Suppose (2.1) fails. Then for every real $y \in A$ the set $B_y = \{\mathbf{r} \in p : \theta(\dot{x}(\mathbf{r}), y)\}$ contains no condition $q \leq p$ in the forcing $\mathbb{S}_\alpha$ as a subset. Since the sets B_y are projective, we can use the dichotomy Lemma 1.4 to find functions $g_y : \mathbb{R}^{<\alpha} \to [\mathbb{R}]^{\aleph_0}$ such that for every real $y \in A$ and every α-sequence $\mathbf{r} \in p$ $\theta(\dot{x}(\mathbf{r}), y)$ implies $\exists \beta \in \alpha\ \mathbf{r}(\beta) \in g_y(\mathbf{r} \restriction \beta)$. Now by transfinite induction on $\beta \in \alpha$ build an α-sequence $\mathbf{r} \in p$ such that for every ordinal $\beta \in \alpha$ $\mathbf{r} \restriction \beta \in p \restriction \beta$ and $\forall y \in A\ \mathbf{r}(\beta) \notin g_y(\mathbf{r} \restriction \beta)$. This is rather easy; at each level $\beta \in \alpha$ use the fact that $\bigcup_{y \in A} g_y(\mathbf{r} \restriction \beta)$ is a set of size $|A| \cdot \aleph_0 < \mathfrak{c}$ while the set $\{t \in \mathbb{R} : (\mathbf{r} \restriction \beta)\langle t\rangle \in p \restriction \beta + 1\}$ is perfect, therefore of size $\mathfrak{c}$ and so must contain a real not in the above union. Now look at the real $\dot{x}(\mathbf{r})$. By the choice of the functions g_y and the sequence z we should have $\forall y \in A\ \neg\theta(\dot{x}(\mathbf{r}), y)$, contradicting the property ψ of A. (2.1) follows.

2.2. *The uncountable case*

The results of the previous subsection can be extended by a rather standard argument to show that for every ordinal α the countable support iteration $\mathbb{S}_\alpha$ of Sacks forcing of length α preserves the statement $\psi(A)$. Just use the following lemma:

Lemma 2.1 (ZFC + projective absoluteness). *Suppose that $\theta(x, y)$ is a projective formula and $A \subset \mathbb{R}$ is a set such that for every ordinal $\beta \in \omega_1$, every condition $p \in \mathbb{S}_\beta$ and every Borel function $f : p \to \mathbb{R}$ there is a condition $q \leq p$ and a real $y \in A$ such that for every sequence $\mathbf{r} \in q$, $\theta(f(\mathbf{r}), y)$ holds. Then for every ordinal α, $\mathbb{S}_\alpha \Vdash \forall x \in \mathbb{R} \exists y \in \check{A}\ \theta(x, y)$.*

Note that the assumptions of the lemma were shown to hold in the model $V[G]$ in the previous subsection.

Proof. First, a small observation. Suppose $\beta \in \omega_1$ and α are ordinals and $\pi : \beta \to \alpha$ is an increasing function. Then π can be naturally extended into an order-preserving map $\pi : \mathbb{S}_\beta \to \mathbb{S}_\alpha$ where $\pi(p)$ is the unique condition in $\mathbb{S}_\alpha$ with support $\pi''\beta$ such that $\forall \gamma \in \beta\ \pi(p) \restriction \pi(\gamma) \Vdash_{\mathbb{S}_{\pi(\gamma)}} (\pi(p))(\pi(\gamma)) = \{t \in \mathbb{R} : \langle \dot{r}_{\pi(\xi)} : \xi \in \gamma\rangle\langle t\rangle \in p \restriction \gamma{+}1\}$, where $\dot{r}_\zeta$ is the ζ-th Sacks generic real. It is not hard to see that $\pi(p) \Vdash_{\mathbb{S}_\alpha} \langle \dot{r}_{\pi(\xi)} : \xi \in \beta\rangle \in p$.

Now suppose that θ, A satisfy the assumptions of the lemma, α is an ordinal, $q_0 \in \mathbb{S}_\alpha$ is a condition and $\dot{x}$ is an $\mathbb{S}_\alpha$-name for a real. I will produce a condition $q_1 \leq q_0$ and a real $y \in A$ such that $q_1 \Vdash \theta(\dot{x}, \check{y})$. This will prove the lemma. Choose a countable elementary submodel M of some large structure containing all relevant objects and let $\beta = o.t.M \cap \alpha$ and $\pi : \beta \to \alpha$ be the inverse of the transitive

