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HAPPY FAMILIES

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0. Definitions and description of the results

This paper is concerned with combinatorial properties of families of infinite subsets of ω, the set of natural numbers $0, 1, 2, \ldots$. Before introducing the notion of a happy family, for the suggestion of which phrase the author is indebted to Professor J.N. Crossley, we state the notational conventions that will be followed throughout the paper and review some familiar concepts.

$0, 1, 2, \ldots$ are identified with finite Neumann ordinals, so that 0 is the empty set, $1 = \{0\}, 2 = \{0, 1\}, \ldots$, and if x is a nonempty subset of ω, $\bigcap x = \inf x$. The variables i, j, k, l, m, n will be used to denote arbitrary members of ω, s, t, u, will denote arbitrary finite subsets of ω, S, T, W, X, Y, Z arbitrary infinite subsets of ω, x, y arbitrary finite or infinite subsets of ω, A, B, C, D, F, G, I arbitrary families of subsets of ω, and finally $\mathscr{C}, \mathscr{F}, \mathscr{I}$ will be used for collections of families of subsets of ω. We define $|s| = \sup\{n + 1 \mid n \in s\}$, so in particular $|0| = 0$. $|s|$ is in fact equal to the set theoretic rank of s. The set theoretical difference of two sets $\mathscr{X}$ and $\mathscr{Y}$ is denoted by $\mathscr{X} \setminus \mathscr{Y}$. With these definitions and conventions $X \setminus |s|$ is the set of those numbers of X exceeding, in the usual ordering of the natural numbers, all members of s. The cardinal of a set $\mathscr{X}$ is denoted by $\bar{\bar{\mathscr{X}}}$ or, where typographically more convenient, $\mathscr{X}^{=}$. For an arbitrary $\mathscr{X}$ we write $[\mathscr{X}]^n$ for the set of n-element subsets of $\mathscr{X}$: that is

$$[\mathscr{X}]^n =_{df} \{w \mid w \subseteq \mathscr{X} \,\&\, \bar{\bar{w}} = n\}.$$

Similarly we write $[\mathscr{X}]^\omega$ for $\{w \mid w \subseteq \mathscr{X}$ and $\bar{\bar{w}} = \omega\}$. We write K for the set of finite subsets of ω and H for the set of infinite subsets of ω, so that $K = \{s \mid s \subseteq \omega\}$ and $H = \{X \mid X \subseteq \omega\}$. The power set of w is denoted by $\mathscr{P}(w)$, so that $\mathscr{P}(\omega) = H \cup K$.

A *filter* on ω is a collection F of subsets of ω with the properties that $x \in F \,\&\, y \in F \to x \cap y \in F$ and that $y \supseteq x \,\&\, x \in F \to y \in F$. If $0 \in F$, F is *improper*; otherwise F is *proper*. We write Fr for the Fréchet filter $\{X \mid \omega \setminus X \in K\}$ of all cofinite subsets of ω. If F is a filter and $F \supseteq \mathrm{Fr}$, F is *free*. If F is a proper filter and $\forall x\, (x \in F$ or $\omega \setminus x \in F)$ then F is an *ultrafilter*; that is equivalent to being a maximal proper filter. For $A \subseteq \mathcal{P}(\omega)$, we write $\tilde{A}$ for $\{x \mid \omega \setminus x \in A\}$. For example, $\tilde{\text{Fr}} = K$. I is an *ideal* if $\tilde{I}$ is a filter, and I is further described as *free*, *proper*, accordingly. I is a *prime ideal* if $\tilde{I}$ is an ultrafilter. For given A and $X_1, \dots, X_n$ we write $\text{fil}(A, X_1, \dots, X_n)$ for the filter generated by $X_1, \dots, X_n$ and the members of $A \cup \text{Fr}$, and $\text{id}(A, X_1, \dots, X_n)$ for the ideal generated by $X_1, \dots, X_n$ and the members of $A \cup K$; so when A is closed under intersection,

$$\text{fil}(A, X_1, \dots, X_n) = \{x \mid \exists y \, \exists z \, (y \in A \ \& \ z \in \text{Fr} \ \& \ x \supseteq y \cap z \cap X_1 \cap \cdots \cap X_n)\},$$

and when A is closed under union,

$$\text{id}(A, X_1, \dots, X_n) = \{x \mid \exists y \, \exists z \, (y \in A \ \& \ z \in K \ \& \ x \subseteq y \cup z \cup X_1 \cup \cdots \cap X_n)\}.$$

$\text{fil}(A, X)$ and $\text{id}(A, X)$ are thus always free, though they may be improper.

We shall assume all the axioms of Zermelo–Fraenkel set theory, ZF. We shall be particularly interested in avoiding the use of the full axiom of choice, AC. Three weak forms of AC will be used occasionally; they are

DC, or Tarski's axiom of dependent choices, which is the statement that given a relation $\mathcal{R}$ on a nonempty set $\mathcal{X}$ such that for all $v \in \mathcal{X}$ there is a $w \in \mathcal{X}$ with $v \mathcal{R} w$, there is a function $f : w \to \mathcal{X}$ such that for all $i \in \omega$, $f(i) \mathcal{R} f(i+1)$;

DCR, or "dependent choices for relations on the reals", which is DC restricted to the special case that $\mathcal{X} = \mathcal{P}(\omega)$; and

ACR, or "choice for relations on the reals", which is the statement that given a relation $\mathcal{R}$ on $\mathcal{P}(\omega)$ such that for all x there is a y with $x \mathcal{R} y$, there is a function $E : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that for all $x, x \mathcal{R} E(x)$.

Of those three, DCR will be used most frequently; it is the weakest of the three, being a consequence in ZF both of DC and of ACR. Attention will be drawn when appropriate to the use or avoidance of these axioms in proofs. Some proofs will use the continuum hypothesis, CH, which is the assertion that $(\mathcal{P}(\omega))^{=} = \aleph_1$, the first uncountable well-ordered cardinal. Some remarks will be made about the possibility of proving certain statements by using Martin's axiom, MA, which is the axiom A defined on page 150 of [15]. MA is weaker than CH being consistent with AC + the negation of CH, and as [15] makes clear, appropriate formulations of certain consequences of CH + AC are derivable from MA + AC. However these remarks are marginal, and familiarity with MA is not required for most of the paper. Familiarity is required, though, with the elementary theory of forcing and Boolean valued models, as expounded for example in [14], for following part of the paper, though in most cases forcing is only used to reduce a theorem to a special case of itself, and can with effort be avoided. The less elementary parts of forcing used are reviewed in Section 3.

We now define the notion of a happy family.

0.0. Definition. X is said to *diagonalize* the family $\{X_s \mid s \in K\}$ if $X \subseteq X_0$ and for all s, if $\sup s \in X$ then $X \setminus |s| \subseteq X_s$.

0.1. Definition. A is a *happy family* if $\mathcal{P}(\omega)\setminus A$ is a free ideal and whenever $\text{fil}(\{X_s \mid s \in K\}) \subseteq A$, there is an $X \in A$ which diagonalizes $\{X_s\}$.

We give three illustrations of that definition; in the first $\mathcal{P}(\omega)\setminus A$ is small, in the second, of medium size, and in the third, large.

0.2. Example. H, the family of all infinite subsets of ω, is happy. In this case $\mathcal{P}(\omega)\setminus A$ is as small as can be, namely the Fréchet ideal $\check{F}r$ which equals K.

To see that H is happy, let $\{X_s \mid s \in K\}$ generate a proper filter $\subseteq H$, and select $n_0 \in X_0$. Choose $n_1 > n_0$ with

$$n_1 \in \bigcap\{X_s \mid |s| \leq n_0 + 1\}$$

and inductively $n_{k+1} > n_k$ with

$$n_{k+1} \in \bigcap\{X_s \mid |s| \leq n_k + 1\}.$$

As $\bigcap\{X_s \mid |s| \leq m\}$ is always infinite, those choices are all possible. Let $X = \{n_k \mid k < \omega\}$. Then $X \subseteq X_0$; given s with $\sup s = n_k \in X$, we have

$$X\setminus|s| = \{n_l \mid l \geq k+1\} \quad \text{and} \quad |s| = n_k + 1,$$

and as $l \geq k+1$ implies $n_l \in X_s$, $X\setminus|s| \subseteq X_s$ as required. □

The next class of examples of happy families requires some preamble.

0.3. Definition. Two subsets x and y of ω are called *almost disjoint* if $x \cap y$ is finite.

0.4. Definition. B is a MAD *family* if B is an infinite maximal collection of pairwise almost disjoint infinite subsets of ω.

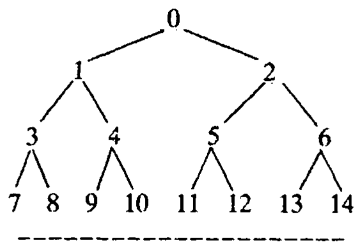
B is required in that definition to be infinite to exclude the trivial case when $B = \{X_0, \ldots, X_k\}$ and $\omega\setminus(X_0 \cup \cdots \cup X_k)$ is finite.

The axiom of choice implies that MAD families exist. No MAD family can be countable for if $B = \{X_i \mid i < \omega\}$, where the X_i are pairwise almost disjoint, there is an X with $X \cap X_i$ finite for each $i \in \omega$: pick

$$n_k \in X_k\setminus\bigcup\{X_j \mid j < k\}$$

and set $X = \{n_k \mid k < \omega\}$. Similarly Martin's axiom with AC implies that each MAD family is of power $2^{\aleph_0}$, whereas Hechler has shown that it is consistent with ZF + AC + $2^{\aleph_0} > \aleph_1$ that there be a MAD family of power $\aleph_1$. The consistency with ZF + DC of the statement that there is no MAD family is established in Section 5, the consistency of the existence of a Mahlo cardinal being assumed.

0.5. An agreeable example of a family of power $2^{\aleph_0}$ of pairwise almost disjoint subsets of ω is this: enumerate the nodes of the binary tree thus:



and for $f:\omega\to 2$ let $X_f=\{i \mid \exists n\, f\restriction n$ is node number $i\}$. Put $A=\{X_f \mid f:\omega\to 2\}$. A is not maximal, as $A\cup\{\{2^n \mid n<\omega\}\}$ is a larger such family. Corollary 4.7 gives a deeper reason for A not being maximal.

The following proposition yields the second class of examples of happy families. With an eye to its future quotation we pause before stating it to repeat a definition from [16].

0.6. Definition. A free ideal I is *tall* if for all X there is a $Y\in I$ with $Y\subseteq X$.

In a sense which is left to the reader's imagination, the free ideals that are not tall are almost as small as the smallest free ideal K.

0.7. Proposition (DCR). *Let B be a* MAD *family, and put $I=\mathrm{id}(B)$. Then I is proper and tall but not prime, and $\mathcal{P}(\omega)\setminus I$ is a happy family.*

Proof. $\omega\notin I$ as, B being infinite, ω is not the union of finitely many elements of B, even up to finite difference. Given X there is a $Z\in B$ with $X\cap Z$ infinite, by the maximality of B; then $X\cap Z\in I$ and $X\cap Z\subseteq X$. So I is proper and tall.

Put $A=\mathcal{P}(\omega)\setminus I$. Then

$$A=\{X \mid \{Y \mid Y\in B\ \&\ X\cap Y \text{ is infinite}\} \text{ is infinite}\}.$$

Given $\{X_s\}_{s\in K}$ with $\mathrm{fil}(\{X_s\})\subseteq A$, let X^0, constructed as in 0.2, diagonalize $\{X_s\}$, and let $Y^0\in B$ have infinite intersection with X^0. Set $X_s^1=X_s\setminus Y^0$. Each X_s^1 is infinite, and $\mathrm{fil}(\{X_s^1\})\subseteq A$, as $Y^0\in I$. Let X^1 diagonalize $\{X_s^1\}$, and let $Y^1\in B$ have infinite intersection with X^1. $Y^1\neq Y^0$ as $X^1\subseteq X_0^1\subseteq\omega\setminus Y^0$. Note that X^1 diagonalizes $\{X_s\}$. Now let $X_s^2=X_s\setminus(Y^0\cup Y^1)\cdots$. A sequence X^i, Y^i may thus be found such that each $Y^i\in B$, $Y^i\neq Y^j$ for $i\neq j$, $X^i\cap Y^i$ is infinite and X^i diagonalizes $\{X_s\}$. Construct a strictly increasing sequence $\{n_k\}_{k<\omega}$ such that $n_0\in X_0$ and for

$$k=2^j(2l+1),\ n_k\in Y^j\cap X^j\cap\bigcup\{X_s \mid |s|\leq n_{k-1}+1\}.$$

Such a sequence may be found because all sufficiently large members of X^j are in $\bigcap\{X_s \mid |s|\leq n_{k-1}+1\}$ and $Y^j\cap X^j$ is infinite. Put $Z=\{n_k \mid k<\omega\}$. Then for each j, $Z\cap Y^j$ is infinite and so $Z\in A$; clearly Z diagonalizes $\{X_s\}_{s\in K}$.

Thus A is happy. To see that I is not prime let $\{Y_i \mid i < \omega\}$ be distinct elements of B, and construct Z so that both Z and $\omega \setminus Z$ have infinite intersection with each Y_i. To do that find a strictly increasing sequence $\{n_k \mid k < \omega\}$ such that for each k, if $k = 2^m(2n+1)$, both n_{2k} and n_{2k+1} are in Y_m, and put $Z = \{n_{2k} \mid k < \omega\}$. Neither Z nor $\omega \setminus Z$ is in I.

That DCR is a sufficiently strong form of choice for this proof is left, as it will be on other occasions, to the reader to verify. □

We now go to the other extreme and consider the case when A is a happy family and $\mathcal{P}(\omega) \setminus A$ is a prime ideal. In that case A will be a free ultrafilter of a certain sort discussed in Booth's paper [3]. To relate the present notions to those in Booth's paper the following is useful:

0.8. Proposition. *The following are equivalent conditions on a family A provided that $\mathcal{P}(\omega) \setminus A$ is a free ideal:*

(i) *A is happy,*

(ii) *given $Y_i \in A$ for $i < \omega$ with $Y_{i+1} \subseteq Y_i$ for each $i < \omega$, there is a function $f : \omega \to \omega$ such that the range or image $W(f)$ of f is a member of A and for all $n, f(n+1) \in Y_{f(n)}$.*

Proof. Assume A is happy and let $\{Y_i\}_{i<\omega}$ be as in (ii). Define for $s \in K$

$$X_s = \bigcap \{Y_i \mid i + 1 \leq |s|\}:$$

so in fact $|s| \geq 1 \to X_s = Y_{|s|-1}$. Then $\{X_s \mid s \in K\}$ generates a filter, contained in A as $\mathcal{P}(\omega) \setminus A$ is an ideal. Let X diagonalize the family $\{X_s\}_{s \in K}$ and let f enumerate X is ascending order. Let $n < \omega$, and set $s = X \cap (f(n)+1)$. Then $|s| = f(n)+1$ and $\sup s \in X$. As $f(n+1) \in X \setminus |s|$ and $X \setminus |s| \subseteq X_s \subseteq Y_{f(n)}$, we have $f(n+1) \in Y_{f(n)}$. But n was rbitrary.

Conversely suppose A has property (ii) and let $\{X_a \mid s \in K\}$ be a family of members of A that generates a filter contained in A. Put

$$Y_i = \bigcap \{X_s \mid |s| \leq i + 1\}.$$

Each Y_i is in A, and $Y_{i+1} \subseteq Y_i$. By (ii) there is an f and an $X \in A$ with $X = W(f)$ and for all n $f(n+1) \in Y_{f(n)}$. Let $\max s = f(n) \in X$ say; let $k \in X \setminus |s|$. Then $k = f(m)$ for some $m > n$, so $k \in Y_{f(n)}$; $|s| = f(n)+1$, as $Y_{f(n)} \subseteq X_s$; so $k \in X_s$. Hence X diagonalizes $\{X_s\}$, as required. □

0.9. Definition. A *Ramsey ultrafilter* is an ultrafilter that is also a happy family.

The reason for that name is this. Ramsey showed that if $\pi : [\omega]^2 \to 2$ then for some X, π is constant on $[X]^2$. Such an X is called *homogeneous* for π; in the notation of Erdös and Rado, Ramsey's theorem is $\omega \to (\omega)^2_2$. Now the following is an immediate consequence of Proposition 0.8 above and Theorem 4.9 on page 20 of

[3], where other equivalent definitions are given: a further list is given in Theorem 9.31 below.

0.10. Proposition. *The following are equivalent conditions on a free ultrafilter F*
(i) *F is happy,*
(ii) *for any $\pi : [\omega]^2 \to 2$ there is an $X \in F$ with π constant on $[X]^2$.*

Ramsey ultrafilters are much easier to work with than happy families as the intersection of arbitrarily chosen elements of a filter is infinite whereas that might not be the case for a happy family. One difficulty is that they need not exist: Kunen has shown that if the universe is the result of adding $\aleph_2$ random reals to L or if there is a real valued measurable cardinal, there is no Ramsey ultrafilter. However it will be shown in Section 4 using DCR that given any happy family A a Boolean extension of the universe may be made which adds no new subsets of ω but which adds a Ramsey ultrafilter as a subset of A. The general combinatorial theorems in this paper about happy families will accordingly be obtained by reduction using Boolean-valued models to the special case that the happy families are Ramsey ultrafilters, though this use of Boolean-valued models may be avoided by more laborious arguments. In the presence of the continuum hypothesis an explicit relationship may be proved:

0.11. Proposition (CH). *Let A be happy. Then there is a Ramsey ultrafilter $F \subseteq A$.*

Taking $A = H$ shows that CH$\to$Ramsey ultrafilters exists. The latter result may be deduced from Martin's axiom, but can the above proposition? In Section 9 MA is used to derive a weaker conclusion from a weaker hypothesis on A.

Proof of the Proposition. Enumerate all sequences $\{X_s \mid s \in K\}$ such that

$$\text{fil}\{X_s\} \subseteq A \quad \text{as} \quad \langle \{X_s^\zeta \mid s \in K\} \mid \zeta < \aleph_1 \rangle.$$

Construct a sequence F^ζ $(\zeta < \aleph_1)$ of countably generated filters such that $F^\zeta \subseteq F^{\zeta+1} \subseteq A$; for each ζ either $\exists s$ with $X_s^\zeta \in \check{F}^{\zeta+1}$ or $\{X_s^\zeta\} \subseteq F^\zeta$ and $\exists Y \in F^{\zeta+1}$ (Y diagonalizes $\{X_s^\zeta\}$); and such that $\forall Z \subseteq \omega\, \exists \zeta\, (Z \in F^\zeta$ or $\omega \setminus Z \in F^\zeta)$. Then $\bigcup \{F^\zeta \mid \zeta < \aleph_1\}$ is the desired Ramsey ultrafilter. The only difficult part of the construction is handled thus: given $F^\zeta \subseteq A$ and $\{X_s^\zeta \mid s \in K\}$, if $\exists s$ $\text{fil}(F^\zeta, \omega \setminus X_s^\zeta) \subseteq A$, let $F^{\zeta+1} = \text{fil}(F^\zeta, \omega \setminus X_s^\zeta)$ for that s; otherwise

$$\forall s\, \forall Y \in F^\zeta (X_s^\zeta \cap Y \in A),$$

and so $\text{fil}(F^\zeta, \{X_s^\zeta \mid s \in K\})$ is contained in A, and is countably generated by $\{Y_i \mid i < \omega\}$ say. Let

$$Z_s = X_s^\zeta \cap \bigcap \{Y_s \mid |s| \leqslant i\}.$$

Then $\text{fil}\{Z_s\} \subseteq A$; as A is happy, $\exists W \in A$ which diagonalises $\{Z_s\}$. Put $F^{\zeta+1} = \text{fil}(\text{Fr}, W)$. Then $F^\zeta \subseteq F^{\zeta+1}$. $\square$

The notion of a Σ^1_1 subset of $\mathcal{P}(\omega)$ will be assumed known: for a discussion see [27] and for some examples, see [16]. Silver in [28] has proved the following generalization of Ramsey's theorem, which by Shoenfield's absoluteness theorem may be proved without the axiom of choice.

0.12. Theorem (Silver). *Let D be a Σ^1_1 subset of $\mathcal{P}(\omega)$. Then there is an X such that*

$$\forall Y\,(Y \subseteq X \to (Y \in D \leftrightarrow X \in D)).$$

The contents of this paper are now summarised: in Section 1 a proof is given, using DCR but with no appeal to forcing, of the following

0.13. Theorem (DCR). *Let D be a Σ^1_1 subset of $\mathcal{P}(\omega)$ and let F be a Ramsey ultrafilter. Then there is an $X \in F$ such that*

$$\forall Y\,(Y \subseteq X \to (Y \in D \leftrightarrow X \in D)).$$

In Section 2, a notion of forcing associated with a Ramsey ultrafilter is studied, the principal result being a criterion for a subset of ω to be generic which is strikingly similar to that in [17], and used to obtain another proof of 0.13 by forcing. In Section 3, certain facts about iterated forcing are reviewed, and Solovay's model in which all sets of reals are Lebesgue measurable is briefly described. In the next section the reduction of theorems about happy families to theorems about Ramsey ultrafilters is discussed and applied to generalise the theorems of Sections 1 and 2. In Section 5, a proof is given that in Solovay's model the partition relation $\omega \to (\omega)^\omega$ holds, that is, that 0.12 holds for every D, Σ^1_1 or not; and that under a further hypothesis there is no MAD family. An axiom scheme of Jensen is mentioned briefly. Section 6 is devoted to the study of functions $E:\mathcal{P}(\omega)\to\mathcal{P}(\omega)$, and a theorem about Borel functions, which generalises Theorem 0.12 and which will hold of all functions if DCR and the truth of $\omega\to(\omega)^\omega$ are assumed, is proved. Roughly the theorem says that any Borel function $E:\mathcal{P}(\omega)\to\mathcal{P}(\omega)$ is, restricted to some set of the form $\{Y \mid Y \subseteq X\}$, primitive recursive in a real. In Section 7 this theorem is used to show that the strong form of the axiom of determinacy implies that if A is happy then $\mathcal{P}(\omega)\setminus A$ is not tall, and hence by Proposition 0.7 there are no MAD families. Another proof of the main result of [18] is given. Section 8 contains a further application of the theorem of Section 6 to the generic reals studied in Section 2 and the proof of my remark quoted in [8]. Finally in Section 9 the notion of a *moderately happy family* is introduced and used to perform certain constructions, announced in [16], of ultrafilters on ω with unusual properties. The paper closes with attributions and acknowledgements.