collapse. A standard countable support iteration argument similar to the proof of Lemma 1.2(5) gives a condition $p_0 \in \mathbb{S}_\beta$ such that $\pi(p_0) \leq q_0$ and for every $\mathbf{r} \in p_0$ the sequence $\mathbf{r} \circ \pi^{-1}$ is M-generic for the poset $\mathbb{S}_\alpha$. Let $f : p_0 \to \mathbb{R}$ be the Borel function defined by $f(\mathbf{r}) = \dot{x}/\mathbf{r} \circ \pi^{-1}$. Thus $\pi(p_0) \Vdash \dot{x} = \dot{f}(\langle \dot{r}_{\pi(\xi)} : \xi \in \beta \rangle)$. The assumptions of the lemma can be now employed to provide a real $y \in A$ and a condition $p_1 \leq p_0$ such that $\forall \mathbf{r} \in p_1\ \theta(f(\mathbf{r}), y))$. By the projective absoluteness and the last sentence of the first paragraph of this proof, setting $q_1 = \pi(p_1)$ we have $q_1 \leq q_0$, $q_1 \Vdash_{\mathbb{S}_\alpha} \theta(\dot{x}, \check{y})$ as desired. □

2.3. *The wrap-up*

To restate the above work, let

$$\chi(A) = \forall \alpha \in \omega_1\ \forall p \in \mathbb{S}_\alpha\ \forall \dot{x} : p \mapsto \mathbb{R} \text{ Borel } \exists y \in A\ \exists q \in \mathbb{S}_\alpha\ q \leq p \wedge \forall \mathbf{r} \in q\ \theta(\dot{x}(\mathbf{r}), y)\,.$$

Note that $\chi(A)$ is a projective statement about the set $A \subset \mathbb{R}$. We proved that $V[G] \models \chi(A)$ and that $\chi(A)$ implies in ZFC + projective absoluteness that for every ordinal α $\mathbb{S}_\alpha \Vdash \psi(\check{A})$. Again,

$$V[G] \models \exists A \subset \mathbb{R}\ \phi(A) \wedge \chi(A)\,.$$

Note that the sentence on the right hand side of the $\models$ sign is Σ^2_1.

Now back to the ground model V. Suppose first that V satisfies the continuum hypothesis. Then by the Σ^2_1 Absoluteness Fact 0.6, $V \models \exists A \subset \mathbb{R}\ \phi(A) \wedge \chi(A)$. Fix a set $A \subset \mathbb{R}$ with $\phi(A) \wedge \chi(A)$ and iterate Sacks reals ω_2 times with countable support to get a model $V[H]$. By the above work,

$$V[H] \models \phi(A) \wedge \psi(A), \mathfrak{x} \leq |A| \leq |\mathfrak{c}^V| = \aleph_1 < \mathfrak{c} = {\aleph_2}^V = \aleph_2$$

as desired. If the continuum hypothesis fails in the ground model V, iterate the Sacks reals $\mathfrak{c}^+$ many times anyway to get the model $V[H]$. Let $V[K] \subset V[H]$ be the intermediate extension given by the first ω_1 many generic reals. As is well known, $V[K] \models CH$ and $V[H]$ is an ω_2 iterated Sacks extension of the model $V[K]$. One can then repeat the above argument with V replaced with $V[K]$ to see that $V[H] \models \mathfrak{x} < \mathfrak{c}$. Theorem 0.2 follows.

3. Other Invariants

Many invariants of the form $\mathtt{cov}(I)$, where I is a Borel generated σ-ideal on the real line, can be isolated by a countable iteration of the forcing Borel($\mathbb{R}$) modulo I, and the proof follows closely the scenario of the previous two sections. It is just sufficient to verify that this forcing is proper (this fact is used in setting up the geometric representation of the iteration as in Sec. 1.1), that under AD every I-positive set of reals has a Borel positive subset (this is needed for the successor step in the proof of Claim 1.6) and that $\mathtt{cov}(I) = \mathtt{cov}(I \restriction B)$ for every positive Borel set B, this is tacitly used in the proof of (2.1) in Sec 2.1. The invariants $\mathfrak{c} = \mathtt{cov}$(countable), $\mathfrak{b}$, $\mathfrak{d}$ and some others conform exactly to this scenario. For the invariants $\mathtt{non}$(strong measure zero) and $\mathfrak{h}$ further changes are necessary.

3.1. *The dominating number*

Clearly the dominating number is the covering number of the ideal of bounded subsets of ω^ω. The countable support iteration of Miller forcing [11] will isolate it as the following two lemmas show.

Lemma 3.1 ([6]). *Every Borel unbounded subset of ω^ω contains all branches of some superperfect tree. Under AD this generalizes to all unbounded sets.*

Thus the Miller forcing is a dense subset of the factor algebra Borel (ω^ω) modulo the bounded sets.

Lemma 3.2. *For every superperfect tree $T \subset \omega^{<\omega}$ there is a continuous function $F : \omega^\omega \to [T]$ such that preimages of bounded sets are bounded.*

Proof. Thinning the tree T out if necessary we may assume that every splitnode of T has in fact infinitely many immediate successors. The natural homeomorphism $F : \omega^\omega \to [T]$ will have the required property. □

Thus $\mathfrak{d} = \mathtt{cov}(\text{bounded}) = \mathtt{cov}(\text{bounded ideal restricted to } B)$ for every Borel unbounded set $B \subset \omega^\omega$. The argument in Secs. 1 and 2 now goes through with the obvious changes, replacing $\mathfrak{c}$ with $\mathfrak{d}$, the countable ideal with the bounded ideal, and the Sacks condition in Definition 1.1 with the obvious Miller condition.

3.2. *The bounding number*

The countable support iteration of Laver reals [7] isolates $\mathfrak{b}$. Consider the σ-ideal I_L on ω^ω generated by the sets $A_g = \{f \in \omega^\omega : \text{for infinitely many } n \in \omega\ f(n) \in g(f \restriction n)\}$ where g varies through all functions from $\omega^{<\omega}$ to ω. We have the almost obvious

Lemma 3.3. $\mathfrak{b} = \mathtt{cov}(I_L)$.

Proof. The map $G : \omega^\omega \to I_L$ defined by $G(f) = A_g$ where $g(t) = f(|t|)$ for every sequence $t \in \omega^{<\omega}$, has the property that preimages of non-covering subsets of I_L are bounded. This proves that $\mathtt{cov}(I_L) \leq \mathfrak{b}$. On the other hand, fixing an enumeration $\{u_n : n \in \omega\}$ of $\omega^{<\omega}$, the map $H : I_L \to \omega^\omega$ sending the set A_g to the function $f : n \mapsto g(u_n)$, has the property that preimages of bounded sets do not cover the whole real line. Thus $\mathfrak{b} \leq \mathtt{cov}(I_L)$. □

As in the previous subsection, I will prove that the Laver forcing is a dense subset of the algebra Borel(ω^ω) modulo the ideal I_L:

Lemma 3.4. *Every Borel I_L-positive set contains all branches of some Laver tree. Under AD this generalizes to all I_L-positive sets.*

Proof. Suppose $A \subset \omega^\omega$ is a set and define an infinite game by letting players Adam and Eve play sequences $t_n \in \omega^{<\omega}$ and bits $b_n \in 2$ respectively, observing the following rules: $b_0 = 1$ and whenever Eve accepts a sequence t_n — that is, plays $b_n = 1$ — then Adam submits one-step extensions t_{n+1}, $t_{n+2}, \dots$ of t_n until Eve accepts one of them. The last number on the sequences t_{n+1}, $t_{n+2}, \dots$ must increase. Adam wins if either Eve accepted only finitely many times or else $\bigcup\{t_n : b_n = 1\} \in A$. The following two claims will complete the proof of the lemma [9]:

Claim 3.5. *Adam has a winning strategy if and only if the set A contains all branches of some Laver tree.*

Proof. For the right to left direction fix a Laver tree T with $[T] \subset A$. Let Adam set $t_0 =$trunk of T, and if $t_n \in T$ has been played and accepted by Eve then let Adam submit immediate successors of the node t_n in the tree T in the increasing order until Eve accepts one of them. This is obviously a winning strategy for Adam.

For the left to right direction let σ be a winning strategy for Adam and let $T \subset \omega^\omega$ be the tree of all sequences that can possibly arise in a run of the game G_A in which Adam follows the strategy σ. Note that for each node $t \in T$ there is a unique shortest run $\tau(t)$ such that it respects σ and t occurs in it, and if $t \subset s$ are both in the tree T then $\tau(t) \subset \tau(s)$. It follows that every branch $f \in [T]$ is a result of the run $\bigcup\{\tau(t) : t \subset f\}$ and therefore must belong to the set A. It is also clear from the definition of the game G_A that T is a Laver tree with trunk $\sigma(0)$. □

Claim 3.6. *Eve has a winning strategy if and only if $A \subset A_g$ for some function $g : \omega^{<\omega} \to \omega$.*

Proof. For the right to left direction fix a function g such that $A \subset A_g$. Let Eve accept a sequence t_n, a one-step extension of some previously accepted sequence t_m as soon as the last number on t_n exceeds $g(t_m)$. The result of such a play must fall outside of the set A_g and therefore this is a winning strategy for Eve.