1. Proof of Theorem 0.13

The proof given here of Theorem 0.13 is modelled on the classical proof that all Σ_1^1 sets of real numbers are Lebesgue measurable. The author was inspired to seek such a proof by a proof by a conversation with Moschovakis at the Cambridge Summer School in Logic in 1971.

The principal ingredients of the classical argument are a σ-algebra $\mathscr{C}$ of subsets of $\mathscr{P}(\omega)$ and a σ-ideal $\mathscr{I}$ in $\mathscr{C}$ such that the factor algebra $\mathscr{C}/\mathscr{I}$ satisfies the countable chain condition, which is to say that there is no uncountable family $\{\mathscr{A}_\zeta \mid \zeta < \aleph_1\}$ of elements of $\mathscr{C}$ with no $\mathscr{A}_\zeta \in \mathscr{I}$ but with $\mathscr{A}_\zeta \cap \mathscr{A}_\xi \in \mathscr{I}$ for all $\zeta < \xi < \aleph_1$. The classical proof, as given for example in [1], shows that for such a $\mathscr{C}$ and $\mathscr{I}$, if $\mathscr{C}$ contains every open set, and thus every Borel set, then every analytic set is equal to a member of $\mathscr{C}$ modulo a member of $\mathscr{I}$.

Examination of the first version of the arguments of Galvin and Prikry in their paper [7] that preceded Silver's paper [28] suggests candidates for $\mathscr{C}$ and $\mathscr{I}$. With a view to extending Theorem 0.13 we give the definition in more generality than is necessary for its proof.

Let A be a family of subsets of ω such that $\mathscr{P}(\omega)\setminus A$ is a free ideal. A partial ordering $\boldsymbol{P}_A$ is associated with A as follows:

1.0. Definition. A *condition* is a pair $\langle s, S\rangle$ where $S \in A$ and $|s| \leqslant \bigcap S$.

The set of conditions is denoted by P_A and is partially ordered by setting

$$\langle s, S\rangle \leqslant \langle t, T\rangle \leftrightarrow_{\mathrm{df}} t \subseteq s \ \&\ S \cup (s\setminus t) \subseteq T.$$

Hence if $\langle s, S\rangle \leqslant \langle t, T\rangle$, then $s = t \cap |s|$. When in Section 2 we come to consider $\boldsymbol{P}_A =_{\mathrm{df}} (P_A, \leqslant)$ as a notion of forcing, the interpretation of the condition $\langle s, S\rangle$ on the generic subset X of ω will be that $s \subseteq X \subseteq s \cup S$. This sort of forcing in the context of measurable cardinals was first considered by Prikry [24]. For the nonce it is convenient to use the vocabulary of Cohen's method and say for $\langle s, S\rangle \in P_A$ and $B \subseteq \mathscr{P}(\omega)$ that

1.1. Definition. $\langle s, S\rangle$ *forces* B iff $\forall X\, (s \subseteq X \subseteq s \cup S \rightarrow X \in B)$,

1.2. Definition. $\langle s, S\rangle$ *decides* B iff $\langle s, S\rangle$ forces B or $\langle s, S\rangle$ forces $\mathscr{P}(\omega)\setminus B$.

Because this is forcing in the real world rather than in some Boolean extension we modify the usual symbolism by writing $\langle s, S\rangle \Vdash_R B$ and $\langle s, S\rangle \|_R B$ for the two notions. It is readily checked that if $\langle s, S\rangle \Vdash_R B$ and $\langle t, T\rangle \leqslant \langle s, S\rangle$, then $\langle t, T\rangle \Vdash_R B$, and that it is impossible that both $\langle s, S\rangle \Vdash_R B$ and $\langle s, S\rangle \Vdash_R \mathscr{P}(\omega)\setminus B$. From now on it will be convenient to use the familiar abbreviations $\forall w : \in A\, \mathfrak{A}$ for $\forall w\, (w \in A \rightarrow \mathfrak{A})$, $\exists w : \in A\, \mathfrak{A}$ for $\exists w\, (w \in A\ \&\ \mathfrak{A})$, $\forall w : \subseteq A\, \mathfrak{A}$ for $\forall w\, (w \subseteq A \rightarrow \mathfrak{A})$ etc.

1.3. Definition.

$$\mathscr{C}_A =_{df} \{B \mid B \subseteq \mathscr{P}(\omega) \ \&\ \forall \langle s, S\rangle : \in P_A \ \exists S' : \subseteq S (S' \in A \ \&\ \langle s, S'\rangle \|_R B\}.$$

1.4. Definition.

$$\mathscr{I}_A =_{df} \{B \mid B \subseteq \mathscr{P}(\omega) \ \&\ \forall \langle s, S\rangle : \in P_A \ \exists S' : \in S (S' \in A \ \&\ \langle s, S'\rangle \Vdash_R \mathscr{P}(\omega) \setminus B\}.$$

Clearly $\mathscr{I}_A \subseteq \mathscr{C}_A$. In the special case of Example 0.2, $\mathscr{C}_H$ is the set of *completely Ramsey* subsets of $\mathscr{P}(\omega)$ in the sense in which that phrase was first used by Galvin and Prikry. It is true in that case, as will be seen when in Section 4, we generalize the results of this section, that $\mathscr{C}_H$ is a σ-algebra containing all the Borel sets (which is in effect the theorem of Galvin and Prikry) and indeed all analytic sets (Silver's theorem) and that $\mathscr{I}_H$ is a σ-ideal in $\mathscr{C}_H$: but $\mathscr{C}_H / \mathscr{I}_H$ does not satisfy the countable chain conditions, for let $\{X_v \mid v \in \mathscr{V}\}$ be an uncountable family of pairwise almost disjoint infinite subsets of ω, as constructed for example in 0.5. Put $B_v = \mathscr{P}(X_v)$. Then each $B_v \in \mathscr{C}_A$, no $B_v \in \mathscr{I}_A$, and $B_v \cap B_{v'} \in \mathscr{I}_A$ for each pair $v \neq v'$ in $\mathscr{V}$. That difficulty may be circumvented by the use of Ramsey ultrafilters. For the rest of this section let F be a Ramsey ultrafilter and $\mathscr{C}_F$ and $\mathscr{I}_F$ be defined as in 1.3 and 1.4.

1.5. Proposition (DCR). *$\mathscr{C}_F$ contains all open sets.*

Here "open" refers to the Cantor topology on 2^ω; so that B is open iff given any

$$x \in B \ \exists n : \in \omega \ \forall y : \subseteq \omega (y \cap n = x \cap n \rightarrow y \in B).$$

The process of diagonalization used in proving Proposition 0.7 is used repeatedly in proving this and other results. Let B be open. The first step is this: for each $s \in K$, pick $X_s \in F$ such that if there is a $Y \in F$ such that $|s| \leq \bigcap Y$ and $\langle s, Y\rangle \|_R B$, then $|s| \leq \bigcap X_s$ and $\langle s, X_s\rangle \|_R B$; if there is no such Y, then let X_s be $\omega \setminus |s|$.

As steps of that type will be frequent, let us abbreviate it by saying "for each $s \in K$, pick $X_s \in F$ such that if possible $\langle s, X_s\rangle \|_R B$."

Let $X \in F$ diagonalize $\{X_s\}_{s \in K}$. Then for all $t \subseteq X$, if

$$\exists Y (\langle t, Y\rangle \in P_F \ \&\ \langle t, Y\rangle \|_R B)$$

then $\langle t, X \setminus |t|\rangle \|_R B$; for $X \setminus |t| \subseteq X_t$, so $\langle t, X \setminus |t|\rangle \leq \langle t, X_t\rangle$ and as $\langle t, X_t\rangle \|_R B$, it follows that $\langle t, X \setminus |t|\rangle \|_R B$. Moreover, as F is a filter, there cannot be both an $X \in F$ such that $\langle t, X \setminus |t|\rangle \Vdash_R B$ and a $Y \in F$ with $\langle t, Y \setminus |t|\rangle \Vdash_R \mathscr{P}(\omega) \setminus B$, for then $\langle t, (X \cap Y) \setminus |t|\rangle$ would achieve the impossible by forcing both B and $\mathscr{P}(\omega) \setminus B$.

For $t \subseteq X$, define

$$\phi(t) = 0 \quad \text{if } \langle t, X \setminus |t|\rangle \Vdash_R B;$$

$$\phi(t) = 1 \quad \text{if } \langle t, X \setminus |t|\rangle \Vdash_R \mathscr{P}(\omega) \setminus B;$$

$$\phi(t) = 2 \quad \text{otherwise.}$$

We assert that

1.6. $\forall t :\subseteq X (\phi(t) = 2 \to \{n \mid n \in X \setminus |t| \ \& \ \phi(t \cup \{n\}) = 2\} \in F)$.

For set $T_i = \{n \mid n \in X \setminus |t| \ \& \ \phi(t \cup \{n\}) = i\}$. $X \setminus |t|$ is the disjoint union of T_0, T_1 and T_2, so precisely one is in F.

$$\{T \mid t \subseteq T \subseteq t \cup T_0\} = \bigcup_{n \in T_0} \{T \mid t \cup \{n\} \subseteq T \subseteq t \cup \{n\} \cup (T_0 \setminus |\{n,\}|)\},$$

and the right hand side of the equation is a subset of B, as for $n \in T_0$, $\langle t \cup \{n\}, X \setminus |t \cup \{n\}| \rangle \Vdash_R B$. Similarly

$$B \cap \{T \mid t \subseteq T \subseteq t \cup T_1\} = 0;$$

but $\phi(t) = 2$ means that for no $S \in F$ does $\langle t, S \rangle$ decide B, and so neither T_0 nor T_1 can be in F: so $T_2 \in F$. (1.6) is thus proved.

Suppose now that $\phi(0) = 2$: for each $t \in K$ let $Y_t = X \setminus |t|$ if $\phi(t) \neq 2$ or $t \not\subseteq X$, and let

$$Y_t = \{n \mid n \in X \setminus |t| \ \& \ \phi(t \cup \{n\}) = 2\}$$

if $t \subseteq X$ and $\phi(t) = 2$. By (1.6) each $Y_t \in F$. Let $Y \in F$ diagonalize $\{Y_t\}_{t \in K}$. We assert that

1.7. $\forall t \subseteq Y \qquad \phi(t) = 2$,

for let $t \subseteq Y$ be a counterexample to (1.7) with $\bar{\bar{t}}$ minimal. $t \neq 0$, as we are assuming that $\phi(0) = 2$: let $n = \max t$ and put $s = t \setminus \{n\}$. Then $\phi(s) = 2$ by the minimality of $\bar{\bar{t}}$; $n \in Y \setminus |s| \subseteq Y_s$, so $\phi(s \cup \{n\}) = 2$, that is, $\phi(t) = 2$, contradicting the choice of t.

The assertion (1.7) rapidly leads to a contradiction: for as $\langle 0, Y \rangle$ does not decide B, there is a $Z \subseteq Y$ with $Z \in B$. But then as B is open, there is an n such that putting $t' = n \cap Z$, we have $\langle t', \omega \setminus |t'| \rangle \Vdash_R B$. For such a t', $\phi(t') = 0$, contradicting (1.7) as $t' \subseteq Y$. Thus the hypothesis on which (1.7) rests, namely that $\phi(0) = 2$, is false and so $\phi(0) = 0$ or 1. That is to say, we have proved that if B is open, then $\exists X :\in F \langle 0, X \rangle \|_R B$. An easy modification of the foregoing argument shows that if B is open then for any $s \in K \ \exists X :\in F(|s| \leq \bigcap X$ and $\langle s, X \rangle \|_R B)$; and thus $B \in \mathscr{C}_F$, as required. $\square$

1.8. Proposition (DCR). *If for each* $i \in \omega$, $B_i \in \mathscr{C}_F$ *then* $\bigcup\{B_i \mid i < \omega\} \in \mathscr{C}_F$.

Proof. Fix $\langle s, S \rangle \in P_F$. For each $t \in K$ and $i \leq |t|$ pick $S_t^i \in F$ such that if $t \subseteq S$, then $S_t^i \subseteq S \setminus |t|$ and $\langle s \cup t, S_t^i \rangle \|_R B_i$, and if $t \not\subseteq S$, then $S_t^i = S \setminus |t|$. Put

$$S_t = \bigcap\{S_t^i \mid i \leq |t|\}.$$

Then for all $t, S_t \in F$ and $S_t \subseteq S \setminus |t|$. Let $T \in F$ diagonalize $\{S_t \mid t \in K\}$. Then $T \subseteq S$.

Suppose i and X are such that $s \subseteq X \subseteq s \cup T$ and $X \in B_i$. Let $n \in T$ be greater than i, and let $t = X \cap (n+1)$. Then $|t| = n+1$; so $i < |t|$ and thus $\langle s \cup t, S_t \rangle \vdash_R B_i$, as

$$t \subseteq T,\ s \cup t \subseteq X \subseteq s \cup t \cup X \setminus |t| \subseteq s \cup t \cup T \setminus |t| \subseteq s \cup t \cup S_t,$$

and we know that $\langle s \cup t, S_t \rangle \|_R B_i$: but then $\langle s \cup T, T \setminus |t| \rangle \Vdash_R B_i$: and so

$$\langle s \cup t, T \setminus |t| \rangle \Vdash_R \bigcup_{i<\omega} B_i.$$

Thus we have shown that for all $X \subseteq T$, if $s \cup X \in \bigcup_{i<\omega} B_i$ then for some n,

$$\langle s \cup t, T \setminus |t| \rangle \Vdash_R \bigcup_{i<\omega} B_i$$

where $t = n \cap X$: but that shows that

$$B =_{\mathrm{df}} \{X \mid X \subseteq T \rightarrow s \cup X \in \bigcup B_i\}$$

is an open subset of $\mathcal{P}(\omega)$ and so by Proposition 1.5 there is a $T' \subseteq T$ such that $\langle s, T' \rangle \|_R B$. But then $\langle s, T' \rangle \|_R \bigcup_{i<\omega} B_i$. $\square$

The next proposition collects four trivialities.

1.9. Proposition. *If $B \in \mathscr{C}_F$ then $\mathcal{P}(\omega) \setminus B \in \mathscr{C}_F$; $\mathscr{I}_F \subseteq \mathscr{C}_F$; if $B \in \mathscr{I}_F$ and $D \subseteq B$ then $D \in \mathscr{I}_F$; if $B \in \mathscr{C}_F$ and the symmetric difference $B \Delta D$ of B and D is in $\mathscr{I}_F$ then D is in $\mathscr{C}_F$.*

1.10. Proposition (DCR). *If $B_i \in \mathscr{I}_F$ for each $i < \omega$ then $\bigcup_{i<\omega} B_i \in \mathscr{I}_F$.*

Proof. By Propositions 1.8 and 1.9, $\bigcup_{i<\omega} B_i \in \mathscr{C}$. Let $\langle s, S \rangle$ be given. Pick S_t as in the proof of Proposition 1.8, but this time requiring that for

$$t \leqslant S,\ i \leqslant |t|,\ \langle s \cup t, S_t \rangle \Vdash_R \mathcal{P}(\omega) \setminus B_i.$$

Let T diagonalize $\{S_t\}_{t \in K}$. Then

$$\langle s, T \rangle \Vdash_R \mathcal{P}(\omega) \setminus \bigcup_{i<\omega} B_i,$$

for if i and X are such that $s \leqslant X \leqslant s \cup T$ and $X \in B_i$, then as before a t may be found of the form $X \cap (n+1)$ such that $\langle s \cup t, S_t \rangle \Vdash_R B_i$, contradicting $B_i \in \mathscr{I}_F$. $\square$

1.11. Proposition. *There is no uncountable set $\{B_v \mid v \in \mathscr{V}\}$ such that each $B_v \in \mathscr{C}_F \setminus \mathscr{I}_F$ and for $v \neq v'$, $B_v \cap B_{v'} \in \mathscr{I}_F$.*

Proof. Let $\{B_v \mid v \in \mathscr{V}\}$ be a counterexample. For $v \in \mathscr{V}$ let S_v be the first, in some natural enumeration of K, element of K such that for some $S \in F$, $\langle s_v, S \rangle \Vdash_R B_v$: such an s_v exists as B_v is not in $\mathscr{I}_F$. As K is countable and $\mathscr{V}$ is not, the map $v \mapsto s_v$

is not 1–1: hence there are v and v' in $\mathcal{V}$ with $v \neq v'$ and $s_v = s_{v'}$. Let $S, T \in F$ be such that $\langle s_v, S\rangle \Vdash_R B_v$ and $\langle s_v, T\rangle \Vdash_R B_{v'}$. Then $\langle s_v, S \cap T\rangle \Vdash_R B_v \cap B_{v'}$, contradicting the hypothesis that for $v \neq v'$, $B_v \cap B_{v'} \in \mathcal{I}_F$. □

It is emphasised that the axiom of choice was not used in that argument. Putting Propositions 1.5, 8, 9, 10 and 11 together we obtain:

1.12. Theorem (DCR). *Let F be a Ramsey ultrafilter and $\mathcal{C}_F$, $\mathcal{I}_F$ be defined as in* 1.3 *and* 1.4. *Then $\mathcal{C}_F$ is a σ-algebra, $\mathcal{I}_F$ is a σ-ideal in $\mathcal{C}_F$, and the quotient algebra $\mathcal{C}_F/\mathcal{I}_F$ satisfies the countable chain condition. Further $\mathcal{C}_F$ contains all open sets, and hence all Borel sets.*

We now follow the classical proof, as expounded in [1] p. 53, in using Lusin sieves to show that

1.13. Corollary (DCR). *$\mathcal{C}_F$ is closed under the operation $(\mathcal{A})$ and hence contains all Σ^1_1 sets.*

The second clause of 1.13 is the required Theorem 0.13. The following abbreviation will become increasingly useful in later sections.

1.14. Definition. $[s, S] =_{\mathrm{df}} \{X \mid s \subseteq X \subseteq s \cup S\}$.

1.15. Proposition (DCR). *If $A \in \mathcal{C}_F$ there is a Borel set $B \in \mathcal{C}_F$ with $B \subseteq A$ and $A \setminus B \in \mathcal{I}_F$.*

Proof. For each $s \in K$ pick $S_s \in F$, such that $\langle s, S_s\rangle \|_R A$. Set

$$B = \bigcup\{[s, S_s] \mid \langle s, S_s\rangle \Vdash_R A\}.$$

Then $B \subseteq A$, as saying $\langle s, T\rangle \Vdash_R D$ is equivalent to saying $[s, T] \subseteq D$. B is a Borel set, being the union of countably many closed sets. Finally, given $\langle t, T\rangle$ either

$$\langle t, S_t \cap T\rangle \Vdash_R \mathcal{P}(\omega) \setminus A \quad \text{or} \quad \langle t, S_t \cap T\rangle \Vdash_R B,$$

so $A \setminus B \in \mathcal{I}_F$. □

In the following discussion, which establishes Corollary 1.13, the axiom of choice is not used save in a trivial reduction and in assuming that ω_1, the least uncountable ordinal is not the supremum of a countable sequence of countable ordinals; but DCR will suffice for both.

Let Q be a countable linearly ordered set and $\mathcal{S}$ a subset of $Q \times \mathcal{P}(\omega)$. For each $x \subseteq \omega$, let $R_x = \{t \mid \langle t, x\rangle \in \mathcal{S}\}$. We define the *inner set determined by the sieve $\mathcal{S}$*, $E(\mathcal{S})$, to be the set of those $x \subseteq \omega$ such that R_x is ill-ordered (that is, is not well-ordered) by the ordering inherited from that of Q, and the *outer set determined* *by the sieve* $\mathcal{S}$, $\bar{E}(S)$, to be $\{x \mid R_x$ is well-ordered$\}$. For $q \in Q$ we define $\mathcal{S}_q = \{x \mid \langle q, x\rangle \in \mathcal{S}\}$. For $x \subseteq \omega$ we define $p(x)$ to be the ordinal of the maximum well-ordered initial segment of R_x. $p(x)$ will be countable as Q is. For each countable ordinal ζ we define

$$E_\zeta(\mathcal{S}) = \{x \mid x \in E(\mathcal{S}) \quad \text{and} \quad p(x) = \zeta\}$$

and

$$\bar{E}_\zeta(\mathcal{S}) = \{x \mid x \in \bar{E}(\mathcal{S}) \quad \text{and} \quad p(x) = \zeta\}.$$

Then $E(\mathcal{S})$ is the disjoint union of the $E_\zeta(\mathcal{S})$, and $\bar{E}(\mathcal{S})$ of the $\bar{E}_\zeta(\mathcal{S})$. We are going to prove the following

1.16. Proposition. *Let* $\mathcal{S}$ *be a sieve such that each* $\mathcal{S}_q \in \mathcal{C}_F$. *Then* $E(\mathcal{S})$ *and* $\bar{E}(\mathcal{S})$ *are both in* $\mathcal{C}_F$.

Proof. As $\mathcal{C}_F$ is closed under complements, it is enough to prove that $E(\mathcal{S})$ is in $\mathcal{C}_F$.

We first make the trivial reduction. For each q pick a Borel set we shall somewhat prematurely call $\mathcal{S}'_q$ such that $\mathcal{S}'_q \subseteq \mathcal{S}_q$ and $\mathcal{S}_q \setminus \mathcal{S}'_q \in \mathcal{I}_F$. Such Borel sets exist by Proposition 1.15. DCR suffices for their choice as Q is countable and Borel sets can be coded by reals. Now set $\mathcal{S}' = \bigcup\{\langle q, x\rangle \mid x \in \mathcal{S}'_q\}$. Then $\mathcal{S}'$ is a sieve which is a subset of $\mathcal{S}$, so $E(\mathcal{S}') \subseteq E(\mathcal{S})$. As

$$E(\mathcal{S}) \setminus E(\mathcal{S}') \subseteq \bigcup\{\mathcal{S}_q \setminus \mathcal{S}'_q \mid q \in Q\},$$

which is in $\mathcal{I}_F$, it is enough to prove that $E(\mathcal{S}') \in \mathcal{C}_F$.

Given a sieve $\mathcal{T}$ we define the derived sieve $\mathcal{T}^-$ to be the result of removing from $\mathcal{T}$ all pairs $\langle q, x\rangle$ where q is the *first* point (in the linear ordering of Q) of R_x. Such a q will exist for given x if and only if $p(x) > 0$. Note that $E(\mathcal{T}^-) = E(\mathcal{T})$.

Define the sequence $\mathcal{S}^\zeta$ for $\zeta \in \omega_1$ of sieves by

$$\mathcal{S}^0 = \mathcal{S}'; \quad \mathcal{S}^{\zeta+1} = (\mathcal{S}^\zeta)^-; \quad \mathcal{S}^\lambda = \bigcap\{\mathcal{S}^\zeta \mid \zeta < \lambda\},$$

for limit ordinals λ. Note that for each $q \in Q$, and $\zeta < \xi < \omega_1$, $\mathcal{S}^\zeta_q \supseteq \mathcal{S}^\xi_q$. Further each $\mathcal{S}^\zeta_q$ is a Borel set, by the lemma on page 50 of [1]. As $\mathcal{C}_F / \mathcal{I}_F$ satisfies the countable chain condition, there is a $\zeta < \omega_1$ such that for

$$\zeta < \xi < \omega_1, \quad \mathcal{S}^\zeta_q \setminus \mathcal{S}^\xi_q \in \mathcal{I}_F.$$

Let ζ_q be the least such. By the regularity of ω_1, there is an $\eta < \omega_1$ which is greater than each ζ_q. Put

$$A = \bigcup\{\mathcal{S}^\eta_q \setminus \mathcal{S}^{\eta+1}_q \mid q \in Q\} \quad A \in \mathcal{I}_F \qquad E(\mathcal{S}') = E(\mathcal{S}^\eta).$$

If $x \in E_\xi(\mathcal{S}^\eta)$ with $\xi > 0$ then $x \in A$; so $E(\mathcal{S}^\eta) = E_0(\mathcal{S}^\eta) \cup A_0$ where $A_0 \subseteq A$, and so $A_0 \in \mathcal{I}_F$. $E_0(\mathcal{S}^\eta)$ is a Borel set by the same lemma in [1]. □

It is a trivial matter using the Kleene–Brouwer ordering to obtain Σ^1_1 sets and more generally applications of the operation $(\mathscr{A})$ to systems of sets in $\mathscr{C}_F$ as inner sets determined by sieves $\mathscr{S}$ with each $\mathscr{S}_q \in \mathscr{C}_F$ and Q an appropriate countable set. Thus the proof of Corollary 1.13 and Theorem 0.13 is complete.

2. P_F generic reals

We now study the partial ordering P_F, where F is a Ramsey ultrafilter, as a notion of forcing. Recall that a subset Δ of P_F is *dense closed* if for all $p \in P_F$ there is a $q \in \Delta$ with $q \leqslant p$ and for all $p \in \Delta$ and all $q \in P_F$, if $q \leqslant p$ then $q \in \Delta$; and that if M is a transitive model of (say) ZF + DCR, $F \in M$ is a Ramsey ultrafilter in M and P_F^M the corresponding partial ordering in M, then a subset x of ω is P_F *generic over* M if for every dense closed subset $\Delta \in M$ of P_F^M there is a condition $\langle s, S\rangle \in \Delta$ with $s \subseteq x \subseteq s \cup S$.

The principal result of this section is

2.0. Theorem. *Let M be a transitive model of* ZF + DCR, *which may be either a set or a class, let $F \in M$ be in M a Ramsey ultrafilter, and let $x \subseteq \omega$. Then x is P_F generic over M if and only if x is infinite but for each $X \in F$, $x \smallsetminus X$ is finite.*

The reader will notice the similarity of this theorem and its proof to the author's characterization [17] of sequences generic with respect to the conditions used by Prikry for changing the cofinality of a measurable cardinal to ω. Indeed, the two theorems were proved together.

Before taking the first step, which we couch as a theorem of ZF + DCR, towards proving Theorem 2.0, we make two definitions.

2.1. Definition. We write s *in* t to mean that s is an initial segment of t, that is, that $s = t \cap |s|$; similarly we write

$$s \text{ in } X =_{\mathrm{df}} s = X \cap |s|.$$

2.2. Definition. Let F be a Ramsey ultrafilter, Δ a dense closed subset of P_F, and $s \in K$. We say X *captures* (s, Δ) if

$$X \in F, |s| \leqslant \bigcap X \quad \text{and} \quad \forall Y: \subseteq X\, \exists t: \text{in } Y\ (\langle s \cup t, X \smallsetminus |t|\rangle \in \Delta).$$

2.3. Proposition (DCR). *Let F be a Ramsey ultrafilter and Δ a dense closed subset of P_F. Then for all s there is an X which captures (s, Δ).*

Proof. Put $Z = \omega \smallsetminus |s|$. For $t \in K$ choose $Y_t \in F$ such that $\langle s \cup t, Y_t\rangle \in \Delta$ if possible; otherwise set $Y_t = Z \smallsetminus |t|$. Let $Y \in F$ diagonalize $\{Y_t\}_{t \in K}$; then for all $t \subseteq Y$, if there is a $Y' \in F$ such that $\langle s \cup t, Y'\rangle \in \Delta$, it must be that $\langle s \cup t, Y \smallsetminus |t|\rangle \in \Delta$. Let B be the set

$$\{y \mid y \subseteq \omega \mathbin{\&} (y \subseteq Y \rightarrow \exists t : in\ y\ \langle s \cup t, Y \setminus |t| \rangle \in \Delta)\}.$$

Then B is open in the Cantor topology, so by Proposition 1.5 there is an $X' \in F$ such that $X' \|_R B$. Put $X = X' \cap Y$. As Δ is dense,

$$\exists t' : \subseteq X\ \exists X'' : \subseteq X (X'' \in F \mathbin{\&} \langle s \cup t', X'' \rangle \leqslant \langle s, X \rangle \mathbin{\&} \langle s \cup t', X'' \rangle \in \Delta),$$

hence

$$\langle s \cup t', Y \setminus |t'| \rangle \in \Delta, \quad t' \cup X'' \in B$$

and $t' \cup X'' \subseteq X \subseteq X'$; so in fact $X' \Vdash_R B$ rather than $X' \Vdash_R \mathscr{P}(\omega) \setminus B$. We assert that X captures (s, Δ): for $X \in F$ and given $X'' \subseteq X$ there is a t *in* X'' such that $\langle s \cup t, Y \setminus |t| \rangle \in \Delta$, so $\langle s \cup t, X \setminus |t| \rangle \in \Delta$ as Δ is closed. □

Proof of Theorem 2.0. First suppose that x is $\boldsymbol{P}_F$ generic over M. For each $n < \omega$, $\{\langle s, S \rangle \mid S \in F \mathbin{\&} \bar{\bar{S}} \geqslant n\}$ is dense closed and in M, and so $\bar{\bar{x}} \geqslant n$: thus x is infinite. If though $X \in F$, $\{\langle s, S \rangle \mid S \in F \mathbin{\&} S \subseteq X\}$ is dense closed and in M, and so for some such $\langle s, S \rangle$, $s \subseteq x \subseteq s \cup S$, whence $x \setminus X \subseteq s$ which is finite.

Now suppose that x is infinite and that for each $X \in F$, $x \setminus X$ is finite. Let $\Delta \in M$ be dense closed. Working in M pick for each $s \in K$ an $X_s \in F$ that captures (s, Δ). Let $X \in K$ diagonalise $\{X_s\}_{s \in K}$. $x \setminus X$ is finite, so let $n \in x$ be such that $x \setminus X \subseteq x \cap n$; put $s = x \cap (n+1)$. $\sup s \in X$, so $x \setminus |s| \subseteq X \setminus |s| \subseteq X_s$; so in M, $X \setminus |s|$ captures (s, Δ). Hence the following statement is true in M:

2.4. $\qquad \forall Y : \subseteq X \setminus |s|\ \exists t : in\ Y\ \langle s \cup t, X \setminus |s \cup t| \rangle \in \Delta.$

But let

$$\mathscr{X} = \{t \mid t \subseteq X \setminus |s| \mathbin{\&} \langle s \cup t, X \setminus |s \cup t| \rangle \notin \Delta\}$$

and give $\mathscr{X}$ the partial ordering $t < t' \leftrightarrow t'\ in\ t$ and $t' \neq t$. Then the relation $\langle \mathscr{X} < \rangle \in M$; and (2.4) is equivalent to saying that $\langle \mathscr{X} < \rangle$ is well-founded: hence by an argument due to Mostowski the above statement is true in the real world, so $\exists t$ *in* $x \setminus |s|$ such that $\langle s \cup t, X \setminus |s \cup t| \rangle \in \Delta$: but $s \cup t \subseteq X \setminus |s \cup t|$. Thus x is indeed $\boldsymbol{P}_F$ generic over M. □

2.5. Corollary. *If X is $\boldsymbol{P}_F$ generic over M and $Y \subseteq X$ then Y is also $\boldsymbol{P}_F$ generic over M.*

Using the notion of capturing defined in 2.2 we can establish convenient criteria for membership of $\mathscr{C}_F$ and $\mathscr{I}_F$:

2.6. Proposition (DCR). *Let F be a Ramsey ultrafilter and let $\mathscr{C}_F$ be defined as in 1.3. Then $A \in \mathscr{C}_F$ if and only if the set $\{p \mid p \in \boldsymbol{P}_F$ and $p \|_R A\}$ is dense closed.*

Proof. If $A \in \mathscr{C}_F$ then by the definitions of $\mathscr{C}_F$ and $\|_R$, the set $\{p \mid p \in \boldsymbol{P}_F$ and $p \|_R A\}$ is dense closed.

Suppose now that A is such that $\Delta =_{df} \{p \mid p \in \boldsymbol{P}_F \mathbin{\&} p \|_R A\}$ is dense closed, and let $s \in K$. By Proposition 2.3 there is an X in F which captures (s, Δ). Let

$$B = \{Y \mid Y \subseteq \omega \setminus |s| \text{ and } \exists t\text{: in } Y\ \langle s \cup t, X \setminus |t|\rangle \Vdash_R A\}$$

and let

$$C = \{Y \mid Y \subseteq \omega \setminus |s|\ \&\ \exists t\text{: in } Y\ \langle s \cup t, X \setminus |t|\rangle \Vdash_R \mathcal{P}(\omega) \setminus A\}.$$

Then both B and C are open. There are therefore $X_1 \subseteq X$ and $X_2 \subseteq X_1$ with $X_2 \in F$, $\langle s, X_1\rangle \|_R B$ and $\langle s, X_2\rangle \|_R C$; that by Proposition 1.5. As $[X]^\omega \subseteq B \cup C$ and $B \cap C = 0$, for D equal to either B or C but not both, $\langle s, X_2\rangle \Vdash_R D$. It is readily checked that if $D = B$ then $\langle s, X_2\rangle \Vdash_R A$ and if $D = C$, $\langle s, X_2\rangle \Vdash_R \mathcal{P}(\omega) \setminus A$. Hence A is in $\mathscr{C}_F$, as required. □

2.7. Proposition (DCR). *Let F be a Ramsey ultrafilter and let $\mathcal{I}_F$ be defined as in 1.4. Then $A \in \mathcal{I}_F$ if and only if the set $\{p \mid p \in P_F\ \&\ p \Vdash_R \mathcal{P}(\omega) \setminus A\}$ is dense closed.*

Proof. The "only if" part is immediate from the definitions. Conversely, if $\{p \mid p \in P_F\ \&\ p \Vdash_R \mathcal{P}(\omega) \setminus A\}$ is dense closed, $A \in \mathscr{C}_F$ by Proposition 2.6, and for no $p \in P_F$ can $p \Vdash_R A$, so that $A \in \mathcal{I}_F$. □

Here is an appropriate place for recording a simple property of $\mathcal{I}_A$:

2.8. Proposition. *If A is a happy family, then given $B \in \mathcal{I}_A$ and $\langle s, S\rangle \in P_A$, there is an $X \in A$ such that $X \subseteq S$ and $\langle 0, s \cup X\rangle \Vdash_R \mathcal{P}(\omega) \setminus B$.*

Proof. Let the subsets of s be enumerated as $t_0, \ldots, t_{n-1}$ where $n = 2^{\bar{s}}$. Choose $X_0, \ldots, X_{n-1} \in A$ such that $S \supseteq X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{n-1}$ and for each $i < n$, $\langle t_i, X_i\rangle \Vdash_R \mathcal{P}(\omega) \setminus B$. Put $X = X_{n-1}$. Then

$$[0, s \cup X] \subseteq \bigcup\{[t_i, X_i] \mid i < n\} \subseteq \mathcal{P}(\omega) \setminus B,$$

so $\langle 0, s \cup X\rangle \Vdash \mathcal{P}(\omega) \setminus B$. □

The following is a counterpart to the lemma of Prikry quoted in [17] as Lemma 2.6.

2.9. Proposition (DCR). *Let F be a Ramsey ultrafilter, let $\mathfrak{A}$ be any sentence of the language of forcing and let $\langle s, S\rangle \in P_F$. Then there is an $S' \subseteq S$ such that $S' \in F$ and either $\langle s, S'\rangle \Vdash \mathfrak{A}$ or $\langle s, S'\rangle \Vdash \neg\mathfrak{A}$.*

Proof. Here of course "$\Vdash$" is the usual notion of forcing and not $\Vdash_R$. However the proof parallels that of Proposition 2.6, so it will only be sketched. Let

$$\Delta = \{p \mid p \Vdash \mathfrak{A} \text{ or } p \Vdash \neg\mathfrak{A}\};$$

Δ is dense closed, and so $\exists T\colon \subseteq S$ with $T \in F$ and T capturing (s, Δ). Let

$$B = \{X \mid X \subseteq \omega \setminus |s| \text{ and for some } t \text{ in } X,\ \langle s \cup t, T \setminus |t|\rangle \Vdash \mathfrak{A}\},$$

and

$$C = \{X \mid X \subseteq \omega \setminus |s| \text{ and for some } t \text{ in } X,\ \langle s \cup t, T \setminus |t|\rangle \Vdash \neg\mathfrak{A}\}.$$

Both B and C are open, and so for some $S' \subseteq T$ either $\langle s, S'\rangle \Vdash_R B$, when $\langle s, S'\rangle \Vdash \mathfrak{A}$ (for otherwise some $\langle s \cup t, T'\rangle \Vdash \neg \mathfrak{A}$ where $t \cup T' \subseteq S'$, and so $s \cup t \cup T' \notin B$) or $\langle s, S'\rangle \Vdash_R C$, when $\langle s, S'\rangle \Vdash \neg \mathfrak{A}$. □

We now give a second proof of Theorem 0.13, using Proposition 1.5 and its consequence Proposition 2.9. The only difficulty in this proof is establishing that DCR is a strong enough form of AC.

Let F be Ramsey and B a Σ^1_1 subset of $\mathcal{P}(\omega)$: say $B = \{x \mid R(x, a)\}$ where $a \subseteq \omega$ and R is Σ^1_1. Let M be a countable transitive model of an appropriate fragment of ZF + DCR with $a \in M$ such that $F \cap M$ is in M and is, in M, a Ramsey ultrafilter. Such an M may readily be found by a Löwenheim–Skolem argument, using DC. To see that DCR suffices, observe that we require that $(\mathcal{P}(\omega) \cap M, F \cap M)$ is an elementary submodel of $(\mathcal{P}(\omega), F)$ with respect to a sufficiently large class of formulae, which can be achieved by putting into M sufficiently many reals, for which DCR is enough. Let $X \in F \cap M$ be such that in M, $\langle 0, X\rangle \| \dot{R}(\dot{x}, \hat{a})$, where $\dot{x}$ is the name in the language of forcing for the proposed real $\boldsymbol{P}_{F\cap M}$ generic over M, and $\hat{a}$ is the name of a. Such an X exists by applying Proposition 2.9 with $s = 0$ inside M. Now suppose that x is a real $\boldsymbol{P}_{F\cap M}$ generic over M with $x \subseteq X$. Then

$$\forall Y : \subseteq x (x \in B \leftrightarrow Y \in B):$$

for let Y be any infinite subset of x. Then Y is also $\boldsymbol{P}_{F\cap M}$ generic over M, by Corollary 2.5 which will hold for those M that are transitive models of appropriate finite fragments of ZF + DCR, and $Y \subseteq X$; so using the absoluteness of Σ^1_1 statements and general properties of forcing we conclude that the following statements are equivalent: $x \in B$; $R(x, a)$; $(R(x, a))_{M[x]}$; $\langle 0, X\rangle \Vdash \dot{R}(\dot{x}, \hat{a})$; $(R(Y, a))_{M[Y]}$: $R(Y, a)$; $Y \in B$.

It remains therefore to show that there is an $x \in F$ which is $\boldsymbol{P}_{F\cap M}$ generic over$M$. Let $\langle X_s \mid s \in K\rangle$ enumerate the members of $F \cap M$, which is a countably infinite set; and let $x \in F$ diagonalise $\{X_s\}$. Then for each $s \in K$, $x \setminus X_s$ is finite, so by Theorem 2.0, which again will hold when M models an appropriate fragment of ZF + DCR, x is $\boldsymbol{P}_{F\cap M}$ generic over M, as required.

A modification of the above argument shows that given $\langle s, S\rangle \in P_F$ there is a $T \subseteq S$ with $T \in F$ and $\langle s, T\rangle \Vdash_R B$. Hence $B \in \mathscr{C}_F$. □

Corollary 2.5 has the following converse:

2.10. Theorem. *Let M be a transitive model, set or class, of* ZF + DCR, *and let $F \in M$ be a free ultrafilter in M. Suppose that there is an $X \subseteq \omega$ such that for all $Y \subseteq X$, Y is $\boldsymbol{P}_F$ generic over M. Then F is a Ramsey ultrafilter in M.*

Proof. Let $\pi \in M$ be such that in M, $\pi : [\omega]^2 \to 2$. We shall see that there is in F an S which is homogeneous for π. As $[\omega]^2 = ([\omega]^2)_M$ $\pi : [\omega]^2 \to 2$. By Ramsey's theorem there is a $Y \subseteq X$ such that Y is homogeneous for π; then, that being an arithmetical predicate of Y and π. "Y is homogeneous for π" is true in $M[Y]$, and hence as Y is P_F generic over M, there is an $\langle s, S\rangle \in P_F$ such that $s \subseteq Y \subseteq s \cup S$ and $\langle s, S\rangle \Vdash \dot{x}$ is homogeneous for $\hat{\pi}$. But then S is homogeneous for π. □

Thus in a sense Corollary 2.5 characterizes Ramsey ultrafilters. A precise formulation of this characterization, using the apparatus of Boolean valued models, is given as Theorem 9.31, clause (iii). We now prove another characterization of Ramsey ultrafilters, which was first stated in [16], from where the following terminology is taken.

2.11. Definition. An ideal I on ω is *gaunt* if is proper, free and a Σ^1_1 set.

2.12. Theorem (DCR). *A free ultrafilter F on ω is Ramsey if and only if $F \cap I \neq 0$ for every tall gaunt ideal I.*

Proof. Suppose F Ramsey and I a tall gaunt ideal. By Theorem 0.13 there is an $X \in F$ such that $\forall Y :\subseteq X (X \in I \leftrightarrow Y \in I)$. As I is tall, there is some $Y \subseteq X$ with $Y \in I$; and hence $X \in I \cap F$.

Conversely let F have the property in question and let $\pi : [\omega]^2 \to 2$. Let

$$I = \mathrm{id}(\{X \mid X \text{ is homogeneous for } \pi\}).$$

I is Σ^1_1; further, by Ramsey's theorem,

$$\forall Y\, \exists X (X \subseteq Y \;\&\; X \text{ homogeneous for } \pi)$$

so I is tall; if I is proper as well, then I is gaunt, and so there is an X in $F \cap I$. There is also such an X if I is improper, namely ω. In either case there are sets $X_1, \ldots, X_k$, each homogeneous for π, and $s \in K$ such that $s \cup X_1 \cup \cdots \cup X_k \in F$. As F is a free ultrafilter, one of the X_i is in F: thus F is Ramsey. □

The possibility of homogeneous sets for partitions $\pi : [\omega]^\omega \to \omega$ will be considered in Section 6. We conclude this section with remarks about another way in which Theorem 0.13 might be generalised, using MA.

Silver has shown [28] that if MA and $2^{\aleph_0} > \aleph_1$ then not only every Σ^1_1 but also every Σ^1_2 subset of $\mathscr{P}(\omega)$ is in $\mathscr{C}_H$. MA implies that Ramsey ultrafilters exist [3]; is it true that if F is Ramsey, A is Σ^1_2, and A is in $\mathscr{C}_H$ then A is in $\mathscr{C}_F$? The answer is that the hypotheses given are not enough to decide:

2.13. Theorem. (i) *If* MA *and every Σ^1_2 set is in $\mathscr{C}_\kappa$ then there is a Ramsey ultrafilter F such that every Σ^1_2 set is in $\mathscr{C}_F$.*

(ii) *If* MA, $2^{\aleph_0} > \aleph_1$ *and* $\exists x\, \aleph_1^{L[x]} = \aleph_1$, *then there is a Σ^1_2 set A and a Ramsey ultrafilter G with $A \not\in \mathscr{C}_G$.*

(iii) *If the universe is the result of collapsing a Mahlo cardinal in* L *in to $\aleph_1$ in the style of Lévy, then for every Ramsey ultrafilter F and every Σ^1_2 set A, $A \in \mathscr{C}_F$. Sadly, in this case $2^{\aleph_0} = \aleph_1$.*

(iv) *If there is a strongly inacessible Rowbottom cardinal, or if* $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ *and Chang's conjecture holds, then for every Ramsey ultrafilter F and every* Σ^1_2 *set* $A, A \in \mathscr{C}_F$.