For the left to right direction let σ be a winning strategy for Eve. For every sequence $s \in \omega^{<\omega}$ let T_s be the tree of all sequences that can be accepted by Eve in some run of the game where he follows the strategy σ and Adam plays $t_0 = s$. It follows that for all sequences $s \subset t$, if $t \in T_s$ then all but finitely many one-step extensions of t must belong to the tree T_s — otherwise Adam could win by first getting to t and then submitting all the one-step extensions of t which do not belong to the tree T_s. Also, $[T_s] \cap A = 0$ for all $s \in \omega^{<\omega}$. To see this, fix a branch $f \in [T_s]$ and define S to be the tree of all partial runs of the game G_A in which Adam set $t_0 = s$, Eve followed the strategy σ and the last move of Adam was accepted and it is an initial segment of the branch f. The tree S is ordered by extension. It follows from the "increasing" rule of the game G_A that the tree S is finitely branching — each run $\tau \in S$ has at most $2^{f(n)}$ immediate successors where n is the length of the last move of τ. Also, the tree S has height ω, so it must be illfounded. Any infinite

branch of the tree S yields a run of the game G_A following the winning strategy σ whose result was the function f. Thus $f \notin A$.

Now define a function $g : \omega^{<\omega} \to \omega$ by setting $g(t) =$ an integer such that for every $s \subset t$, if $t \in T_s$ then $g(t)$ is larger than all of the finitely many numbers n such that $t\langle n\rangle \notin T_s$. I claim that $A \subset A_g$. If this were not true then there would be a function $f \in A$ such that for some $n \in \omega$, for all larger numbers m necessarily $g(f \restriction m) \in f(m)$. But then $f \in [T_{f\restriction n}]$ by the definition of the function g, so by the previous paragraph $f \notin A$. A contradiction! □

The last thing that must be verified before unleashing the technology developed in Secs. 1 and 2 is that $\mathtt{cov}(I_L) = \mathtt{cov}(I_L \restriction B)$ for every Borel I_L positive set $B \subset \omega^\omega$:

Lemma 3.7. *For every Laver tree T there is a continuous function $F : \omega^\omega \to [T]$ such that preimages of I_L-small sets are I_L-small.*

Proof. The natural homeomorphism $F : \omega^\omega \to [T]$ has the required property. □

3.3. *The uniformity of the strong measure ideal*

This invariant has a definition that is not suitable for our purposes for syntactical reasons. I will use the following combinatorial characterization of this invariant. For a function $g \in \omega^\omega$ let $I_{ie}(g)$ be the σ-ideal on $\Pi_n g(n)$ generated by the sets $A_f = \{h \in \Pi_n g(n) : h \cap f \text{ is finite}\}$. Then

Lemma 3.8 ([2, 8.1.14, 10]). $\mathtt{non}(\textit{strong measure zero}) = \min\{\mathtt{cov}(I_{ie}(g)) : g \in \omega^\omega\}$.

Thus a natural attempt at isolating $\mathtt{non}$(strong measure zero) is the countable support iteration of the forcings Borel($\Pi_n g(n)$) modulo the ideal $I_{ie}(g)$ for all possible (names for) functions $g \in \omega^\omega$. The following two lemmas show that this attempt will actually work. Lemma 3.9 gives us the representation of the forcings suitable to prove that they satisfy Axiom A, and yields the crucial dichotomy. Lemma 3.12 provides the necessary homogeneity in the covering number.

Fix a function $g \in \omega^\omega$. A nonempty tree $T \subset \omega^{<\omega}$ will be called *g-thick* if the sequences in T are everywhere dominated by the function g, and for every sequence $t \in T$ there is a natural number n such that for every $m \in g(n)$ there is an extension $s \in T$ of the sequence t such that $s(n) = m$. It is quite obvious that if T is a g-thick tree then $[T] \subset \Pi_n g(n)$ is an $I_{ie}(g)$-positive set. In fact,

Lemma 3.9. *For every function $g \in \omega^\omega$, every Borel $I_{ie}(g)$-positive set contains all branches of some g-thick tree. Under AD this generalizes to all $I_{ie}(g)$-positive sets.*

Proof. Let $g \in \omega^\omega$ be a function and let $A \subset \Pi_n g(n)$ be a set. Define a game G_A by setting

Adam	$t_0,\ n_0$		$t_1,\ n_1$		$t_2,\ n_2$		$\cdots$
Eve		m_0		m_1		m_2	$\cdots$

where $n_0, n_1, \ldots$ is an increasing sequence of natural numbers, $m_i \in g(n_i)$ and $0 = t_0 \subset t_1 \subset \ldots$ are sequences of natural numbers dominated by the function g, $\mathrm{dom}(t_i) \in n_i$ and $t_{i+1}(n_i) = m_i$. Adam wins if $\bigcup t_n \in A$. The following two claims will complete the proof of the lemma [9]:

Claim 3.10. *Adam has a winning strategy if and only if the set A contains all branches of some g-thick tree.*

Proof. For the right to left direction fix a g-thick tree T with $[T] \subset A$. Adam will easily win by making sure that for each of his moves $t_i \in T$, and that n_i is such that for every $m \in g(n_i)$ there is an extension $s \in T$ of the sequence t_i such that $s(n_i) = m_i$.