Part (iii) of that theorem will be proved in Section 5; the rest will be proved in Section 9 using results of Booth [3], Martin and Solovay [15] and Solomon [29]. For a statement of Chang's conjecture and a definition of the notion of a Rowbottom cardinal, see Chapter 7 Section 3 of [4]. We conclude this section by sketching an alternative proof of Silver's result.

Let $A = \{x \mid R(x, a)\}$ where $a \subseteq \omega$ and R is Σ^1_2. Let $\langle s, S\rangle$ be given, and let F be, in $L[a, S]$, a Ramsey ultrafilter. Let $T \in F$ be such that in $L[a, S]$, $\langle s, T\rangle \Vdash \dot{R}(\dot{x}, \hat{a})$. As $\bar{\bar{F}} \leq \aleph_1 < 2^{\aleph_0}$ and we are assuming MA, there is an X such that for all $Y \in F$, $X \setminus Y$ is finite, by Theorem 4.10 of [3]. Such an X is $\boldsymbol{P}_F$ generic over $L[a, S]$ by Theorem 2.0; we may also suppose that $s \subseteq X \subseteq s \cup T$. Using the absoluteness of Σ^1_2 predicates we see as in the second proof of Theorem 0.13 that $\langle s, X \setminus |s|\rangle$ decides A. Hence $A \in \mathscr{C}_H$. □

3. Review of Solovay's model

In this section we list some well-known facts about Boolean valued models, and sketch briefly some ideas from Solovay [30]. For details see [30] and [31].

3.0. We use arithmetical notation for Boolean algebras, so that a Boolean algebra is a structure $\boldsymbol{B} = \langle \mathscr{B}, \boldsymbol{0}, \boldsymbol{1}, +, \cdot, - \rangle$, where $\boldsymbol{0} \cdot b = \boldsymbol{0}$, $\boldsymbol{1} + b = \boldsymbol{1}$, $\boldsymbol{0} = -\boldsymbol{1}$, etc. The canonical partial ordering of $\mathscr{B}$ is given by setting $b < c \leftrightarrow b \cdot c = b$. $\boldsymbol{B}$ is *complete* if every non-empty subset $\mathscr{X}$ of $\mathscr{B}$ has an upper bound, $\Sigma^{\boldsymbol{B}}\mathscr{X}$. If $\boldsymbol{B}$ and $\boldsymbol{C}$ are complete Boolean algebras, $\boldsymbol{B}$ is a *regular subalgebra* of $\boldsymbol{C}$, in symbols $\boldsymbol{B} \lhd \boldsymbol{C}$, if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{C}$ and for each nonempty subset $\mathscr{X}$ of B, $\Sigma^{\boldsymbol{B}}\mathscr{X} = \Sigma^{\boldsymbol{C}}\mathscr{X}$. The notion of a *regular embedding* is defined in a similar spirit.

3.1. Two elements p and q of a partial order $\boldsymbol{P} = \langle P, \leq \rangle$ are *incompatible* if there is no $r \in P$ with $r \leq p$ and $r \leq q$. $\boldsymbol{P}$ is a *suitable* partial ordering if the following three conditions hold:

(a) there is an element of P, call it $\mathbb{1}$, such that $\forall p : \in P\ p \leq \mathbb{1}$;

(b) $\forall p : \in p\ \exists q : \in P\ (p \neq q\ \&\ q \leq p)$;

(c) $\forall p : \in P\ \forall q : \in P (p \leq q$ or $\exists r : \leq p (q$ and r are incompatible$))$.

The elements of a suitable partial orering are often called *conditions*, for historical reasons.

3.2. Definition. Let $\boldsymbol{P}$ be a suitable partial ordering. The *canonical topology* on P is that with basis $\{O_p \mid p \in P\}$, where $O_p =_{\mathrm{df}} \{q \mid q \leq p\}$.

The essential feature of this topology is that given p there is a smallest open set containing p, namely O_p. Hence for $\mathscr{X} \subseteq p$, the interior of $\mathscr{X}$, int $\mathscr{X}$, is equal to $\{p \mid O_p \subseteq \mathscr{X}\}$; the closure, cl($\mathscr{X}$), to $\{p \mid O_p \cap X \neq 0\}$; and $\mathrm{int\,cl}\,\mathscr{X} = \{p \mid \forall q : \leq p\, \exists r : \leq q\; r \in \mathscr{X}\}$.

3.3. Let $(\mathscr{X}, \tau)$ be a topological space. Then the algebra of *regular open* sets of $(\mathscr{X}, \tau)$, defined by $\mathscr{B} = \{\mathscr{Y} \mid \mathscr{Y} \subseteq \mathscr{X}$ and $\mathscr{Y} = \mathrm{int\,cl}\,\mathscr{Y}\}$; $\mathbf{0} = 0$; $\mathbf{1} = \mathscr{X}$; $\mathscr{Y} + \mathscr{Z} = \mathrm{int\,cl}(\mathscr{Y} \cup \mathscr{Z})$; $\mathscr{Y} \cdot \mathscr{Z} = \mathscr{Y} \cap \mathscr{Z}$; $-\mathscr{Y} = \mathrm{int}(\mathscr{X} \setminus \mathscr{Y})$; is a complete Boolean algebra, and for $\mathscr{A} \subseteq \mathscr{B}$, $\Sigma\mathscr{A} = \mathrm{int\,cl} \bigcup \mathscr{A}$. $\boldsymbol{B} = \langle \mathscr{B}, \mathbf{0}, \mathbf{1}, +, \cdot, - \rangle$ is called the *regular open algebra* of $(\mathscr{X}, \tau)$.

3.4. Proposition. *Let $\boldsymbol{P} = \langle P, \leq \rangle$ be a suitable partial ordering. Then each O_p is a regular open set in the canonical topology, and the map $p \mapsto O_p$ embeds $\boldsymbol{P}$ as a dense subset of the regular open algebra $\boldsymbol{B}$. Further $O_1 = \mathbf{1}$.*

In the last clause "1" is used in two senses. The $\boldsymbol{B}$ of 3.4 is called the *algebra over* $\boldsymbol{P}$. A particular case is of interest:

3.5. Proposition. *Let $\boldsymbol{B}$ be an atomless possibly incomplete Boolean algebra: then $\langle \mathscr{B} \setminus \{\mathbf{0}\}, \leq \rangle$ is a suitable partial ordering; let $\boldsymbol{C}$ be the algebra over it. Then $\boldsymbol{C}$ is a complete Boolean algebra containing as a dense subalgebra an isomorphic copy of $\boldsymbol{B}$.*

The $\boldsymbol{C}$ of 3.5 is called the *regular minimal completion* of $\boldsymbol{B}$ and is characterized up to isomorphism by the last sentence of 3.5. If $\boldsymbol{B}$ is complete the embedding is onto.

3.6. Given a complete Boolean algebra $\boldsymbol{B}$, we define its associated Boolean valued universe by the recursion

$$V_0^{\boldsymbol{B}} = 0; \quad V_{\zeta+1}^{\boldsymbol{B}} = \{v \mid v \text{ is a function with domain a subset of } V_\zeta^{\boldsymbol{B}} \text{ and range a subset of } B\};$$

$$V_\zeta^{\boldsymbol{B}} = \bigcup\{V_{\xi+1}^{\boldsymbol{B}} \mid \xi < \zeta\} \quad \text{for limit } \zeta;$$

$$V^{\boldsymbol{B}} = \bigcup\{V_\zeta^{\boldsymbol{B}} \mid \zeta \text{ an ordinal}\}.$$

Note that if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{C}$ then $V^{\boldsymbol{B}} \subseteq V^{\boldsymbol{C}}$; so in particular if $\mathbf{2}$ is the two-element algebra $\{\mathbf{0}, \mathbf{1}\}$, $V^{\mathbf{2}} \subseteq V^{\boldsymbol{B}}$ for each $\boldsymbol{B}$.

3.7. We associate with $V^{\boldsymbol{B}}$ a language $\mathscr{L}^{\boldsymbol{B}}$ with the primitive predicate symbols $\boldsymbol{\in}$ and $\boldsymbol{=}$ (corresponding to $\in$ and $=$), the usual connectives, a special constant $\hat{V}$, and for each $v \in V^{\boldsymbol{B}}$ a name which we shall also denote by v. We shall not take much trouble to distinguish formulae of $\mathscr{L}^{\boldsymbol{B}}$ from assertions of the language of set theory: the context will usually do so for us: when we do, it will usually be through the convention that if $\mathfrak{A}$ is a wff of ZF, $\dot{\mathfrak{A}}$ is the corresponding expression in $\mathcal{L}^B$, the wffs of which are of course Gödelized as sets in ZF.

3.8. Definition. The embedding $\hat{}: V \to V^2$ is defined by recursion on the epsilon relation thus: $\hat{x} = \{\langle \mathbf{1}, \hat{y}\rangle \mid y \in x\}$.

3.9. A function $[\![\cdot]\!]^B$ from the sentences of $\mathcal{L}^B$ to B is defined by a schema: First $[\![v \in w]\!]^B$ and $[\![v \equiv w]\!]^B$ are defined by double recursion:

$$[\![v \equiv w]\!]^B = \Pi^B\{v(v') \Rightarrow [\![v' \in w]\!]^B \mid v' \in D(v)\} \cdot \Pi^B\{w(w') \Rightarrow [\![w' \in v]\!]^B \mid w' \in D(w)\}$$

$$[\![v \in w]\!]^B = \Sigma^B\{w(w') \cdot [\![w' \equiv v]\!]^B \mid w' \in D(w)\}.$$

Then define $[\![v \in \hat{V}]\!]^B = \Sigma^B\{[\![v \equiv \hat{w}]\!]^B \mid w \in V\}$. Finally the definition is extended to all sentences of $\mathcal{L}^B$ in the obvious way, as in [14].

3.10. Let $\mathfrak{A}$ be a Δ_0 wff in the sense of Lévy [12], with the free variables $v_1 \cdots v_n$. Then if $B \lhd C$ and $v_1 \cdots v_n$ are in V^B,

$$[\![\dot{\mathfrak{A}}(v_1 \cdots v_n)]\!]^B = [\![\dot{\mathfrak{A}}(v_1 \cdots v_n)]\!]^C.$$

Again if $\mathfrak{A}$ is a Δ_0 wff with the free variables $v_1 \cdots v_n$, then for any $v_1 \cdots v_n$,

$$\mathfrak{A}(v_1, \ldots, v_n) \leftrightarrow [\![\dot{\mathfrak{A}}(\hat{v}_1, \ldots, \hat{v}_n)]\!]^B = \mathbf{1}.$$

3.11. In V^B there are no new members of ω, and indeed no new ordinals:

$$[\![v \in \hat{\omega}]\!]^B = \Sigma^B\{[\![v \equiv \hat{n}]\!]^B \mid n \in \omega\}$$

and

$$[\![v \in \dot{On}]\!]^B = \Sigma^B\{[\![v \equiv \hat{\zeta}]\!] \mid \zeta \in On\}.$$

3.12. Proposition. *Let F_B in V^B as the function with domain $\{\hat{b} \mid b \in \mathcal{B}\}$ such that $F_B(\hat{b}) = b$ for each b. Then $[\![F_B \subseteq \hat{\mathcal{B}}]\!]^B = \mathbf{1}$; for all $b \in B$, $[\![\hat{b} \in F_B]\!]^B = b$; and $[\![F_B$ is a $\hat{V}$-complete ultrafilter in $\hat{B}]\!]^B = \mathbf{1}$.*

A generalisation of that will be useful.

3.13. Proposition. *Let $B \lhd C$. There is an $F^+_{B,C}$ in V^B such that*

$$F^+_{B,C}(\hat{c}) = [\![\hat{c} \in F^+_{B,C}]\!] = \Sigma\{b \mid b \in B \ \&\ b \leq c\}$$

and

$$[\![F^+_{B,C} \text{ is a } \hat{V} \text{ complete filter in } \hat{C}]\!]^B = \mathbf{1}.$$

If $B = C$, then $F^+_{B,C}$ is F_B. $F^+_{B,C}$ is, in V^B, the filter in C generated by F_B.

In the relative consistency results of Section 5 we shall make use of a device due to McAloon: that of considering the inner model of sets hereditarily definable from ordinal and real parameters. We define ROD to be the class of those sets v such that for some ordinal ζ and some $x \subseteq \omega$, v is definable in V_ζ, admitting x as a parameter. That this definition encompasses all those sets which naively are definable from ordinals and reals may be established using the reflection principle as illustrated in the paper of Myhill and Scott on ordinal definability [21]. We further define HROD to be the class of those sets v such that every member of the transitive closure of $\{v\}$ is in ROD.

3.14. Proposition. HROD *is an inner model, and* $\mathcal{P}(\omega) \subseteq$ HROD. *If* DCR *holds, then in* HROD, DC *holds.*

We shall use repeatedly the following trivial principle:

3.15. Proposition. *Let $\Phi(A)$ be a formula of set theory with one free variable such that every bound variable of Φ is restricted to range over ω or over $\mathcal{P}(\omega)$. Suppose that for every $A \subseteq \mathcal{P}(\omega)$ with $A \in$ ROD, $\Phi(A)$ holds. Then in* HROD, $\forall A : \subseteq \mathcal{P}(\omega) \Phi(A)$.

We now sketch a proof of a technical lemma, Theorem 3.21, from [30], reasoning in ZF + AC.

3.16. Lemma. *Let κ be a strongly inaccessible cardinal: that is, $\kappa > \omega$, κ is regular and $\lambda < \kappa \to 2^\lambda < \kappa$. Let $\boldsymbol{A} \lhd \boldsymbol{B}$, $\boldsymbol{A} \lhd \boldsymbol{C}$, where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are complete Boolean algebras of cardinality $< \kappa$. Then there is a complete Boolean algebra $\boldsymbol{D}$ of power $< \kappa$ and regular embeddings π, π^- such that the diagram below commutes.*

$$\begin{array}{ccc} \boldsymbol{B} & \overset{\pi}{\cdots\cdots>} & \boldsymbol{D} \\ \triangledown & & \wedge \\ & & \vdots \quad \pi^- \\ \boldsymbol{A} & \lhd & \boldsymbol{C} \end{array}$$

Proof. Let $\boldsymbol{E}$ in $V^{\boldsymbol{A}}$ be such that

$[\![\boldsymbol{E}$ is the algebra over the product partial ordering

$$(\boldsymbol{B}_{/F^+_{A,B}} \setminus \{0\}) \times (\boldsymbol{C}_{/F^+_{A,C}} \setminus \{0\})]\!]^{\boldsymbol{A}} = \mathbf{1},$$

and let $\boldsymbol{D}$ be the Boolean composition of $\boldsymbol{A}$ and $\boldsymbol{E}$ in the sense of Definition 5.2 of [31], so that $V^{\boldsymbol{D}}$ is "isomorphic" to $(V^{\boldsymbol{A}})^{\boldsymbol{E}}$. □

Using Lemma 3.16 and Jónsson's method for constructing homogeneous universal models, we may prove the following, first proved by explicitly exhibiting an example of the required isomorphism class of algebras:

3.17. Theorem (Jensen) (AC). *Let κ be strongly inaccessible. There is a complete Boolean algebra $\boldsymbol{B}$ of cardinality κ which is characterized up to isomorphism by the following two properties:*

(i) *$\boldsymbol{B} = \bigcup\{\boldsymbol{B}_\zeta \mid \zeta < \kappa\}$ where each $\boldsymbol{B}_\zeta$ is a complete Boolean algebra of cardinality $< \kappa$ and whenever $\zeta < \eta < \kappa$, $\boldsymbol{B}_\zeta \lhd \boldsymbol{B}_\eta \lhd \boldsymbol{B}_\kappa$, and for all limit ordinals $\lambda < \kappa$, $\boldsymbol{B}_\lambda$ is the regular minimal completion of $\bigcup\{\boldsymbol{B}_\zeta \mid \zeta < \lambda\}$;*

(ii) *for any complete Boolean algebras $\boldsymbol{A}, \boldsymbol{C}$ with $\boldsymbol{A} \lhd \boldsymbol{B}$, $\boldsymbol{A} \lhd \boldsymbol{C}$ and $\boldsymbol{C}$ of power $< \kappa$ there is a regular embedding of $\boldsymbol{C}$ into $\boldsymbol{B}$ which is the identity on $\boldsymbol{A}$.*

3.18. Definition. Write $\phi(\boldsymbol{B}, \kappa)$ if κ is strongly *inaccessible* and $\boldsymbol{B}$ has properties (i) and (ii) of Theorem 3.17.

3.19. Lemma (AC). *If $\phi(\boldsymbol{B}, \kappa)$, then $\boldsymbol{B}$ has the κ chain condition, and is homogeneous in the sense that if $\mathfrak{A}$ is any wff of the language of set theory with the free variables $v_1, \ldots, v_n$, then for all $v_1, \ldots, v_n$,*

$$[\![\dot{\mathfrak{A}}(\hat{v}_1, \ldots, \hat{v}_n)]\!]^{\boldsymbol{B}} = \boldsymbol{0} \text{ or } \boldsymbol{1}.$$

3.20. Lemma (AC). *Suppose that $\phi(\boldsymbol{B}, \kappa)$, $\boldsymbol{A} \lhd \boldsymbol{B}$ and $\boldsymbol{A}$ is of power $< \kappa$. Then*

$$[\![\dot{\phi}(\hat{\boldsymbol{B}}_{/F^+_{\boldsymbol{A},\boldsymbol{B}}}, \hat{\kappa})]\!]^{\boldsymbol{A}} = \boldsymbol{1}.$$

3.21. Theorem (Solovay). *Let $\mathfrak{A}(x, a, \xi)$ be a wff with the free variables shown; let $\mathfrak{B}(x, a, \xi, \kappa)$ be the wff "for all $\boldsymbol{B}$, if $\phi(\boldsymbol{B}, \kappa)$, then $[\![\dot{A}(\hat{x}, \hat{a}, \hat{\xi})]\!]^{\boldsymbol{B}} = \boldsymbol{1}$"; and let $\mathfrak{C}(\kappa)$ be the wff*

$$\forall a : \subseteq \omega\ \forall \xi : \in On(\{x \mid x \subseteq \omega\ \&\ \mathfrak{A}(x, a, \xi)\} =$$
$$= \{x \mid x \subseteq \omega\ \&\ (\mathfrak{B}(x, a, \xi, \kappa))_{L[x,a]}\})$$

Then the following is provable in ZF + $V = L$:

$$\text{if } \phi(\boldsymbol{A}, \kappa), \text{ then } [\![\dot{\mathfrak{C}}(\hat{\kappa})]\!]^{\boldsymbol{A}} = \boldsymbol{1}.$$

It is immediate from that and some trivial coding of pairs of ordinals as single ordinals that

3.22. Theorem (Solovay). *If $V = L$ and $\phi(\boldsymbol{B}, \kappa)$, then in $V^{\boldsymbol{B}}$ with truth value $\boldsymbol{1}$, every* ROD *set of reals is of the form $\{x \mid (\mathfrak{D}(x, a, \zeta))_{L[x,a]}\}$ for some ordinal ζ and some $a \subseteq \omega$.*

3.23. Proposition (AC). *If $\phi(\boldsymbol{B}, \kappa)$, then $[\![\hat{\kappa} = \aleph_1]\!]^{\boldsymbol{B}} = \boldsymbol{1}$, and*

$$[\![\textit{for all } x \subseteq \hat{\omega}, \hat{\kappa} \textit{ is inaccessible in } L[x]]\!]^{\boldsymbol{B}} = \boldsymbol{1}.$$

The proof of that rests on two points and on Lemma 3.20: if $\zeta < \kappa$, then the canonical collapsing algebra for making ζ countable is of power $< \kappa$ and so isomorphic to a regular subalgebra of $\boldsymbol{B}$; and if $[\![x \dot{\subseteq} \hat{\omega}]\!]^{\boldsymbol{B}} = \mathbf{1}$ then there is a $\zeta < \kappa$ and a $y \in V^{\boldsymbol{B}_\zeta}$ (where $\boldsymbol{B}$ is the union of the ascending regular chains $\boldsymbol{B}_\xi\,(\xi < \kappa)$) with $[\![x \equiv y]\!]^{\boldsymbol{B}} = \mathbf{1}$: let ζ be large enough so that $\boldsymbol{B}_\zeta$ contains each truth value $[\![\hat{n} \in x]\!]$ for $n \in \omega$.

We end this review by proving something relevant to Section 7:

3.24. Proposition (AC). *If $\phi(\boldsymbol{B}, \kappa)$ and $V = L$, then*

$$[\![\text{ACR } \textit{is false in } \text{HROD}]\!]^{\boldsymbol{B}} = \mathbf{1}.$$

Proof. In $V^{\boldsymbol{B}}$, let $\mathscr{R}$ be $\{\langle x, y\rangle \mid y$ is Cohen generic over $L[x]\}$. $\mathscr{R}$ is in fact a Π^1_2 set. It is trivial from 3.23 that $\forall x\, \exists y\, \langle x, y\rangle \in \mathscr{R}$. We assert that there is no ROD function $E : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ such that for all x, $\langle x, E(x)\rangle \in \mathscr{R}$. For let E be a counterexample: by 3.22 there is a wff $\mathfrak{D}(a, x, y, \zeta)$ such that for some $a \subseteq \omega$ and $\zeta \in On$, we have that for all x, y, $y = E(x)$ if and only if $(\mathfrak{D}(a, x, y, \zeta))_{L[a,x,y]}$. Let $y_0 = E(a)$; as y_0 is Cohen generic over $L[a]$, some Cohen condition forces the statement $\dot{\mathfrak{D}}(\hat{a}, \hat{a}, \dot{y}, \hat{\zeta})$; but then there are lots of Cohen generic reals $y_1 \neq y_0$ which satisfy the same condition, and for which we could have $(\mathfrak{D}(a, a, y_1, \zeta))_{L[a,y_1]}$, contradicting the uniqueness of y_0. An application of Proposition 3.15 completes the proof, as $\mathscr{R}$ is in HROD. □

That argument shows incidentally that it is consistent with ZFC that both Π^1_3 and Σ^1_3 uniformisation fail.