For the left to right direction fix a winning strategy σ for Adam. Let T be the closure under initial segment of the set of all sequences arising in partial runs of the game G_A in which Adam follows the strategy σ. It is immediately clear that T is a g-thick tree and if h is a branch through T then there is a unique run of the game in which Adam follows the strategy σ and obtains the function h. Ergo, $[T] \subset A$. □

Claim 3.11. *Eve has a winning strategy if and only if $A \subset \bigcup_k A_{f_k}$ for some functions $f_k \in \Pi_n g(n), k \in \omega$.*

Proof. For the right to left direction let $A \subset \bigcup_k A_{f_k}$. Eve will easily win by fixing a bookkeeping function $b : \omega \to \omega$ such that for every number k the set $b^{-1}\{k\}$ is infinite, and then playing $m_i = g_{b(i)}(n_i)$.

For the left to right direction let σ be a winning strategy for Eve. For each partial run τ of the game G_A where Eve followed the strategy and Adam made the last move t_i let $f_\tau \in \Pi_n g(n)$ be the function defined by $f_\tau(n) = \sigma(\tau\langle n\rangle)$. Then necessarily $A \subset \bigcup_\tau A_{f_\tau}$. If this failed, then there would be a function $h \in A$ with infinite intersection with each f_τ. And then Adam could beat the strategy σ by inductively constructing a run of the game which respects the strategy σ and results in the function h. Assuming that the partial run τ_i has been constructed so that Adam made a last move $t_i \subset h$ in it, he finds a number n_i such that $h(n_i) = f_{\tau_i}(n_i)$ and the game continues into $\tau_{i+1} = \tau_i\langle n_i, h(n_i) = \sigma(\tau_i\langle n_i\rangle), h \restriction n_i + 1\rangle$. □

Thus for every function $g \in \omega^\omega$ the forcing $\mathrm{Borel}(\Pi_n g(n))/I_{ie}(g)$ has a dense set consisting of the g-thick trees. It follows easily that the forcing satisfies Axiom A. The following lemma shows that $\min\{\mathtt{cov}(I_{ie}(g)$ restricted to an arbitrary Borel

positive set), $g \in \omega^\omega\} = \min\{\mathtt{cov}(I_{ie}(g)) : g \in \omega^\omega\}$, which will be used in the proof of the relevant variation of (2.1) in Sec. 2.1.

Lemma 3.12. *For every function $g \in \omega^\omega$ and every g-thick tree T there is a function h and a continuous map $F : \Pi_n h(n) \to [T]$ such that the preimages of $I_{ie}(g)$-small sets are $I_{ie}(h)$-small.*

Proof. Fix a function $g \in \omega^\omega$ and a g-thick tree T. By induction on $n \in \omega$ construct finite sets $X_n \subset T$ so that $X_0 = \{0\}$, for each node $t \in X_n$ there is an integer k such that the set $X_{n+1}(t) = \{s \in X_{n+1} : t \subset s\}$ consists of sequences of length $k+1$ and for every $m \in g(k)$ there is a unique $s \in X_{n+1}$ with $s(k) = m$. Moreover make sure that $X_{n+1} = \bigcup_{t \in X_n} X_{n+1}(t)$. This is not hard to do; the sequences in any of the sets X_n will be pairwise incompatible and the union in the last sentence will always be a union of disjoint sets.

It will be convenient to define the function h so that its range consists of finite sets rather than natural numbers. Simply let $h(n) = \{Y \subset X_{n+1} : \forall t \in X_n \ |Y \cap X_{n+1}(t)| = 1\}$. The map $F : \Pi_n h(n) \to [T]$ will be defined by $F(f) =$ the unique function $e \in [T]$ such that for all numbers n the set $f(n)$ contains an initial segment of e. It is not hard to check the required properties for the function F. □

To compare the forcing PT_g of [1] with the forcing Borel($\Pi_n g(n)$) modulo $I_{ie}(g)$ note that the former is a somewhere dense subset of the latter. A moment's thought will then reveal that a suitable iteration of the PT_g forcings must isolate the invariant $\mathtt{non}$(strong measure zero) as well.