4. The reduction of happy families to Ramsey ultrafilters

We present in this section a general method for reducing the problem of proving combinatorial results about happy families to the special case when they are Ramsey ultrafilters. We give first a discussion leading to a proof of the main technical result, Proposition 4.2, and then illustrate its use by generalising Theorem 0.13 and by showing that no $\boldsymbol{\Sigma}^1_1$ set can be a MAD family.

In this technical discussion we reason in ZF + DCR. Let A be a happy family; put $I = \mathscr{P}(\omega) \setminus A$, and let $\boldsymbol{B}$ be the regular minimal completion of the quotient algebra $\mathscr{P}(\omega)/I$. $\boldsymbol{B}$ may be regarded as the algebra over the partial order $\boldsymbol{P}$ of the non-zero elements of $\mathscr{P}(\omega)/I$ ordered by inclusion mod I.

4.0. Lemma (DCR). *Given a sequence $p_0, p_1, p_2 \cdots$ of elements of P such that $p_i > p_{i+1}$, there is a $q \in P$ such that for all i, $q \leq p_i$. Hence*

$$[\![\widehat{\mathscr{P}(\omega)} \equiv \mathscr{P}(\omega)]\!]^{\boldsymbol{B}} = \mathbf{1}.$$

Proof. Using DCR pick $X_i \in p_i$; then $X_{i+1} \setminus X_i \in I$ for each i. Put $Y_s = \bigcap\{X_j \mid j \leq |s|\}$. Then $\mathrm{fil}\{Y_s \mid s \in K\} \subseteq A$; as A is happy there is a $Z \in A$ which diagonalises $\{Y_s \mid s \in K\}$; for such a Z, each $Z \setminus X_i$ is finite. Let q be the equivalence class of Z in P.

The second part is quite standard: it is only necessary to remark that DCR is an adequate form of AC for the usual proof. □

4.1. Lemma (DCR). $[\![\mathrm{DCR}]\!]^{\boldsymbol{B}} = \mathbf{1}$.

Proof. Again standard: let $p \Vdash [\mathcal{R}$ is a relation on $\mathcal{P}(\omega)$ such that $\forall x\, \exists y\, x\mathcal{R}y]$. Define a relation $\mathcal{S}$ on $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$: $\langle x_1, x_2\rangle \mathcal{S} \langle y_1 y_2\rangle$ if and only if $x_1 \notin I$, $x_1 \setminus y_1 \in I$, and $(x_1)_I \Vdash \hat{y}_2 \mathcal{S} \hat{x}_2$, where $(x_1)_I$ is the equivalence class of x_1. Using Lemma 4.0 we see that for all $\langle y_1, y_2\rangle$ with $y_1 \notin I$ and $(y_1)_I \leq p$, there is an $\langle x_1, x_2\rangle$ with $\langle x_1, x_2\rangle \mathcal{S} \langle y_1 y_2\rangle$. Using DCR and the encodability of pairs of reals as reals, we obtain a sequence $\langle x_i, y_i\rangle$ descending in the relation $\mathcal{S}$ with $x_0 \in p$. Put $p_i = (x_i)_I$, and let $q \leq$ each p_i. Then $q \leq p$ and $q \Vdash [\langle y_i \upharpoonright i < \omega\rangle$ ascends in the relation $\mathcal{R}]$. □

4.2. Theorem (DCR). *Let A be a happy family, $\boldsymbol{B}$ defined as above, and $F_{\boldsymbol{B}}$ as in Proposition* 3.12 *so that*

$$[\![F_{\boldsymbol{B}} \text{ is a } \hat{V}\text{-complete ultrafilter on } \hat{\boldsymbol{B}}]\!]^{\boldsymbol{B}} = 1.$$

Let G be that member of $V^{\boldsymbol{B}}$ with domain $\{\hat{x} \mid x \subseteq \omega\}$ and for $x \subseteq \omega$, $G(\hat{x}) = (x)_I$. Then

(i) $[\![G \subseteq \hat{A}$ *and G is a Ramsey ultrafilter*$]\!]^{\boldsymbol{B}} = 1$;

(ii) $[\![G \in \hat{V}]\!]^{\boldsymbol{B}} = \mathbf{1}$ *if and only if for all $Y \in A$ there is an $X \in A$ with $X \subseteq Y$ such that* $\mathrm{id}(I, \omega \setminus X)$ *is a proper prime ideal*;

(iii) *for each $\mathcal{X} \subseteq \mathcal{P}(\omega)$, $\mathcal{X} \in \mathcal{C}_A$ if and only if* $[\![\hat{\mathcal{X}} \in \dot{\mathcal{C}}_G]\!]^{\boldsymbol{B}} = \mathbf{1}$;

(iv) *for each $\mathcal{X} \subseteq \mathcal{P}(\omega)$, $\mathcal{X} \in \mathcal{I}_A$ if and only if* $[\![\hat{\mathcal{X}} \in \dot{\mathcal{I}}_G]\!]^{\boldsymbol{B}} = \mathbf{1}$.

Proof. (i) It is readily checked using Proposition 3.12 and the fact that $[\![G = \{x \mid (x)_I \in F_{\boldsymbol{B}}\}]\!]^{\boldsymbol{B}} = \mathbf{1}$, that $[\![G$ is an ultrafilter on $\omega]\!]^{\boldsymbol{B}} = \mathbf{1}$. That $[\![G \dot{\subseteq} \hat{A}]\!]^{\boldsymbol{B}} = \mathbf{1}$, and hence that $[\![G$ is nonprincipal$]\!]^{\boldsymbol{B}} = \mathbf{1}$, follows from the last part of Lemma 4.0 and the fact that for $x \in I$, $(x)_I = \mathbf{0}$ and so $[\![\hat{x} \in G]\!]^{\boldsymbol{B}} = \mathbf{0}$. Let $p \Vdash f : \hat{K} \to G$. As K is countable, Lemma 4.0 shows that there is a $q \leq p$ and a $g : K \to A$ such that $q \Vdash f = \hat{g}$. Note that for $p' \in P$, $x \subseteq \omega$, $p' \Vdash \hat{x} \in G \leftrightarrow p' \leq (x)_I$, as $[\![\hat{x} \in G]\!]^{\boldsymbol{B}} = (x)_I$. Let $Y \in q$, and for each $s \in K$ put $Y_s = Y \cap g(s)$: as $Y \setminus g(s) \in I$, each $Y_s \in A$, and as $p \Vdash$ [the range of f generates a filter $\subseteq G$], the $\{Y_s \mid s \in K\}$ generate a filter $\subseteq A$. As A is happy, there is a $Z \in A$ which diagonalizes the family $\{Y_s \mid s \in K\}$. Let $r = (Z \cap Y)_I$. Then $r \in P$, and $r \Vdash$ [there is something in G which diagonalizes the range of f]. Thus $[\![G$ is Ramsey$]\!]^{\boldsymbol{B}} = \mathbf{1}$.

(ii) If $\mathrm{id}(I, \omega \setminus X)$ is a proper prime ideal J, say, then $(X)_I \Vdash G = \hat{\tilde{J}}$, and so $(X)_I \Vdash G \in \hat{V}$. Hence the density condition of (ii) implies that $[\![G \in \hat{V}]\!]^{\boldsymbol{B}} = \mathbf{1}$. Conversely if $[\![G \in \hat{V}]\!]^{\boldsymbol{B}} = \mathbf{1}$, then for all $Y \in A$ there is an $X \subseteq Y$ with $X \in A$ such that for some ultrafilter G', $(X)_I \Vdash G = \hat{G}'$; but then as $(X)_I \Vdash \hat{X} \in G$ and $[\![\hat{\tilde{I}} \subseteq G]\!]^{\boldsymbol{B}} = \mathbf{1}$, we must have $\tilde{G}' = \mathrm{id}(I, \omega \setminus X)$, which is therefore a proper prime ideal.

(iii) and (iv) Let $\mathscr{X}$ be in $\mathscr{C}_A$, let $s \in K$ and let $p \models x \in G$. The there is a $q \leq p$ and an $X \in A$ such that $q \Vdash \hat{X} \equiv x$. $q \Vdash \hat{X} \in G$, so let $Z \in q$; then $Z \setminus X \in I$, so $Z \cap X \in A$. As $\mathscr{X} \in \mathscr{C}_A$ there is a $Y \in A$ with $Y \subseteq Z \cap X$ such that $\langle s, Y\rangle$ decides $\mathscr{X}$ in the sense of Definition 1.2. Let $r = (Y)_I$. Then $r \in P$, and $r \leq q$. Further $r \Vdash \hat{Y} \in G$, $r \Vdash Y \subseteq x$ and $r \Vdash \langle s, Y\rangle$ decides $\mathscr{X}$, essentially by the last part of Lemma 4.0. By the generality of s, p and x, and using Proposition 3.11, $[\![\hat{\mathscr{X}} \in \dot{\mathscr{C}}_G]\!]^B = \mathbf{1}$.

If further $\mathscr{X} \in \mathscr{I}_A$, then in the above argument, it would always be the case that $\langle s, Y\rangle$ forces $\mathscr{P}(\omega)\setminus\mathscr{X}$, and so $[\![\hat{\mathscr{X}} \in \dot{\mathscr{I}}_G]\!]^B = \mathbf{1}$.

Suppose now that $\mathscr{X} \subseteq \mathscr{P}(\omega)$ is such that $[\![\hat{\mathscr{X}} \in \dot{\mathscr{C}}_G]\!]^B = \mathbf{1}$, and let $s \in K$ and $X \in A$ with $|s| \leq \bigcap X$. Put $p = (X)_I$. Then there is a $q \leq p$ and a Y such that $q \Vdash [\hat{Y} \in G$ and $\hat{Y} \subseteq \hat{X}$ and $\langle s, Y\rangle$ decides $\hat{\mathscr{X}}]$. Hence $Y \subseteq X$; $Y \in A$ as $[\![\hat{Y} \in G]\!]^B \neq \mathbf{0}$ and $\langle s, Y\rangle$ decides $\mathscr{X}$, by Lemma 4.0 and Proposition 3.10. Thus $\mathscr{X} \in \mathscr{C}_A$.

Finally if $[\![\hat{\mathscr{X}} \in \dot{\mathscr{I}}_G]\!]^B = \mathbf{1}$ then we could always have in the above that $q \Vdash (\langle s, Y\rangle$ forces $\mathscr{P}(\omega)\setminus\hat{\mathscr{X}})$, and so actually $\langle s, Y\rangle$ forces $\mathscr{P}(\omega)\setminus\mathscr{X}$; and so $\mathscr{X} \in \mathscr{I}_A$. □

We now use Theorem 4.2 in proving

4.3. Theorem (DCR). *Let A be a happy family and C a Σ^1_1 subset of $\mathscr{P}(\omega)$. Then*

$$\exists X{:}\in A\ \forall Y{:}\subseteq X(X \in C \leftrightarrow Y \in C);\ \textit{and indeed } C \in \mathscr{C}_A.$$

Proof. Let B, G be as in Theorem 4.2. C will be a Σ^1_1 set in V^B, defined by the same formula as in V. Theorem 1.13 holds in V^B with B-value $\mathbf{1}$, by Theorem 4.2(i) and Lemma 4.1, so $[\![\hat{C} \in \dot{\mathscr{C}}_G]\!]^B = \mathbf{1}$; by Theorem 4.2(iii), $C \in \mathscr{C}_A$. □

In fact Theorem 4.2 can be used to prove the following

4.4. Theorem (DCR). *Let A be a happy family. Then $\mathscr{C}_A$ is closed under countable unions, complements and the operation* ($\mathscr{A}$).

The proof will also use Theorem 1.13 and the absoluteness of the definition of an inner set by a sieve. A direct proof of 4.4 can be given along the lines of Section 1, but one always has to pick the X_i's successively, to satisfy the condition that they generate a filter $\subseteq A$ whereas when Ramsey ultrafilters are used they may be picked simultaneously, which is a helpful simplification.

4.5. Remark. As H is a happy family, Theorem 4.2 has as a special case the following:

If DCR holds, then there is a Boolean extension of the universe containing no new subsets of ω but containing a new Ramsey ultrafilter.

Theorem 4.3 has a consequence for MAD families:

4.6. Proposition. *If A is Σ_1^1 and a happy family, then $\exists X \forall Y :\subseteq X \; Y \in A$; consequently $\mathscr{P}(\omega) \setminus A$ is not tall.*

Proof. Take $C = A$ in Theorem 4.3. □

4.7. Corollary. *No Σ_1^1 set is a* MAD *family.*

Proof. By 4.6 and 0.7. □

In obtaining a general metatheorem from Theorem 4.2 some care must be exercised: let $\Phi(A)$ be a wff with all variables restricted to reals expressing the predicate "A is a Ransey ultrafilter". Then $\vdash_{ZF}(A$ is a Ramsey ultrafilter$\rightarrow \Phi(A))$ but if ZF is consistent, (A is a happy family $\rightarrow \Phi(A)$) is not provable. The following is not perhaps the most general possible theorem but will be useful in Sections 6 and 7. The notion of an A-smooth function is defined in 6.0.

4.8. Metatheorem. *Let $\Phi(X, E)$ be a wff with precisely two free variables, all bound variables of which are restricted to range over the members or subsets of ω.*

If $\vdash_{ZF+DCR} A$ is a Ramsey ultrafilter & E an A-smooth function $\rightarrow \exists X :\in A$ $\Phi(X, E)$, then $\vdash_{ZF+DCR} A$ is a happy family & E an A-smooth function $\rightarrow \exists X :\in A$ $\Phi(X, E)$.

Proof. With the hypotheses in mind we reason in ZF + DCR. Let A be a happy family and E an A-smooth function. With $\boldsymbol{B}$, G as before, we have by 4.0 and 4.2(iii) that

$$[\![G \text{ is a Ramsey ultrafilter \& DCR \& } \hat{E} \text{ is a } G\text{-smooth function}]\!]^{\boldsymbol{B}} = \mathbf{1}.$$

So by hypothesis, $[\![\exists X :\in G \; \dot{\Phi}(X, \hat{E})]\!]^{\boldsymbol{B}} = \mathbf{1}$. So for some $X \in A$, $[\![\dot{\Phi}(\hat{X}, \hat{E})]\!]^{\boldsymbol{B}} \neq \mathbf{0}$, and so $[\![\dot{\Phi}(\hat{X}, \hat{E})]\!]^{\boldsymbol{B}} = \mathbf{1}$ by 3.10, 4.0 and the restricted nature of Φ. By 3.10 $\Phi(X, E)$ as required. □

Finally we use the discussion of this section to generalise the results of Section 2. In stating the next theorem it is convenient to blur the slight distinction between $F_{\boldsymbol{B}}$ and G as defined in Theorem 4.2.

4.9. Theorem. *Let M be a transitive model of* ZF + DCR, *and A a happy family in M.*

(i) *Suppose that x is $\boldsymbol{P}_A$ generic over M, and set $G = \{X \mid X \in M \;\&\; x \setminus X \text{ is finite}\}$. Then G is a Ramsey ultrafilter in $M[G]$ and is generic over M with respect to the partial ordering $\boldsymbol{P}$ defined above; further x is $\boldsymbol{P}_G$ generic over $M[G]$.*

(ii) *Conversely let G be $\boldsymbol{P}$ generic over M, and x $\boldsymbol{P}_G$ generic over $M[G]$; then x is $\boldsymbol{P}_A$ generic over M.*

Proof. (i) It is trivial from the definition of $\boldsymbol{P}_A$ generic that if x is $\boldsymbol{P}_A$ generic over M and $Y \in M \setminus A$ then $x \cap Y$ is finite; consequently $G \subseteq A$. Let $\Delta \in M$ be dense closed in $\boldsymbol{P}$; put $\Delta' = \{\langle s, S\rangle \mid (S)_I \in \Delta\}$, where $I = (\mathcal{P}(\omega) \cap M) \setminus A$. Then Δ' is dense closed in $\boldsymbol{P}_A$, and $\Delta' \in M$, so there is an $\langle s, S\rangle \in \Delta'$ with $x \in [s, S]$; hence $S \in G$. Thus G is M generic over $\boldsymbol{P}$. By Lemma 4.0, $\mathcal{P}(\omega) \cap M[G] \subseteq M$, so x is $\boldsymbol{P}_G$ generic over $M[G]$ by Theorem 2.0.

(ii) By the $\boldsymbol{P}$ genericity of $G, \mathcal{P}(\omega) \cap M[G] \subseteq M$, and $G \subseteq A$. Hence, again using Theorem 2.0,

$$G = \{X \mid X \in M \ \& \ x \setminus X \text{ is finite}\}.$$

Let $\Delta \in M$ be a dense closed subset of $\boldsymbol{P}_A$, and set

$$\Delta' = \{\langle s, S\rangle \mid \langle s, S\rangle \in \Delta \text{ and } S \in G\}$$

and

$$\Delta'' = \{(S)_I \mid \exists s \langle s, S\rangle \in \Delta\}.$$

$\Delta' \in M[G]$ and Δ' is closed in $\boldsymbol{P}_G$; it is dense in $\boldsymbol{P}_G$ as Δ'' is dense closed in $\boldsymbol{P}, \Delta'' \in M$ and G is $\boldsymbol{P}$ generic over M. As x is $\boldsymbol{P}_G$ generic over $M[G]$, $x \in [s, S]$ for some $\langle s, S\rangle \in \Delta' \subseteq \Delta$. Thus x meets Δ; and as Δ was arbitrary, x is $\boldsymbol{P}_A$ generic over M. □

4.10. Corollary. *Let M and A be as in Theorem 4.9, and let x be $\boldsymbol{P}_A$ generic over M. Then*

(i) $\aleph_1^M$ *is uncountable in* $M[x]$,
(ii) *every infinite subset of x is $\boldsymbol{P}_A$ generic over M.*

Proof. (i) $\aleph_1^M$ is preserved in the passage from M to $M[G]$, which adds no new subsets of ω; and the extension from $M[G]$ to $M[x]$ is by the algebra over $\boldsymbol{P}_G$, which satisfies the countable chain condition.

(ii) is immediate from Theorem 4.9 and Corollary 2.5. □

A criterion for $\boldsymbol{P}_A$ genericity on the lines of Theorem 2.0 may be formulated: in the case of the happy family H it is this:

4.11. Proposition. *Let M be a transitive model of* ZF + AC. *A subset x of ω is $\boldsymbol{P}_H$ generic over M if and only if x is infinite but for every* MAD *family A in M, there is an $X \in A$ with $x \setminus X$ finite.*

The proof is left to the reader, as is the proof of

4.12. Proposition. 2.6, 2.7 *and* 2.9 *hold in the more general case that F is a happy family and not necessarily a Ramsey ultrafilter.*

5. Some relative consistency results

5.0. Theorem. *If $\aleph_1$ is inaccessible in $L[X]$ for each X then every set of reals of the form $\{x \mid \Re(x, a, \zeta))_{L[x,a]}\}$, where $a \subseteq \omega$ and ζ is an ordinal, is in $\mathscr{C}_H$.*

Proof. The hypothesis is equivalent to saying that for all $X, \aleph_1^{L[X]} < \aleph_1$. Given $\langle s, S\rangle$ let F be in $L[a, S]$ a Ramsey ultrafilter with S in F. Let $T \in F$ be such that $T \subseteq S$ and

$$\langle s, T\rangle \Vdash (\dot{\Re}(\dot{x}, \hat{a}, \hat{\zeta}))_{L[\dot{x}, \hat{a}]}$$

in the sense of $\boldsymbol{P}_F$. Let x be $\boldsymbol{P}_F$ generic over $L[a, S]$ with $S \subseteq x \subseteq s \cup T$; such an x exists by Theorem 2.0 as F is countable in the real world. Then for $Y \in [s, x \setminus |s|]$, Y is also $\boldsymbol{P}_F$ generic over $L[a, S]$ and satisfies the condition $\langle s, T\rangle$: hence x is in the set concerned if and only if Y is. $\square$

5.1. Metatheorem. *If* ZF + AC + "*there is a strongly inaccessible cardinal*" *is consistent, so are* ZF + AC + "*every* ROD *set of reals is in* $\mathscr{C}_H$" *and* ZF + DC + $\omega \to (\omega)^\omega$.

Proof. By 3.22 and 3.23 the hypothesis shows that the theory ZF + AC + "$\aleph_1$ is inaccessible in every $L[X]$" + "every ROD set of reals is of the form $\{x \mid (\Re(x, a, \zeta))_{L[x,a]}\}$" is consistent. Applying Theorem 5.0 in that theory gives the first part; for the second, it is enough to consider the inner model HROD in that theory which by 3.14 and 3.15 will be a model of $\omega \to (\omega)^\omega$ and of DC. $\square$

Before stating the next metatheorem we remind the reader of a definition from cardinal arithmetic.

5.2. Definition. κ is a *Mahlo cardinal* if $\kappa > \omega$ and every closed unbounded subset of κ contains a regular cardinal.

5.3. Metatheorem. *If* ZF + AC + "*there is a Mahlo cardinal*" *is consistent, so is* ZF + DC + "*there are no MAD families*".

It is probable that the hypothesis is unnecessarily strong: indeed the author conjectures that it is a theorem of ZF + DCR + $\omega \to (\omega)^\omega$ that there are no MAD families. Some further remarks on that will be found in Section 7.

It will be convenient in proving 5.3 to consider the following hypothesis, which, containing as it does a bound class variable, is not a formula of Zermelo–Fraenkel set theory: in the context in which it is used, that of Theorem 5.8, that difficulty disappears and will accordingly now be blandly ignored.

5.4. Hypothesis. Given any $a \subseteq \omega$ and any $A \subseteq \mathscr{P}(\omega)$, there is an inner model N of ZF + AC with $a \in N$, $A \cap N \in N$, and $(2^{2^\omega})_N$ countable and such that $\langle \mathscr{P}(\omega) \cap N, A \cap N \rangle$ is an elementary submodel of $\langle \mathscr{P}(\omega), A \rangle$ with respect to formulae of second-order arithmetic with a distinguished unary predicate denoting membership of A.

5.5. Lemma. *Let A be a happy family and let B be a set of reals of the form $\{x \mid (\mathfrak{R}(x, a, \zeta))_{L[x,a]}\}$ where ζ is an ordinal, $a \subseteq \omega$, and R is an arbitrary formula of set theory. If Hypothesis 5.4 holds, then $B \in \mathscr{C}_A$.*

Proof. Let $\langle s, S \rangle$ be given with $S \in A$, and let N be an inner model such that $A \cap N \in N$, $a \in N$, $S \in N$ and $(2^{2^\omega})_N$ is countable. Then $A \cap N$ is in N a happy family, and there is a $G \subseteq A \cap N$ which is generic over N in the sense of Section 4 such that G is in $N[G]$ a Ramsey ultrafilter. By Proposition 2.9 applied in $N[G]$, there is a $T \subseteq S$ such that $T \in N[G]$, and in $N[G]$,

$$\langle s, T \rangle \Vdash (\dot{\mathfrak{R}}(\dot{x}, \hat{a}, \hat{\zeta}))_{L[\dot{x}, \hat{a}]}.$$

G is countable, as $A \cap N$ is, and forms a filter $\subseteq A$, so there is, as A is happy, an $X \in A$ such that $X \in [s, T]$ and for each $S' \in G$, $X \setminus S'$ is finite. By Theorem 2.0, such an X is P_G generic over $N[G]$, and by a by now familiar argument $\langle s, X \rangle$ decides B. Thus $B \in \mathscr{C}_A$. $\square$

5.6. Theorem. *If Hypothesis 5.4 holds, and A is a happy family of the form $\{x \mid (\mathfrak{R}(x, a, \zeta))_{L[x,a]}\}$, then $\mathscr{P}(\omega) \setminus A$ is not tall.*

Proof. Take $B = A$ in Lemma 5.5: then there is an $X \in A$ such that $\forall Y : \subseteq X (Y \in A)$. $\square$

5.7. Corollary. *If hypothesis 5.4 holds, there are no* MAD *families of the form $\{x \mid \mathfrak{R}(x, a, \zeta))_{L[x,a]}\}$.*

Proof. By 5.6 and 0.7. $\square$

5.8. Theorem. *Suppose that $V = L$, that κ is a Mahlo cardinal, and that $\phi(\boldsymbol{B}, \kappa)$, where ϕ is as in Definition 3.18. Then in $V^{\boldsymbol{B}}$ Hypothesis 5.4 holds with truth value* **1**.

The difficulty about formulating 5.4 vanishes in this case, as the inner model N will always be of the form $V^{\boldsymbol{A}}$, where $\phi(\boldsymbol{A}, \lambda)$ for some inaccessible $\lambda < \kappa$, and $\boldsymbol{A} \lhd \boldsymbol{B}$. From 5.8, 5.7 and 3.22 it follows that, with the hypotheses of 5.8,

$$[\![\text{no ROD subset of } \mathscr{P}(\omega) \text{ is a MAD family}]\!]^{\boldsymbol{B}} = \mathbf{1},$$

and Metatheorem 5.3 follows at once by Proposition 3.15. We have shown too that with the hypotheses of 5.8,

$$[\![\text{If } B \text{ is a ROD subset of } \hat{\omega} \text{ and } A \text{ is a happy family, then } B \in \mathscr{C}_A]\!]^B = 1,$$

and

$$[\![\text{In HROD, if } A \text{ is a happy family then } \mathscr{P}(\mathscr{P}(\omega)) = \mathscr{C}_A]\!]^B = 1.$$

Theorem 5.6 shows that what is happening in HROD is that there are only rather trivial happy families. Of course assuming that $V = L$ in 5.8 is for tidiness rather than necessity.

Proof of Theorem 5.8. As GCH holds in L, we need not distinguish between strong and weak inaccessibility. As κ is Mahlo, κ is itself inaccessible, and the set $\{\lambda \mid \lambda < \kappa$ and λ is inaccessible$\}$ is *stationary* in that it meets each closed unbounded subset of κ.

Suppose that $\phi(B, \kappa)$ and that as in 3.17(i) $\boldsymbol{B}$ is the union of the ascending regular chain $\{\boldsymbol{B}_\zeta \mid \zeta < \kappa\}$. We shall content ourselves with proving 5.9: standard techniques will then complete the proof of 5.8.

5.9. Let $A \in V^B$, $a \in V^B$ be such that

$$[\![A \subseteq \mathscr{P}(\hat{\omega})]\!]^B = 1 \quad \text{and} \quad [\![a \subseteq \hat{\omega}]\!]^B = 1:$$

then there is an inaccessible $\lambda < \kappa$ such that $\phi(\boldsymbol{B}_\lambda, \lambda)$ and if N is the inner model of V^B defined by

$$[\![x \in N]\!]^B = \Sigma^B\{[\![x \equiv y]\!]^B \mid y \in V^{B_\lambda}\},$$

so that N "is" V^{B_λ}, then

$$[\![a \in N]\!]^B = 1, \qquad [\![A \dot{\cap} N \in N]\!]^B = 1,$$

$$[\![(2^{2^\omega})_N \text{ is countable}]\!]^B = 1,$$

$$[\![\langle \mathscr{P}(\omega) \cap N, A \cap N\rangle \text{ is an elementary submodel of } \langle \mathscr{P}(\omega), A\rangle \text{ in the desired sense}]\!]^B = 1.$$

To prove 5.9, we first let $\zeta_0 < \kappa$ be such that each $[\![\hat{n} \in a]\!]^B$ is in $\boldsymbol{B}_{\zeta_0}$. Then there is a $v_0 \in V^{B_{\zeta_0}}$ such that $[\![a \equiv v_0]\!]^B = 1$: namely $v_0 = \{\langle [\![\hat{n} \in a]\!]^B, \hat{n}\rangle \mid n \in \omega\}$.

Now let $\zeta < \kappa$ and let $\mathscr{X}(\zeta)$ be a set of elements of V^{B_ζ} such that for all $v \in \mathscr{X}(\zeta)$ $[\![v \subseteq \hat{\omega}]\!]^B = 1$; for all $v \in V^{B_\zeta}$, if $[\![v \subseteq \hat{\omega}]\!]^B = 1$ then for some $v' \in \mathscr{X}(\zeta)$, $[\![v \equiv v']\!]^B = 1$; and for each pair v, v' of distinct elements of $\mathscr{X}(\zeta)$, $[\![v \equiv v']\!]^B \neq 1$. The cardinality of such an $\mathscr{X}$ is necessarily $\leq \{f \mid f : \omega \to B_\zeta\}^-$ which, by the inaccessibility of κ is less than κ. Call such an $\mathscr{X}(\zeta)$ a set of representatives of the reals of V^{B_ζ}. For each $v \in V^{B_\zeta}$ there will be a v' in $\mathscr{X}(\zeta)$ such that $[\![v \equiv v']\!]^B = [\![v \subseteq \hat{\omega}]\!]^B$.

Let $\mathfrak{A}(v_1, \ldots, v_n, w)$ be a wff of the language of second order arithmetic with a unary predicate denoting A, and let $v_1, \ldots, v_n$ be in $\mathscr{X}(\zeta)$. For $v_1, \ldots, v_n \in \mathscr{X}(\zeta)$ let $\chi(v_1, \ldots, v_n, \mathfrak{A})$ be the least ordinal η, $\zeta < \eta < \kappa$ such that there is a w' in V^{B_η} with

$$[\![\exists w : \subseteq \check{\omega}\ \mathring{\mathfrak{A}}(v_1, \ldots, v_n, w)]\!]^B = [\![\mathring{\mathfrak{A}}(v_1, \ldots, v_n, w') \ \&\ w' \subseteq \check{\omega}]\!]^B.$$

Such an η exists by Scott's maximum principle. For $v_1, \ldots, v_n w \in \mathscr{X}(\zeta)$ let $\psi(v_1, \ldots, v_n, w, \mathfrak{A})$ be the least ordinal η with $\zeta < \eta < \kappa$ such that the truth value $[\![\mathring{\mathfrak{A}}(v_1, \ldots, v_n, w)]\!]^B$ is in $\mathbf{B}_\eta$.

As κ is a Mahlo cardinal, there is a strongly inaccessible cardinal $\lambda, \zeta_0 < \lambda < \kappa$ such that for all $\zeta < \lambda$, all wff's $\mathring{\mathfrak{A}}(v_1, \ldots, v_n, w)$, and all $v_1, \ldots, v_n, w \in \mathscr{X}(\zeta)$, the ordinals $\chi(v_1, \ldots, v_n, \mathring{\mathfrak{A}})$ and $\psi(v_1, \ldots, v_n, w, \mathring{\mathfrak{A}})$ are both less than λ; moreover in view of the way the property $\phi(\mathbf{A}, \theta)$ is defined in terms of closure properties, λ can be chosen so that $\phi(\mathbf{B}_\lambda, \lambda)$. Then λ has the properties stated in 5.9: as $\zeta_0 < \lambda$, $[\![u \in N]\!]^B = \mathbf{1}$; as λ is a closure point of ψ,

$$[\![A \,\dot{\cap}\, N \in N]\!]^B = \mathbf{1};$$

$$[\![(2^{2^\omega})_N \text{ is countable}]\!]^B = \mathbf{1},$$

as

$$[\![(2^{2^\omega})^- \equiv \widehat{\lambda^+}]\!]^{B_\lambda} = \mathbf{1},$$

and $\lambda^+ < \kappa$; and the required elementary submodel property follows from the fact that λ is a closure point of the function $\mathscr{X}$ and hence Tarski's criterion, (Proposition 3.1.2 of [4]) applies. We leave the reader to supply the details. □

It is of interest to note that Hypothesis 5.4 is as strong as the existence of a Mahlo cardinal:

5.10. Proposition. *If Hypothesis 5.4 holds, then $\aleph_1$ is a Mahlo cardinal in L.*

Proof. Let A be a closed unbounded subset of $\aleph_1$, with $A \in L$. Let $\mathbf{A}$ be a subset of $\mathscr{P}(\omega)$ such that each member x of $\mathbf{A}$ codes an ordinal, denoted by $\rho(x)$, and furthermore $A = \{\rho(x) \mid x \in \mathbf{A}\}$. Let N be as in 5.4 such that $\mathbf{A} \cap N \in N$, etc., and let λ be $(\aleph_1)_N$. A is unbounded, so

$$\forall x : \subseteq \omega \ (x \text{ codes an ordinal}) \rightarrow \exists y : \in \mathbf{A} \rho(x) \leq \rho(y).$$

That is expressible in the language of second order arithmetic with a unary predicate for $\mathbf{A}$, so as $\langle \mathscr{P}(\omega) \cap N, \mathbf{A} \cap N \rangle$ is an elementary submodel of $\langle \mathscr{P}(\omega), \mathbf{A} \rangle$, we have that for all $x \in N$, if x codes an ordinal, then there is a $y \in \mathbf{A} \cap N$ with $\rho(x) \leq \rho(y)$. As $y \in \mathbf{A} \cap N \rightarrow \rho(y) < \lambda$, we have that $A \cap \lambda$ is unbounded in λ. As A is closed, $\lambda \in A$. λ is regular in N, and so is regular in L which is a subclass of N. Thus in L every closed unbounded subset of the real $\aleph_1$ contains a regular cardinal, and so $\aleph_1$ is Mahlo in L. □

We end this section with two remarks. The first is that Theorem 2.13(iii) is an immediate consequence of Lemma 5.5, Theorem 5.8 and Shoenfield's theorem on the absoluteness of Σ^1_2 predicates. The second is that the proofs of 5.9 and 5.10 contain the germ of the proof of the following unpublished theorem of Jensen, which is quoted with his permission. It may be left to the reader to supply a detailed proof.

5.11. Metatheorem (Jensen). *These two systems of set theory are equiconsistent: the first system is* ZF + AC + *the Mahlo schema that every closed unbounded class of ordinals contains a regular cardinal; the second is* ZF + AC + *the following schema:*

Let $\boldsymbol{B}$ *be the class Boolean algebra that makes every ordinal countable, which may be specified by saying that* $\phi(\boldsymbol{B}, On)$. *Then the set HC of hereditarily countable sets is an elementary submodel of* $V^{\boldsymbol{B}}$.

6. The theorem on functions

In this section we shall study functions with this property:

6.0. Definition. Let A be a happy family. E is an A*-smooth function* if $E : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ and for each n $\{Y \mid n \in E(Y)\} \in \mathscr{C}_A$.

The foundation of our discussion is

6.1. Theorem (DCR). *Let A be a happy family, and E an A-smooth function. Then there is an* $X \in A$ *and a family* $\{t_s \mid s \subseteq X\}$ *of finite subsets of* ω *indexed by the finite subsets of* X *such that first,* $t_s \subseteq |s|$ *for each* $s \subseteq X$*; second, if* $s \cup s' \subseteq X$ *and* $s = s' \cap |s|$, *then* $t_s = t_{s'} \cap |s|$: *and third, for any* $Y \subseteq X$,

$$E(Y) = \bigcup\{t_s \mid s \text{ in } Y\};$$

in other words, if $k \in Y \subseteq X$, *then* $E(Y) \cap (k+1) = t_{Y \cap (k+1)}$.

6.2. Example. Let E be a Borel function and A any happy family: then E is A-smooth by Theorem 4.3, and so the conclusion of Theorem 6.1 holds of E and A. That should be contrasted with the classical theorem of Baire proved in Kuratowski's book [11] Chapter II, Section 32, that every Borel function on a metric space is continuous on a comeagre set. Theorem 5 of [11] Chapter III, Section 39 states further that if f is continuous on the Borel set E, there is a $\boldsymbol{\Pi}^1_1$ set $C \subseteq E$ such that $f \restriction C$ is 1–1 and $f''C = f''E$: compare also the theorem of Gandy and Sacks cited in Kechris [9], page 381. Theorem 6.1 cannot be improved, though, to say that given an A-smooth function E, there is an $X \in A$ such that $E \restriction [0, X]$ is either constant or 1–1: define E by $E(\{n_i \mid i < \omega\}) = \{n_{2i} \mid i < \omega\}$.

6.3. Example. If $\omega \to (\omega)^\omega$ then every function $E : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is H-smooth.

Proof of Theorem 6.1. By the Metatheorem 4.8 it suffices to prove Theorem 6.1 for the special case when A is a Ramsey ultrafilter, which we shall now call F. Let us call that case Theorem 6.1′. The argument given can be adapted to prove 6.1 without recourse to Boolean extensions.

So let F be a Ramsey ultrafilter and E an F-smooth function. For each $s, t \in K$ pick $X_t^s \in F$ such that $X_t^s \subseteq \omega \setminus |t|$ and for all $n < |t|$ and all $Y \subseteq X_t^s$,

$$n \in E(s \cup Y) \leftrightarrow n \in E(s \cup X_t^s).$$

That is possible as $\{Y \mid n \in E(Y)\} \in \mathscr{C}_F$. Let $X^s \in F$ diagonalise $\{X_t^s \mid t \in K\}$, and let $X \in F$ diagonalise $\{X^s \mid s \in K\}$. Define for all s,

$$t_s = E(s \cup (X \setminus |s|)) \cap |s|.$$

Evidently $t_s \subseteq |s|$. We have to show that if $Y \subseteq X$, $k \in Y$ and $s = Y \cap (k+1)$, then $E(Y) \cap |s| = t_s$. Let Y, k, s be as above. Note that $|s| = k+1$: let $n \leqslant k$. $Y \setminus |s| \subseteq X \setminus |s| \subseteq X_s^s \setminus |s|$: so

$$n \in E(s \cup (Y \setminus |s|)) \leftrightarrow n \in E(s \cup X_s^s)$$

and

$$n \in E(s \cup X_s^s) \quad \leftrightarrow n \in E(s \cup (X \setminus |s|)); \quad s \cup (Y \setminus |s|) = Y,$$

so

$$n \in E(Y) \quad \leftrightarrow n \in E(s \cup (X \setminus |s|)).$$

so

$$E(Y) \cap |s| = E(s \cup (X \setminus |s|)) \cap |s| = t_s, \text{ as required.}$$

If s *in* $s' \subseteq X$, *put* $Y = s' \cup (X \setminus |s'|)$. Then

$$t_{s'} \cap |s| = E(Y) \cap |s'| \cap |s| = t_s. \quad \square$$

We continue the assumptions on E and F, and suppose that $X, \{t_s \mid s \subseteq X\}$ are as in Theorem 6.1′. We define a function $\chi(s,k)$ for $s \subseteq X$, $k \in X \setminus |s|$ by

6.4. $\qquad \chi(s,k) =_{\mathrm{df}} t_{s \cup \{k\}} \setminus t_s.$

Note that $\chi(s,k)$ is equal to $t_{s \cup \{k\}} \setminus |s|$, and it may be empty. We use the function χ to examine E. Define for $s \subseteq X$, $n, k \in X \setminus |s|$, $n < k$,

$$\pi_s(n,k) = 0 \quad \text{if} \quad n \in \chi(s,k)$$

$$\qquad\quad = 1 \quad \text{if} \quad n \notin \chi(s,k).$$

Then $\pi_s : [X \setminus |s|]^2 \to 2$. Let $Y_s \subseteq X$, $Y_s \in F$, be homogeneous for π_s, and let $Y \in F$ diagonalize $\{Y_s \mid s \subseteq X\}$. $Y \subseteq X$ and Y has the following property:

6.5. Given $s \subseteq Y$, $|s| \leqslant n < n'$, $\{n, n'\} \subseteq Y$, $W \subseteq Y \setminus (n+1)$, and $W' \subseteq Y \setminus (n'+1)$, we have

$$n \in E(s \cup W) \leftrightarrow n' \in E(s \cup W').$$

For let $k = \inf W$, and $k' = \inf W'$. Then $n < k$, $n' < k'$, $\{n, n', k, k'\} \subseteq Y - |s| \subseteq Y_s$, and $s \cup W \subseteq X$, so we have this chain of equivalences:

$$n \in E(s \cup W) \leftrightarrow n \in \chi(s,k) \leftrightarrow \pi_s(n,k) = 0$$
$$\leftrightarrow \pi_s(n',k') = 0 \leftrightarrow n' \in E(s \cup W').$$

The above leads to the following, which by 4.8 immediately generalises:

6.6. Proposition (DCR). *Let F be a Ramsey ultrafilter and E an F-smooth function. Then there is a $Z \in F$ such that*

$$\forall W : \subseteq Z(Z \neq W \rightarrow E(W) \neq Z).$$

Proof. Let Y be as in the above discussion, and let $\{n_i \mid i < \omega\}$ enumerate Y monotonically. Let Z be that one of $\{n_{2i} \mid i < \omega\}$ $\{n_{2i+1} \mid i < \omega\}$ which lies in F. Then if $W \subseteq Z$, $W \neq Z$, let $n_i \in Z \setminus W$. Put $s = W \cap n_i$; $W \setminus |s| \subseteq Z \setminus n_{i+2}$, and so by 6.5

$$n_i \in E(W) \leftrightarrow n_{i+1} \in E(W);$$

as $n_i \in Z$ and $n_{i+1} \notin Z$, $E(W) \neq Z$. □

The arguments behind that proposition will be developed in Section 8. We now examine further the function $\chi(s,k)$. As before let F be a Ramsey ultrafilter, E an F-smooth function and $X \in F$, $\{t_s \mid s \subseteq X\}$ such that

$$\forall Y : \subseteq X \, \forall k : \in Y E(Y) \cap (k+1) = t_{Y \cap (k+1)}.$$

Now $\chi(s,k)$ for $s \subseteq X$, $k \in X \setminus |s|$, is a finite set of integers, which may be empty. Let us introduce the function $\lambda(n,s,k)$ as follows: if $n \in \omega$, $s \subseteq X$, $k \in X \setminus |s|$ and $\chi(s,k)$ has exactly p members then $\lambda(n,s,k)$ is undefined if $n \geq p$, whereas for $n < p$, $\lambda(n,s,k)$ is defined by the requirements

6.7. $$\lambda(0,s,k) < \lambda(1,s,k) < \cdots < \lambda(p-1,s,k)$$

and

$$\chi(s,k) = \{\lambda(n,s,k) \mid n < p\}.$$

Thus $\lambda(n,s,k)$ is the $(n+1)$-st element of $\chi(s,k)$ if it exists, and is undefined otherwise. It is notationally convenient to say "$\lambda(n,s,k) = *$" rather than "$\lambda(n,s,k)$ is undefined" in the sequel.

For $s \subseteq X$, $n \in \omega$ we define a partition $\pi_s^n : [X \setminus |s|]^2 \rightarrow 6$ by saying for $k, l \in X \setminus |s|$ $k < l$:

$$\begin{aligned}
\pi_s^n(k,l) &= 0 \quad \text{if } \lambda(n,s,k) = * = \lambda(n,s,l) \\
&= 1 \quad \text{if } \lambda(n,s,k) = * \neq \lambda(n,s,l) \\
&= 2 \quad \text{if } \lambda(n,s,k) \neq * = \lambda(n,s,l) \\
&= 3 \quad \text{if } * \neq \lambda(n,s,k) < \lambda(n,s,l) \neq * \\
&= 4 \quad \text{if } * \neq \lambda(n,s,k) = \lambda(n,s,l) \neq * \\
&= 5 \quad \text{if } * \neq \lambda(n,s,k) > \lambda(n,s,l) \neq *.
\end{aligned}$$

Let $X_s^n \in F$ be homogeneous for π_s^n, so that $X_s^n \subseteq X \setminus |s|$, and let $\tau_s(n)$ be the common value of π_s^n on $[X_s^n]^2$. As X_s^n is infinite, only three values are possible for $\tau_s(n)$: 0, 3 and 4. Furthermore if $\tau_s(n) = 0$, then $\tau_s(n+1) = 0$, and if $\tau_s(n+1) = 4$, then $\tau_s(n) = 4$. Let us call s *free* if there is an n such that $\tau_s(n)$ is not 4; otherwise if for all n $\tau_s(n) = 4$, call s *captive*.