3.4. *The distributivity of the algebra Power(ω) modulo finite*

It is well known that an iteration of Mathias forcing will increase the invariant $\mathfrak{h}$ defined as the minimum cardinality of a collection of open dense subsets of the algebra Power(ω) modulo finite with empty intersection. Actually $\mathfrak{h}$ is isolated through this iteration. The proof of this fact is a little different from the previous cases since Mathias forcing cannot be written as Power($\mathbb{R}$) modulo a Borel generated ideal under any determinacy hypothesis. It is necessary to settle for a more complicated representation of the forcing. First, some notation. For sets $a, b \subset \omega$ let $a \subset^* b$ mean that a is included in b up to a finite number of elements. $[a]$ then denotes the equivalence class of the set a in the algebra Power(ω) modulo finite, for a set $A \subset$ Power(ω) write $[A] = \{[a] : a \in A\}$ and let I_M be the σ-ideal on Power(ω) consisting of those sets A for which $[A]$ is nowhere dense in the algebra Power(ω) modulo finite.

Lemma 3.13.

(1) *($ZF + DC + AD\mathbb{R}$) Let $T \subset (2 \times Ord)^{<\omega}$ be a tree. Then either $p[T] \in I_M$ or there is a condition $p \in \mathbb{M}$ so that $p \Vdash$ the generic real is in $p[\check{T}]$.*

(2) *(ZFC) Let $T \subset (2 \times Ord)^{<\omega}$ be $a < \delta$-weakly homogeneous tree, where δ is a supremum of ω Woodin cardinals. Then the same dichotomy as in* (1) *holds.*

With some additional work, (1) could be restated to say that under ZF + DC + AD$\mathbb{R}$ Mathias forcing is naturally forcing isomorphic to the algebra PowerPower(ω) modulo the ideal I_M. It is methodologically important to observe that I_M is not a Borel generated ideal.

Proof of Theorem 3.13(2). The following well known geometric condition for Mathias genericity will be used: □

Claim 3.14 ([14]). *Suppose that $a \subset \omega$ is an external V-generic Mathias real and $b \subset^* a$ is an infinite external set. Then b is a V-generic Mathias real.*

Now let T be a suitably weakly homogeneous tree such that $p[T] \notin I_M$. Then there is an infinite set $c \subset \omega$ such that the set $[p[T]]$ is dense below $[c]$ in the algebra Power(ω) modulo finite. Let $a \subset c$ be a V-generic Mathias real. By the weakly homogeneous absoluteness there is an infinite set $b \in V[a]$ such that $b \in p[T]$ and $b \subset^* a$. By the above claim, the set b is a V-generic Mathias real and by a wellfoundedness argument involving the tree T $V[b] \models b \in p[T]$. So there must be a condition $p \in \mathbb{M}$ such that $p \Vdash$ the generic real is in $p[\check{T}]$.

On the other hand, suppose that some condition $p \in \mathbb{M}$ forces the generic real into $p[\check{T}]$. Choose a countable elementary submodel M of a large enough structure containing all the relevant objects and consider the set $A = \{a \subset \omega : a$ is a Mathias M-generic real meeting the condition $p\}$. This set is nonempty, Borel by Claim 1.3 and its projection into the algebra Power(ω) modulo finite is open by Claim 3.14. By the choice of the condition p we also have $A \subset p[T]$. Lemma 3.13(2) follows. □

The proof of the previous lemma also yields

Claim 3.15. *Every suitably weakly homogeneous set not in I_M has a Borel subset not in I_M.*

With the above facts in hand, the geometric analysis of countable iterations of Mathias forcing proceeds just as in Sec. 1.1 replacing the countable ideal by the ideal I_M everywhere, and with the Sacks condition in Definition 1.1 replaced by the Mathias condition-splitting into an I_M-positive set. The reader is urged to use Lemma 3.13(2) to prove on his own that the Mathias forcing is forcing isomorphic to the algebra Borel(Power(ω)) modulo I_M. The dichotomy Lemma 1.4 must be reformulated. Let $\alpha \in \omega_1$ and let $A \subset \mathbb{R}^\alpha$ be a projective set. Consider the game G_A of α many rounds where at round $\beta \in \alpha$ Eve plays an infinite set $t_\beta \subset \omega$, Adam plays an infinite set $s_\beta \subset^* t_\beta$ and Eve plays a set $r_\beta \subset^* t_\beta$ in this order. Eve wins if the sequence $\langle r_\beta : \beta \in \alpha \rangle$ belongs to the set A.

Under the assumption of proper class many Woodin cardinals (actually ω_1 many suffice) the game is determined and there are two possibilities:

(1) Adam has a winning strategy.
(2) Eve has a weakly homogeneous winning strategy and then by an argument essentially identical to that in Sec.1.2 using Claim 3.15, there is a condition $p \in \mathbb{M}_\alpha$ with $p \subset A$.

Note that we could not use a game similar to the original one because there it is important that Adam can play arbitrarily large sets in the relevant ideal. Here the ideal is not Borel generated and so we would not get a real game and the determinacy of the game would be open to question.