Now if t is free *either* there is a largest $n = n_t$ such that $\tau_t(n) = 4$, when we define $\zeta(t) = \lambda(n_t, t, k) + 1$ where $k \in X_t^{n_t}$: the value of $\zeta(t)$ is independent of the choice of k; *or* there is no n with $\tau_t(n) = 4$, when we let n_t be undefined, and set $\zeta(t) = |t|$. Again if t is free *either* there is a least $m = m_t$ for which $\tau_t(m_t)$ is 3, when we define $\eta(t,k) = \lambda(m_t, t, k)$ for $k \in X_t^{m_t}$, and then $\eta(t,k)$ is, for fixed t, a strictly increasing function of k; *or* there is no such n, when m_t and $\eta(t,k)$ are undefined. If n_t and m_t are both defined, $n_t + 1 = m_t$. Define

$$f(t) = \sup\{m_t, |t|\} \quad \text{if } m_t \text{ is defined}$$
$$= |t| \quad \text{if } m_t \text{ is undefined.}$$

Let $X^n \in F$ diagonalise $\{X_s^n\}$. Put

$$X^{(s)} = \bigcap\{X^n \mid n \leq f(s)\}$$

for $s \subseteq X$ and let $Y \in F$ diagonalise the family $\{X^{(s)} \mid s \subseteq X\}$. Then Y has the following property:

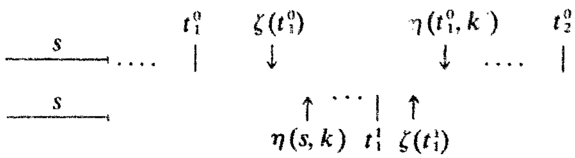
6.8. If $s \subseteq Y$, $n \leq f(s)$, $k < l$, and $\{k, l\} \subseteq Y \setminus |s|$, then $\pi_s^n(k) = \pi_s^n(l)$; and if further m_s is defined, then $\eta(s,k) < \eta(s,l)$; that is because

$$Y \setminus |s| \subseteq X^{(s)} \setminus |s| \subseteq X^n \setminus |s| \subseteq X_s^n.$$

We are now in a position to prove the following

6.9. Theorem. *If F is a Ramsey ultrafilter, Ξ an F-smooth function, Y obtained as in 6.8 and there is an $s \subseteq Y$ such that s is free and $\{t \mid s \text{ in } t \subseteq Y \text{ and } t \text{ is free}\}$ is dense in $\{t \mid s \text{ in } t \subseteq Y\}$, then there are subsets Y^0, Y^1 of Y such that $E(Y^0) \cap E(Y^1)$ is finite.*

Proof. We shall construct Y^0 and Y^1, as the union of sets $t_i^0, t_j^1 (i, j < \omega)$. Set $t_0^0 = t_0^1 = s$. Let t_1^0 be a proper extension of t_0^0 that is free, with $|t_1^0| > \zeta(s)$.



Choose $k > \zeta(t_1^0)$, $k \in Y$ such that $\eta(s,k)$ if defined is greater than $\zeta(t_1^0)$, and let t_1^1 be free and extend $s \cup \{k\}$. Choose $k' > \zeta(t_1^1)$ such that $\eta(t_1^0, k')$ if defined is greater than $\zeta(t_1^1)$ and let t_2^0 extend $t_1^0 \cup \{k'\}$ and be free. Repeat this process picking k'', k''' successively until all t_i^0, t_j^1 have been chosen, and put

$$Y^0 = \bigcup\{t_i^0 \mid i < \omega\},$$

$$Y^1 = \bigcup\{t_j^1 \mid j < \omega\}.$$

We assert that $E(Y^0) \cap E(Y^1) \subseteq \zeta(s)$; the essential point in proving that being that no element of $E(Y^0)$ lies between (say) $\zeta(t_1^0)$ and $\eta(t_1^0, k')$ whereas the only elements of $E(Y^1)$ less than $\eta(t_1^1, k'')$ other than those less than $\zeta(s)$ lie strictly between $\eta(s,k)-1$ and $\zeta(t_1^1)$. □

That proof shows in fact that $(Y^0 \cup E(Y^0)) \cap (Y^1 \cup E(Y^1))$ is finite.

6.10. The case not covered by Theorem 6.9 for an F-smooth function E is when there is a $Y \in F$ and an s, where we may assume $|s| \leq \bigcap Y$, such that for all $t \subseteq Y$ $s \cup t$ is captive. Define for $t \subseteq Y$ the partition $\sigma_t : [Y \setminus |t|]^3 \to 2$ by

$$\sigma_t(l,n,m) = 0 \quad \text{if } \chi(s \cup t, n) \cap (l+1) = \chi(s \cup t, m) \cap (l+1)$$

$$= 1 \quad \text{otherwise; where } l < n < m \text{ and } \{l,n,m\} \subseteq Y \setminus |t|.$$

Let $Y_t \in F$ be homogeneous for σ_t, and let $Z \in F$ diagonalise $\{Y_t \mid t \subseteq Y\}$. We describe the restriction of E to $[s, Z]$. Enumerate Z in ascending order as $\{n_i \mid i < \omega\}$. Note that for $t \subseteq Z$ $[Z \setminus |t|]^3$ is a set on which σ_t takes only the value 0, the value 1 being impossible as for each $l \in Z \setminus |t|$ there are only finitely many possiblities for $\chi(s \cup t, n) \cap (l+1)$. For $t \subseteq Z$, $n_k = \bigcap(Z \setminus t)$, $i \geq k$ we define

$$g_t(n_k) = 0, q_t(n_{i+1}) = \chi(s \cup t, n_{i+1}) \cap (n_i + 1),$$

$$h_t(n_k) = \chi(s \cup t, n_k);\ h_t(n_{i+1}) = \chi(s \cup t, n_{i+1}) \setminus (n_i + 1),$$

$$S_t = \bigcup\{g_t(n_i) \mid i \geq k\}.$$

Note that $k \leq i < j$ implies that $g_t(n_i)$ *in* $g_t(n_j)$ by the definition of σ_t and the value σ_t takes on $[Z \setminus |t|]^3$. $E \upharpoonright [s, Z]$ is now defined by the sets $S_t (t \subseteq Z)$ and the functions $h_t (t \subseteq Z)$: e.g. if $W \subseteq Z$, $n_{i+1} \in W$, then setting $t = W \cap n_{i+1}$ we have

$$E(s \cup W) \cap \{l \mid |s \cup t| \leq l \leq n_{i+1}\} = (S_t \cap n_i) \cup h_t(n_{i+1}).$$

Here of course $|t|$ may be very much less than n_i.

6.11. That appears to be as attractive a description of a general E for which 6.9 fails as is possible to get. There are counterexamples showing that one cannot hope for a Y such that every $t \subseteq Y$ is captive, in much the same way that given an A-smooth partition $E : \mathcal{P}(\omega) \to \omega$ of $\mathcal{P}(\omega)$ into countably many parts there is (by 6.1) an $[s, Y]$ with $Y \in A$ on which E is constant but there need be no $[0, Y]$ on which E is constant: let $E(X)$ be the third member of X.

6.12. Finally we consider the conjecture that if $C \in \mathscr{C}_A$ and E is A-smooth, then $\{X \mid E(X) \in C\} \in \mathscr{C}_A$. If C is Σ_1^1 then the conjecture is true, as given $\langle s, S \rangle \in P_A$, there is by Theorem 6.1 a $T \subseteq S$, $T \in A$ such that $E \upharpoonright [s, T]$ is continuous, and hence $\{X \mid E(X) \in C \ \& \ X \in [s, T]\}$ is a Σ^1_1 set, so there will be a $T' \subseteq t$, $T' \in A$, such that $\langle s, T' \rangle$ decides $\{X \mid E(X) \in C\}$. If A is H and $\omega \to (\omega)^\omega$ then the conjecture is true as then $\mathscr{C}_H = \mathscr{P}(\mathscr{P}(\omega))$. If though A is H and $\omega \not\to (\omega)^\omega$, the conjecture is false: let

$$E(x) = \{2n \mid n \in x\} \cup \{2n+1 \mid n \in x\}.$$

Then the range of E is

$$\{x \mid \forall n\, (2n \in x \leftrightarrow (2n+1) \in x)\},$$

which is in $\mathscr{I}_H$, and so every subset of it is. Further E is 1–1, so if $D \notin \mathscr{C}_H$ there will be a $C \in \mathscr{I}_H$ such that $\{X \mid E(X) \in C\} = D$.

7. Deductions from $\omega \to (\omega)^\omega$ and the strong axiom of determinacy

In this section we apply the arguments developed in Section 6 to derive some consequences of the partition relation $\omega \to (\omega)^\omega$ using some form of the axiom of choice, and of an axiom of Mycielski [20].

A filter F on ω is *rare* if $F \supseteq \mathrm{Fr}$ and given a partition of ω into nonempty finite sets s_i $(i \in \omega)$ there is an $X \in F$ with $(X \cap s_i)^= = 1$ for each i. Rare filters can be constructed using the continuum hypothesis. In [18] an elementary proof is given of the following

7.0. Theorem. *If $\omega \to (\omega)^\omega$ then there are no rare filters.*

No form of choice is used in that proof, which also shows the following:

7.1. Theorem. *No Σ^1_1 filter is rare.*

We present here an argument which embodies the ideas in our first proof of 7.1, to obtain a weaker form of 7.0. The interest lies in its being an application of the discussion of Section 6.

7.2. Theorem. *If* ACR *and $\omega \to (\omega)^\omega$, then there is no rare filter.*

Proof. Suppose that G is a rare filter on ω, and let $I = \check{G}$. Call X *thin* if $X \in I$, $0 \notin X$ and $\forall n\, (n \in X \to n+1 \notin X)$. If $Y \subseteq X$ and X is thin so is Y. If X is thin, then there is a $Z \in G$ such that $Z \cap X = 0$, there is exactly one member of Z less than $\inf X$ and between any two adjacent members of X there is exactly one member of Z. We say for short that such a Z *interleaves* X. To see that such a Z exists, let $X = \{n_i \mid i \in \omega\}$ in ascending order, and consider the partition given by

$$s_0 = \{k \mid 0 \leq k < n_0\}, \quad s_{i+1} = \{k \mid n_i \leq k < n_{i+1}\};$$

obtain a choice set $Z' \in G$ for that partition of ω; then as $\omega \setminus X \in G$, $Z' \cap (\omega \setminus X) \in$

G; the desired Z is some appropriate superset of $Z' \cap (\omega \setminus X)$. By ACR there is a function E such that for X thin $E(X) \in G$, $E(X) \cap X = 0$ and $E(X)$ interleaves X; and for X not thin $E(X) = X$. Let now X_0 be thin. Remembering 6.3, obtain as in 6.1 an $X \subseteq X_0$ and a tree $\{t_s \mid s \subseteq X\}$ and note now that by definition of interleaving, for $s \subseteq X$, $k \in X \setminus |s|$, $\chi(s, k)$ has precisely one element and so $\lambda(n, s, k)$ is defined if and only if $n = 0$, where χ and λ are as in 6.4 and 6.7. Consequently $\tau_s(n)$ will be 0 for $n > 0$, and so every s is free. It follows from Theorem 6.9, which holds of course for all H-smooth functions as well, by 4.8 that there are $Y^0, Y^1 \subseteq X$ such that $E(Y^0) \cap E(Y^1)$ is finite, which cannot be if G is a free filter. □

The following is an amusing consequence of 6.6:

7.3. Theorem (DCR). *Let E be an H-smooth function of $\mathcal{P}(\omega)$ onto $\mathcal{P}(\omega)$ such that $\forall X E(X) \supseteq X$. Then $\exists X \forall Y : \subseteq X E(Y) = Y$.*

Proof. Proposition 6.6 implies via 4.8 that

$$\forall X\, \exists Y \subseteq X\, \forall W \subseteq Y (E(W) \neq Y \text{ or } W = Y);$$

as E is onto, $E(s) = s$ for all s, and $\exists Z : \subseteq Y\, E(Z) = Y$; accordingly for such a Y, $E(Y) = Y$. As $\{Y \mid Y = E(Y)\}$ is not a counterexample to $\omega \rightarrow (\omega)^\omega$, $\exists X\, \forall Y \subseteq X$ $E(Y) = Y$. □

In connection with that, the following problem of Čech may be mentioned: Is there a function $E : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ such that E is onto, $\forall x E(x) \supseteq x$, $\forall x\, \forall y E(x \cup y) = E(x) \cup E(y)$ and $\exists Y E(Y) \neq Y$?

In a recent paper [25] Prikry showed that

7.4. Theorem (Prikry). *If* ADR *and* DC *hold, then* $\omega \rightarrow (\omega)^\omega$.

ADR is Mycielski's strong axiom of determinacy, defined in [20], which asserts the determinacy of all games where two players define a sequence $\langle r_n \mid n \in \omega \rangle$ of real numbers by picking them alternately. In fact as Mycielski shows in [20] ADR implies ACR, which implies DCR; and we shall see that DCR is a form of the axiom of choice adequate for Prikry's argument. The next two theorems should be compared with 5.5, 5.6, 5.7, and the remarks following the statement of 5.8.

7.5. Theorem. *If* ADR *then there are no* MAD *families, moreover if A is a happy family, then $\mathcal{P}(\omega) \setminus A$ is not a tall ideal.*

We have failed to prove that $\omega \rightarrow (\omega)^\omega$ + DCR implies that there are no MAD families; perhaps a proof can be found using the notion of a feeble filter defined at the end of [18].

The theorem follows immediately from the next theorem in the same way that 5.5 led sucessively to 5.6 and 5.7.

7.6. Theorem. *If* ADR *and A is a happy family, then every subset of* $\mathcal{P}(\omega)$ *is in* $\mathcal{C}_A$.

The proof is modelled on that of Prikry [25], where the case $A = H$ is proved using DC. We go into some detail to show that DC is not necessary.

Let $P \subseteq \mathcal{P}(\omega)$. Define a game $G(P)$ as follows: players I and II pick alternately conditions $\langle s_i, S_i \rangle \in P_A$ $(i > \omega)$ with $\langle s_{i+1}, S_{i+1} \rangle \leqslant \langle s_i, S_i \rangle$ and $s_{i+1} \neq s_i$. The first player to fail to observe those requirements loses. If $\langle \langle s_i, S_i \rangle \mid i < \omega \rangle$ is a sequence satisfying those rules or, as we shall say, of *legitimate moves*, then it converges to a set $X = \bigcup \{ s_i \mid i < \omega \}$. I *wins* if $X \in P$ and II *wins* if $X \notin P$.

By ADR, one of the players has a winning strategy. Suppose first that player II has a winning strategy, specified by the function E. Let $p, q, \ldots$ denote elements of P_A, and let $\mu_1, \mu_2 \cdots$ denote finite sequences of legitimate moves $(p_0, p_1, \ldots, p_{2n+1})$ starting with a move by player I, and in which p_{2k+1} is dictated by player II's strategy E: say

$$P_{2k+1} = E(p_0, \ldots, p_{2k}).$$

If $\mu = (p_0 \cdots p_{2n+1})$, write $\dot{\mu} = p_{2n+1}$. Call such μ's *partial plays*.

Prikry proceeds by moving to a Boolean extension in which the continuum is well-ordered and then quoting a result of Oxtoby [22]. In fact Oxtoby uses AC to obtain maximal sets of a certain kind. We shall proceed more directly by adding these maximal sets (rather than a well ordering) generically.

For each partial play μ we seek a set X_μ of pairs $(q, E(\mu, q))$ such that

(i) q is a legitimate move by player I after μ;

(ii) $E(\mu^\frown q)$ is player II's response to that move according to his strategy;

(iii) for each couple $(q_1, E(\mu, q_1)), (q_2, E(\mu, q_2))$ in X_μ, q_1 and q_2 are incompatible;

(iv) for any p compatible with $\dot{\mu}$, there is a $(q, E(\mu^\frown q)) \in X_\mu$ such that p is compatible with $E(\mu^\frown q)$.

(iv) is a maximality condition; to achieve it it is evidently sufficient to have

(iv′) for any $p \leqslant \dot{\mu}$ there is a $(q, \mu^\frown q)) \in X_\mu$ such that p is compatible with $E(\mu^\frown q)$.

To obtain such sets X_μ (which may not exist in the standard universe) we make a Boolean extension of the universe as follows. A *condition* is to be a pair $(\mathcal{K}, \mathcal{X})$ where $\mathcal{K}$ is a countable set of partial plays μ, $\mathcal{X}$ is a function with domain $\mathcal{K}$ and for each $\mu \in \mathcal{K}$, $\mathcal{X}(\mu)$ is a countable set of pairs $(q, E(\mu^\frown q))$ satisfying conditions (i) (ii) and (iii). The partial ordering of conditions is given by $(\mathcal{K}, \mathcal{X}) \leqslant (\mathcal{K}', \mathcal{X}')$ if $\mathcal{K}' \subseteq \mathcal{K}$ and for each $\mu \in \mathcal{K}'$, $\mathcal{X}'(\mu) \subseteq \mathcal{X}(\mu)$. Each condition can be coded by a real number; so DCR is enough to conclude that given a descending sequence $(\mathcal{K}_i, \mathcal{X}_i) \geqslant (\mathcal{K}_{i+1}, \mathcal{X}_{i+1}) \cdots$ of conditions, $(\mathcal{K}, \mathcal{X})$ is a condition below them all where $\mathcal{K} = \bigcup \{ \mathcal{K}_i \mid i < \omega \}$ and for $\mu \in \mathcal{K}$,

$$\mathcal{X}(\mu) = \bigcup \{ \mathcal{X}_i(\mu) \mid i < \omega, \mu \in \mathcal{K}_i \},$$

and thus by the obvious analogue of Lemma 4.0 no new reals are added in the associated Boolean extension. Given a partial play μ, a condition $(\mathscr{H}, \mathscr{X})$ and a $p \in P_A$ compatible with $\dot{\mu}$, if p is compatible with some $E(\mu, q)$ where $(q, E(\mu, q)) \in X_\mu$, then let $(\mathscr{H}', \mathscr{X}') = (\mathscr{H}, \mathscr{X})$; if not, let $q \leq p$ be a legitimate move for player I after $\dot{\mu}$, and let $(\mathscr{H}', \mathscr{X}')$ be an extension of $(\mathscr{H}, \mathscr{X})$ such that $\mu \in \mathscr{H}'$, and $(q, E(\mu, q)) \in \mathscr{X}'(\mu)$: such $(\mathscr{H}', \mathscr{X}')$ exists as $E(\mu \,{}^\frown q) \leq p$, and so is incompatible with each r, if any, such that for some q' $(q', r) \in \mathscr{X}(\mu)$. Thus by standard density arguments there is in the Boolean extension a function that assigns to each μ a set X_μ satisfying (i)–(iv). We now work in this extended universe, where E will still be a winning strategy for player II. This notation will be convenient: write $[\dot{\mu}]$ for that $[s, S]$ for which $\langle s, S \rangle = \dot{\mu}$. Define for each μ,

$$B_\mu = \{(\mu, q, E(\mu, q)) \mid (q, E(\mu, q)) \in X_\mu\}.$$

Now set

$$Q_0 = B_0 \quad \text{(0 here being the empty partial play);}$$

$$Q_{n+1} = \bigcup\{B_\mu \mid \mu \in Q_n\};$$

$$R_n = \bigcup\{[\dot{\mu}] \mid \mu \in Q_n\}.$$

Then $R_{n+1} \subseteq R_n$. Following Oxtoby we show that

$$\bigcap\{R_n \mid n < \omega\} \subseteq \mathscr{P}(\omega) \setminus P.$$

For let Y be in $\bigcap\{R_n \mid n < \omega\}$. Then for each n there is a μ in Q_n such that $Y \in [\dot{\mu}]$; we see by induction on n using condition (iv) that this μ is unique: put $\mu = \mu_n$ accordingly. Then the sequence $\langle \mu_n \mid n < \omega \rangle$ specifies a sequence $\langle p_i \mid i < \omega \rangle$ of plays (where $\mu_n = (p_0 \cdots p_{2n-1})$) of the game in which II has been using his strategy and which converges to Y. Accordingly Y is in $\mathscr{P}(\omega) \setminus P$. We show now that (in the extended universe) $\bigcap \{R_n \mid n < \omega\}$ is in $\bar{\mathscr{F}}_A$ and so $\mathscr{P}(\omega) \setminus P$ is in $\bar{\mathscr{F}}_A$; it will then follow by an easy absoluteness argument that in the original universe $\mathscr{P}(\omega) \setminus P$ is in $\bar{\mathscr{F}}_A$.

As $\bar{\mathscr{F}}_A$ is a σ-filter, it is enough to show that each R_n is in $\bar{\mathscr{F}}_A$. To do that it is enough, by Proposition 2.7 applied in the extended universe, to show that given p $\exists q \leq p$ such that $q \Vdash_R R_n$. So let p be given. $\exists \mu_0 \cdots \mu_n$ with $\mu_i \in Q_i$ and p compatible with each $\dot{\mu}_i$ (by condition (iv)); further μ_{i+1} extends μ_i in the obvious sense. Let $q \leq p$, $q \leq \dot{\mu}_i$. Then $q \Vdash_R R_n$.

Back to the original universe. We have now said enough to show that if II wins, $\mathscr{P}(\omega) \setminus P$ is in $\bar{\mathscr{F}}_A$, and so $P \in \mathscr{C}_A$. But player I winning is equivalent to II winning a derived game. Straightforward arguments now show that $\forall p\, \exists q \leq p\; q \|_R P$; by Proposition 2.6, $P \in \mathscr{C}_A$. $\square$

8. Further properties and an application of P_F generic reals

In this section we discuss further properties of the notion of forcing P_F introduced in Section 2. Throughout we assume DCR and suppose that F is a Ramsey ultrafilter.

Let E be an F-smooth function. The proof of Proposition 6.6 relativises to give the following: for any $\langle s, S\rangle \in P_F$ there is a $Z \subseteq S$, $Z \in F$ such that for all $W \subseteq Z$, if $W \neq Z$, then $F(s \cup W) \neq Z$; applying that for each finite subset s of t we have

8.0. For any $\langle t, S\rangle \in P_F$ there is a $Z \subseteq S$, $Z \in F$, such that for all $W \subseteq t \cup Z$, if $Z \not\subseteq W$ then $E(W) \neq Z$.