The argument for the $\mathfrak{h}$ version of Theorem 0.2 then proceeds exactly as in Sec. 2 except that the proof of (2.1) has to be changed. Let me recall the setup there. There is a tame invariant $\mathfrak{x} = \min\{|A| : A \subset \mathbb{R}, \phi(A) \wedge \psi(A)\}$ where the quantifiers of $\phi(A)$ is are restricted to the set A and the natural numbers and $\psi(A) = \forall x \in \mathbb{R}\ \exists y \in A\ \theta(x, y)$ where θ is a formula whose quantifiers range over natural and real numbers only. We work in a model where $\mathfrak{x} < \mathfrak{h}$ and A is a witness for it, that is $|A| < \mathfrak{h}, \phi(A) \wedge \psi(A)$, also we have $\alpha \in \omega_1$ and a condition $p \in \mathbb{M}_\alpha$ and a Borel function $\dot{x} : p \to \mathbb{R}$. We want to show that $\exists y \in A \exists q \leq p \forall \mathbf{r} \in q\ \theta(\dot{x}(\mathbf{r}), y)$.

For each real $y \in A$, let $B_y = \{\mathbf{r} \in p : \theta(\dot{x}(\mathbf{r}), y)\}$. If Eve had a weakly homogeneous winning strategy for one of the games G_{B_y} then by (2) above for that real $y \in A$ there would be a condition $q \leq p$ such that $q \subset B_y$ and we would be done. So it is enough to derive a contradiction from the assumption that Adam has a winning startegy σ_y for every game $G_{B_y}, y \in A$. By a simultaneous transfinite induction on $\beta \in \alpha$ build partial plays of games G_{B_y} for all $y \in Y$ played according to the strategies σ_y so that

- writing $t_{\beta y}, s_{\beta y}, r_{\beta y}$ for the moves at the β-th round of the partial play we build for $y \in A$, the set $r_{\beta y} \subset \omega$ does not depend on y. We can write r_β to denote this set,
- $\langle r_\gamma : \gamma \in \beta \rangle \in p \restriction \beta$.

To find the moves $t_{\beta y}, s_{\beta y}, r_\beta$ under the assumption that the partial plays $\langle t_{\gamma y}, s_{\gamma y}, r_\gamma : \gamma \in \beta \rangle$ were constructed for all $y \in A$, let $D_y = \{r \subset \omega :$ there are sets $t, s \subset \omega$ such that the sequence $\langle t_{\gamma y}, s_{\gamma y}, r_\gamma : \gamma \in \beta \rangle \langle t, s, r \rangle$ is a legal partial play of G_{B_y} observing the strategy $\sigma_y\}$. It follows from the definitions that the sets D_y, $y \in A$ are closed under finite changes of their elements and that they are all open dense in the algebra Power(ω) modulo finite. Since $|A| < \mathfrak{h}$, the intersection of all of these sets is open dense as well and has an element r_β in common with the somewhere dense set $\{r \subset \omega : \langle r_\gamma : \gamma \in \beta \rangle \langle r \rangle \in p \restriction \beta + 1\}$. The induction step is concluded by finding sets $t_{\beta y}, s_{\beta y} \subset \omega$ witnessing that $r_\beta \in D_y$, for all $y \in A$.

Now look at the sequence $\mathbf{r} = \langle r_\beta : \beta \in \alpha \rangle$. Since the strategies σ_y were winning for Adam and in the previous paragraph we produced plays following these strategies whose outcome was the sequence $\mathbf{r}$, it must be that $\theta(\dot{x}(\mathbf{r}), y)$ fails for every real $y \in A$. This contradicts the property ψ of the set A.

4. The Tower Number

The proof of Theorem 0.5 is really just a variation on an argument of Woodin concerning the maximization of Σ_2 theory of the model $\langle H_{\aleph_2}, \in, \omega_1\rangle$. Let δ be a measurable Woodin cardinal such that for every tame invariant $\mathfrak{x}$, if there is a forcing extension satisfying $\aleph_1 = \mathfrak{x} < \mathfrak{t}$ then there is such an extension of size $< \delta$. Without loss of generality assume that $2^\delta = \delta^+$. Let $P_\mathfrak{t}$ be a partial order with the following definition. It is a two step iteration $P_0 * \dot{P}_1$ where P_0 is again a two step iteration $R_0 * \dot{R}_1$. Here R_0 is just the Levy collapse of δ to ω_1 and $R_1 = \{\langle c, D\rangle : c \subset \delta$ is a closed bounded set such that every limit point κ of it is a weakly compact cardinal of V and the set $c \cap \kappa$ diagonalizes the weakly compact filter on κ. The set $D \subset \delta$ belongs to the weakly compact filter on δ as computed in $V\}$. The poset R_1 is ordered by $\langle c_1, D_1\rangle \leq \langle c_0, D_0\rangle$ if c_1 end-extends c_1, $D_1 \subset D_0$ and $c_1 \setminus c_0 \subset D_0$. Having defined the poset P_0, P_1 is just an arbitrary σ-centered forcing of size $\delta^+ = \aleph_2$ in the model V^{P_0} making MA(σ-centered) and $\mathfrak{c} = \aleph_2$ true.

The forcing $P_\mathfrak{t}$ deserves an aside. The first step in the iteration defining the poset P_0 collapses δ to $\aleph_1$ and the second step adds a rather mysterious club subset of δ without adding reals or collapsing $\delta = \aleph_1$ or $\delta^+ = \aleph_2$. In some sense the model V^{P_0} is supposed to be the most generic model of $\diamondsuit$ and the model $V^{P_0 * P_1}$ should be the most generic model of MA(σ-centered) + $\mathfrak{c} = \aleph_2$. The key properties of the forcing P_0 are summed up in the following lemma.

Lemma 4.1 ([17] (Woodin)). *Let γ be any Woodin cardinal above δ. For every poset Q of size less than δ there are external V-generic filters $G_0 * G_1 \subset Q * \dot{\mathbb{P}}_{<\gamma}$ and $H_0 \subset P_0$ so that $G_1 \cap \dot{\mathbb{Q}}_{<\delta}$ is a $V[G_0]$-generic filter, $V[G_0][G_2] \subset V[H_0] \subset V[G_0][G_1]$ and $V[G_0][G_1] \models \delta = \aleph_1 = |(\mathrm{Power}(\delta^+))^V|$.*

Let $j : V[G_0] \to M$ and $i : V[G_0] \to N$ be the elementary embeddings derived from the filters G_2, G_1 respectively. There is a natural factor embedding $k : M \to N$ such that $i = j \circ k$ and since $\delta = \aleph_1^M = \aleph_1^N$, necessarily $\mathrm{crit}(k) > \delta$. The point in the definition of the forcing P_0 is that the model $V[H_0]$ can be sandwiched between the elementarily equivalent models M and N as far as subsets of ω_1 are concerned.

Back to the proof of Theorem 0.5, let $\mathfrak{x} = \min\{|A| : A \subset \mathbb{R}, \phi(A) \wedge \psi(A)\}$ be a tame invariant and let Q be a forcing of size less than δ such that $Q \Vdash \aleph_1 = \mathfrak{x} < \mathfrak{t}$. I must prove that $P_0 * \dot{P}_1 \Vdash \aleph_1 = \mathfrak{x} < \mathfrak{t}$. Choose a Woodin cardinal $\gamma > \delta$ and find the external objects G_0, G_1, G_2, H_0 as in Lemma 4.1 and write $j : V[G_0] \to M$ and $i : V[G_0] \to N$ for the elementary embeddings derived from the filters G_2, G_1 respectively. The model N is elementarily equivalent to $V[G_0]$ and so $\mathfrak{t} > \aleph_1$, $\mathfrak{p} > \aleph_1$ and by a theorem of Bell [3], $MA_{\aleph_1}(\sigma\text{-centered})$ are all true there. Moreover, since the model is closed under $< \gamma$ sequences in $V[G_0][G_1]$, the set $\mathrm{Power}(\delta^{+V}) \cap V[H_0]$ is in N and has size $\aleph_1$ there. In particular, the forcing $P_1 \in V[H_0]$ is in N, it is σ-centered and by an application of Martin Axiom there is a $V[H_0]$-generic filter $H_1 \subset P_1$ in N. I will show that $V[H_0][H_1] \models \mathfrak{x} = \aleph_1$ and that will complete the proof of Theorem 0.5.

Let $A \subset \mathbb{R} \cap V[G_0]$ be a witness to $\mathfrak{x} = \aleph_1$ in $V[G_0]$, that is $V[G_0] \models |A| = \aleph_1$, $\phi(A) \wedge \psi(A)$. Look at the set $jA \in M$. Since the set has size $\aleph_1$ in the model and the critical point of the factor embedding $k : M \to N$ is above $\aleph_1^M = \delta$, it must be that $iA = kjA = jA \in M \subset V[H_0] \subset V[H_0][H_1]$. Now the set iA has the properties ϕ and ψ in the model N by elementarity of the embedding i, and it can be argued that it has these properties in the smaller model $V[H_0][H_1]$ as well. For the property ϕ is certainly absolute, and if $\psi(iA) = \forall x \in \mathbb{R} \exists y \in iA\ \theta(x, y)$ failed in the model $V[H_0][H_1]$ as witnessed by a real x then it would fail in the model N for the same real x, since the reals of both models are generic extensions of the ground model V and therefore agree on the truth of the projective formula $\theta(x, y)$. Thus $V[H_0][H_1] \models iA$ is a witness to $\aleph_1 = \mathfrak{x}$ as desired.

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