8.1. Proposition. *Let $\langle E_i \mid i < \omega\rangle$ be a sequence of F-smooth functions. Then*

$$\{X \mid \exists i\, \exists Y \subseteq X(X \setminus Y \text{ infinite and } X = E_i(Y)\} \in \mathscr{I}_F.$$

Proof. The set in question is $\bigcup\{P_i \mid i < \omega\}$ where

$$P_i = \{X \mid \exists Y \subseteq X(X \setminus Y \text{ infinite and } X = E_i(Y)\}.$$

As $\mathscr{I}_F$ is a σ-ideal it is enough to show that each $P_i \in \mathscr{I}_F$. But that is clear from 8.0. □

8.2. Theorem. *Let F be a Ramsey ultrafilter in L and let X be P_F generic over L. Then for no $Z \subseteq X$ with $X \setminus Z$ infinite is $X \in L[Z]$.*

8.3. Corollary. *Under the hypotheses of the theorem, if $Z_1, Z_2 \subseteq X$ and $Z_1 \setminus Z_2$, $Z_2 \setminus Z_1$ are both infinite, $Z_1 \notin L[Z_2]$ and $Z_2 \notin L[Z_1]$. Hence there are in $L[X]$ $2^{\aleph_0}$ incomparable degrees of constructibility.*

Proof of Theorem 8.2. The statement that $\exists Z \subseteq X(X \setminus Z$ infinite and $X \in L[Z])$ is a Σ^1_2 predicate of X and so if true is true in $L[X]$, and is hence forced by some $\langle s, S\rangle \in P_F$ with $X \in [s, S]$. The partial ordering $\mathbf{P}_F$ satisfies the countable antichain condition and so preserves cardinals. Hence there is an ordinal $\zeta < \omega_1^L$, and $\langle s', S'\rangle \leqslant \langle s, S\rangle$ such that in L,

$$\langle s', S'\rangle \Vdash \exists Z : \subseteq X\, X$$

is the ζ^{th} real constructible from Z. Let $Y \in L$ code ζ. Then "X is the ζ^{th} real constructible from Z" is a Δ^1_1 predicate of X, Y and Z. Consider the function E defined by $E(W) =$ the ζ^{th} real constructible from W. For each n, $\{W \mid n \in E(W)\}$ is $\Sigma^1_1(Y)$, and E is an F-smooth function (in L). So by 8.1 there is a $T \subseteq S'$ such that $T \in F$ and

8.4. $$\forall W \subseteq s' \cup T \neg \exists Y \subseteq W\, W \setminus Y \text{ infinite and } W = F(Y).$$

8.4 is a Π^1_2 predicate of T and so is true in all extensions of L; hence $\langle s', T\rangle \Vdash \neg \exists Y : \subseteq X\, X \setminus Y$ infinite and X is the ζ^{th} constructible from Y. Standard density arguments now lead to a contradiction. □

Proof of Corollary 8.3. If $Z_1 \in L[Z_2]$, then $Z_1 \cup Z_2 \in L[Z_2]$, contradicting the theorem and the fact that $Z_1 \setminus Z_2$ is infinite. Given X, let C be a family of $2^{\aleph_0}$ pairwise almost disjoint infinite subsets of X. Then any two elements of C are mutually nonconstructible. □

The above theorem and corollary also hold for $\boldsymbol{P}_A$-generic reals where A is happy in L.

We now prove the theorem of the author stated in Jockusch and Soare [8]. X is called *hyperarithmetically encodable*, or h.e. for short, if $\forall Y\ \exists Z :\subseteq Y$, X is hyp in Z. Let X be h.e., then $\{Y \mid X \text{ is hyp in } Y\}$ is Π^1_1 in X, and so is in $\tilde{\mathscr{F}}_H$, as every set has a subset in it, rather than the reverse. Now let Z be $\boldsymbol{P}_H$ generic over the universe. The X is hyp in Z, as given any condition $\langle s, S\rangle\ \exists T \subseteq S\ \forall T' \subseteq T$; X is hyp in T'; so $\langle s, T\rangle \Vdash \hat{X}$ is hyp in $\dot{Z}$, by Shoenfield's absoluteness lemma. Thus the set of standard h.e. sets is countable in the extended universe, but by 4.10 the extension preserves $\aleph_1$. Hence

8.5. Theorem. *There are only countably many h.e. sets.*

Solovay has improved that to "Every h.e. set is Δ^1_1". The author's proposed proof of that broke down for lack of an answer to the following problem.

8.6. Is there a criterion similar to that of Theorem 2.0 for a pair $\langle X, Y\rangle$ of reals to be $\boldsymbol{P}_F \times \boldsymbol{P}_F$ generic over L?

We close this section by recording the following

8.7. Proposition. *Let M be a transitive model of* ZF − DCR *and suppose that F is in M a Ramsey ultrafilter. Let*

$$I = K \cup \{X \mid X \text{ is } \boldsymbol{P}_F \text{ generic over } M\}.$$

Then I is a proper ideal.

Proof. If x and y are in I, then by Theorem 2.0, for all $A \in F$, $(x \setminus A)$ and $(y \setminus A)$ are finite and so therefore is $(x \cup y) \setminus A$; hence by 2.0 $x \cup y$ is either finite or $\boldsymbol{P}_F$ generic over M. If $x \in I$ and $y \subseteq x$, $y \in I$, being either finite or $\boldsymbol{P}_F$ generic by 2.5. Hence I is an ideal, and is proper as ω is not $\boldsymbol{P}_F$ generic. □

9. Moderately happy families

There is interest among analysts in certain ultrafilters, known as p-points, which are defined by a property rather more general than that of being Ramsey. There is a corresponding generalisation of the notion of a happy family which we study in this final section.

9.0. Definition. A is a *moderately happy family*, or MHF for short, if $\mathscr{P}(\omega) \setminus A$ is a free proper ideal and whenever $\mathrm{nl}\{X_i \mid i < \omega\} \subseteq A$, there is an $X \in A$ such that for all i, $X \setminus X_i$ is finite.

Note that in Definition 9.0 we may without loss of generality restrict attention to those families $\{X_i\} \subseteq A$, where $X_{i+1} \subseteq X_i$ for all i.

9.1. Definition. I is an MH *ideal* if $\mathcal{P}(\omega) \setminus I$ is a moderately happy family.

That definition will be useful as we shall in fact work more in terms of MH ideals than of their complementary moderately happy families. An ideal first used by Kunen furnishes our first example.

9.2. Example. Let $\{\alpha_i\}$ be a sequence of positive real numbers decreasing to 0 but with $\Sigma\alpha_i = \infty$. Put

$$I_{\{\alpha_i\}} = \{x \mid \Sigma\{\alpha_i \mid i \in x\} < \infty\}.$$

Then $I_{\{\alpha_i\}}$ is MH and tall.

Proof. $I_{\{\alpha_i\}}$ is an ideal. Let $X_i \notin I$, $X_{i+1} \subseteq X_i$, for $i < \omega$. Pick

$$n_0, \ldots, n_{i_0} \in X_0 \quad \text{with } \Sigma\{\alpha_{n_i} \mid i \leq i_0\} > 1,$$

$$n_{i_0+1}, \ldots, n_{i_1} \in X_1 \quad \text{with } \Sigma\{\alpha_{n_i} \mid i_0 < i \leq i_1\} > 1,$$

and so on. Put $X = \{n_i \mid i < \omega\}$. Then $\Sigma\{\alpha_i \mid i \in X\}$ is infinite so $X \notin I_{\{\alpha_i\}}$, but $X \setminus X_i$ is finite for each i. I is tall as the $\{\alpha_i\}$ converge to 0. $\square$

9.3. Example. Let π be a partition of ω into infinitely many finite pieces $\pi_0, \pi_1, \ldots$ such that for all $k \in \omega$ there is a π_i with at least k elements. Put

$$I'_\pi = \{x \mid \exists k \; \forall i \in \omega (x \cap \pi_i)^= \leq k\}.$$

Then I'_π is MH and tall.

Proof. Given $A_i \notin I'_\pi$ with $A_i \supseteq A_{i+1}$, pick $n_0 \in A_0$, $n_1, n_2 \in A_1$ such that for some i $\{n_1, n_2\} \subseteq \pi_i$; $n_3, n_4, n_5 \in A_2 \cap \pi_{i_2}$, for some i_2, and so on. Then $\{n_i \mid i \in \omega\}$ is up to finite differences contained in each A_i but is not a member of I'_π. That I'_π is tall is readily verified. $\square$

9.4. Remarks. Let

$$\mathcal{R} =_{\mathrm{df}} \{I'_\pi \mid \pi \text{ a partition as in 9.3}\}.$$

Then an ultrafilter F is rare if and only if for all $I \in \mathcal{R}$, $F \cap I \neq 0$. If F is rare then $F \cap I_{\{\alpha_i\}}$ is non-empty for every sequence $\{\alpha_i\}$ of the type described in Example 9.2, but if $2^{\aleph_0} = \aleph_1$, there is an ultrafilter which meets each $I_{\{\alpha_i\}}$ but is nevertheless not rare.

9.5. Definition. A *p-point* is an ultrafilter that is also an MHF.

9.6. Definition. If $f:\omega\to\omega$ and $A\subseteq\mathcal{P}(\omega)$, $f_*A =_{\text{df}}\{x \mid f^{-1''}x\in A\}$.

9.7. Proposition. *If A is an ideal, a prime ideal, a filter or an ultrafilter, then so accordingly is f_*A. If F and G are ultrafilters, and for some f and g, $F=f_*G$ and $G=f_*F$ then for some permutation h of ω, $F=h_*G$.*

9.8. Definition. The Rudin–Keisler ordering of *ultrafilters* is given by

$$F\leqslant_{\text{RK}} G \leftrightarrow_{\text{df}} \exists f\,(f:\omega\to\omega\ \&\ F=f_*G).$$

Proposition 9.7 shows that $\leqslant_{\text{RK}}$ is strictly only a pre-partial ordering, $\{G \mid F\leqslant_{\text{RK}} G$ and $G\leqslant_{\text{RK}} F\}$ being precisely the set of ultrafilters of the form h_*F for some permutation h of ω. For more on this ordering see [2], [10] and [26]. The following proposition lists some known properties.

9.9. Proposition. (i) *If F is a principal ultrafilter for all ultrafilters G, $F\leqslant_{\text{RK}} G$.*

(ii) *If F is Ramsey and $G\leqslant_{\text{RK}} F$, then G is principal or $F\leqslant_{\text{RK}} G$.*

(iii) *If F is a p-point and $G\leqslant_{\text{RK}} F$, then G is a p-point or principal.*

(iv) *If $2^{\aleph_0}=\aleph_1$, then above each ultrafilter G is a rare ultrafilter, so that rarity need not be transmitted downwards.*

(v) *If F is not Ramsey, then there is a free ultrafilter G strictly below F in the Rudin–Keisler ordering.*

(vi) *A free ultrafilter is Ramsey if and only if it is a p-point and rare.*

(vii) *If $2^{\aleph_0}=\aleph_1$, then strictly above each p-point is another one.*

The clauses (ii) and (v) together show that the Ramsey ultrafilters may be characterised as those that are minimal among free ultrafilters in the Rudin–Keisler ordering. Clause (vii) which is due to Mrs. Rudin and has been considerably extended by Blass [2] shows taken with (ii) that the continuum hypothesis implies that there are p-points which are not Ramsey. The same conclusion may be obtained by coupling Example 9.2 or 9.3, and the first remark of 9.4 with the following general principle:

9.10. Proposition (CH). *Let I be an* MH *ideal. Then there is a p-point $F\supseteq \check{I}$.*

That may be proved by imitating the proof of Proposition 0.11. We shall however be longer-winded.

9.11. Lemma. *Let I be an* MH *ideal, and $\omega\setminus X\notin I$. Then* $\text{id}(I,X)$ *is* MH.

Proof. Let $X_i\notin \text{id}(IX)$ with $X_i\supseteq X_{i+1}$. Then $X_i\setminus X\notin I$ for each i and $X_i\setminus X\supseteq$

$X_{i+1} \setminus X$; so there is a $Y \subseteq \omega \setminus X$ with $Y \notin I$ and $Y \setminus (X_i \setminus X)$ finite for each i. Then $Y \notin \mathrm{id}(I, X)$ and $Y \setminus X_i$ is finite for each i. $\square$

9.12. Lemma (DCR). *If $\{I_i \mid i < \omega\}$ is an ascending sequence of* MH *ideals, so that $I_j \subseteq I_{j+1}$, then $\bigcup\{I_i \mid i < \omega\}$ is an* MH *ideal, which is not prime if none of the I_i is.*

Proof. We may by removing repetitions assume that the sequence is properly ascending. Set $J = \bigcup\{I_i \mid i < \omega\}$: J is trivially an ideal. Let $X_i \supseteq X_{i+1}$ with no $X_i \in J$. For $j \in \omega$, pick $Y_i \notin I_j$ such that for each i, $Y_j \setminus X_i$ is finite. Put $Z = \bigcup\{Y_j \cap X_j \mid j \in \omega\}$. Then for each i, $Z \setminus X_i$ is contained in $\bigcup\{Y_j \setminus X_i \mid j < i\}$, which is finite. As for each j, $Y_j \cap X_j \notin I_j$ (for $Y_j \notin I_j$ and $Y_j \setminus X_j$ is finite) and $Z \supseteq Y_j \cap X_j$, $Z \notin I_j$ and so $Z \notin J$. Hence J is an MH ideal.

Suppose finally that no I_i is prime, as will be the case for all but trivial sequences. Pick $X_i \in \tilde{I}_{i+1} \setminus \tilde{I}_i$: let $Y \notin J$ with $Y \setminus X_i$ finite for each i. Then $Y \notin \tilde{J}$; so J is not prime, as $Y \notin J \cup \tilde{J}$. $\square$

9.13. Lemma. *Let $\mathscr{A}$ be a non-empty collection of proper free ideals such that*

(i) *for each $I \in \mathscr{A}$ and each $X \notin \tilde{I}$, $\mathrm{id}(I, X) \in \mathscr{A}$ and*

(ii) *whenever $\lambda < 2^{\aleph_0}$ and $\langle I_\nu \mid \nu < \lambda \rangle$ is a sequence of members of $\mathscr{A}$ such that $\nu < \nu' < \lambda \to I_\nu \subseteq I_{\nu'}$, then $\bigcup\{I_\nu \mid \nu < \lambda\} \in \mathscr{A}$.*

Let $\mathscr{B}$ be a second collection of ideals such that,

(iii) *$\bar{\bar{\mathscr{B}}} \leqslant 2^{\aleph_0}$ and*

(iv) *for each $I \in \mathscr{A}$ and $J \in \mathscr{B}$, $J \not\subseteq I$.*

Then if there is a well-ordering of the continuum there is an ultrafilter F such that for all $J \in \mathscr{B}$, $J \cap F \neq 0$. Moreover F may be chosen to include $\tilde{I}$ for any preassigned $I \in \mathscr{A}$.

Proof. Enumerate with repetitions if necessary the members of $\mathscr{B}$ as $\langle J_\nu \mid \nu < 2^{\aleph_0}\rangle$, and the infinite subsets of ω as $\langle X_\nu \mid \nu < 2^{\aleph_0}\rangle$. Define an ascending sequence $\langle I_\nu \mid \nu < 2^{\aleph_0}\rangle$ of members of $\mathscr{A}$ as follows. Let I_0 be any member of $\mathscr{A}$. Suppose that for all $\nu' < \nu$, $I_{\nu'}$ has been defined, $I_{\nu'} \in \mathscr{A}$ and $\nu' < \nu'' < \nu \to I_{\nu'} \subseteq I_{\nu''}$. If ν is a limit set $I_\nu = \bigcup\{I_{\nu'} \mid \nu' < \nu\}$. Then $I_\nu \in \mathscr{A}$ by (ii) and $\nu' < \nu \to I_{\nu'} \subseteq I_\nu$.

Now suppose that ν is a successor ordinal, say $\nu = \zeta + 1$. By (iv) there is an $X \in J_\zeta \setminus I_\zeta$: then $\omega \setminus X \notin \tilde{I}_\zeta$, so by (i) the ideal $I'_\zeta =_{\mathrm{df}} \mathrm{id}(I_\zeta, \omega \setminus X)$ is in $\mathscr{A}$. Now set $I_\nu = I'_\zeta$ if $X_\zeta \in I'_\zeta$, and $I_\nu = \mathrm{id}(I'_\zeta, \omega \setminus X_\zeta)$ otherwise. In either case $I_\nu \in \mathscr{A}$, by (i), and $I_\zeta \subseteq I_\nu$, so for $\nu' \leqslant \zeta$, $I_{\nu'} \subseteq I_\nu$. Let

$$I = \bigcup\{I_\nu \mid \nu < 2^{\aleph_0}\},$$

and set $F = \tilde{I}$. Then F is the desired ultrafilter. F is proper, as each I_ν is; and for each ζ,

$$F \cap J_\zeta \supseteq \tilde{I}_{\zeta+1} \cap J_\zeta \neq 0.$$

We remark finally that it is not assumed that the members of $\mathscr{B}$ are free or proper. □

The last three lemmata provide a general method for constructing p-points which we illustrate by proving Proposition 9.10.

9.14. Definition. For $f:\omega\to\omega$, let

$$I_f =_{\mathrm{df}} \{x \mid \exists k\, \forall n :> k\,(x \cap f^{-1''}\{n\} \text{ is finite})\}.$$

9.15. Definition. Let $\mathscr{P} =_{\mathrm{df}} \{I_f \mid f:\omega\to\omega\}$.

9.16. Definition. (i) *For each* $f:\omega\to\omega$, I_f *is a possibly improper free tall* Σ^1_1 *ideal, which is generated by the sets* $f^{-1''}\{n\}$, *for* $n<\omega$, *and those* X *which meet each* $f^{-1''}\{n\}$ *in a finite set.*

(ii) *A free ultrafilter* F *is a* p*-point if and only if for each* $I\in\mathscr{P}$, $I\cap F\neq 0$.

Proof. That I is a free ideal is immediate from its definition, which shows too that I is Σ^0_4 in f, and hence Σ^1_1. Given X, either $f''X$ is finite, when for some n, $f^{-1''}\{n\}$ is an infinite subset of X, or $f''X$ is infinite, when there is some infinite $Y\subseteq X$ on which f is finite-to-one. Thus I is tall. The last clause of (i) is easily checked, and (ii) follows from it. □

Proof of 9.10. Let I_0 be an MH ideal and suppose that $2^{\aleph_0}=\aleph_1$. In Lemma 9.13, let

$$\mathscr{A}=\{I \mid I_0\subseteq I \text{ and } I \text{ an MH ideal}\}.$$

Then conditions (i) and (ii) of 9.13 hold by 9.11, 9.12 and CH. Let $\mathscr{B}=\mathscr{P}$. Condition (iii) is clearly satisfied; as for condition (iv) let $I\in\mathscr{A}$ and $I_f\subseteq I$. for $i<\omega$, set $X_i=\omega\setminus f^{-1''}i$. Then

$$i<j<\omega\to X_i\supseteq X_j;$$
$$f^{-1''}i=\bigcup\{f^{-1''}\{n\}\mid n<i\},$$

which is in I_f; and so each $X_i\in\tilde{I}_f\subseteq\tilde{I}$. As I is MH, there is a Y not in I such that for each i, $Y\setminus X_i$ is finite; but the restriction of f to such a Y is finite-to-one, and so $Y\in I_f\subseteq I$, a contradiction. Thus if $I\in\mathscr{A}$ and $J\in\mathscr{B}$, $J\not\subseteq I$, as required.

So conditions (i)–(iv) are satisfied, and CH implies of course that there is a well ordering of the continuum. We may therefore conclude by 9.13 that there is an ultrafilter $F\supseteq\tilde{I}_0$ such that for all $J\in\mathscr{P}$, $F\cap J\neq 0$. Such an F is free, as $K\subseteq I_0$, and so by 9.16 is a p-point. □

Remark. It might seem that the argument above would work were the definition of MH ideal to be weakened to

9.17. Given $X_i\supseteq X_{i+1}$, with each $X_i\in\tilde{I}$, there is an $X\notin I$ such that $\forall i\,(X\setminus X_i$ is finite).

Regrettably the union of a countable ascending chain of ideals with Property 9.17 need not share Property 9.17: let ω be the disjoint union of infinite sets Z_i $(i<\omega)$. Put

$$I_0 = \{x \mid \forall i\,(x \cap Z_i \text{ is finite})\},$$

$$I_1 = \mathrm{id}(I_0, Z_0), \ldots, I_{k+1} = \mathrm{id}(I_k, Z_k) \cdots.$$

Then $\bigcup\{I_j \mid j<\omega\}$ fails to have Property 9.17; for consider the sequence $X_i = \omega \setminus \bigcup\{Z_j \mid j<i\}$. But each I_j has Property 9.17.

The notion of a moderately happy family was originally investigated by the author with a view to proving

9.18. Theorem (Pitt [23]; Solomon [29]) (CH). *There is a p-point F such that for no f is f_*F Ramsey,*
which by 9.4, 9.9(iii), 9.9(vi) and 9.16(ii) is equivalent to proving that

9.19. if CH, then there is a free ultrafilter F such that for all $I \in \mathcal{P}$, $F \cap I \neq 0$, but all $f:\omega \to \omega$ with f_*F free there is a $J \in \mathcal{R}$ with $\tilde{J} \subseteq f_*F$.

Unfortunately the author's proposed proof contains a gap, to which the reader will now be led in the hope that he may see how to bridge it.

9.20. Lemma. *Let $\omega \setminus X$ be infinite. Then there is a tall Σ^1_1 MH ideal I containing X.*

Proof. Partition ω into finite pieces such that X meets each piece in precisely one point and such that the size of the pieces is unbounded. Let π be the partition and take I to be I'_π. □

Until the discussion of 9.19 is complete, let $\mathcal{A}_M$ be the set of tall Σ^1_1 MH ideals, and assume CH. Then $\mathcal{A}_M$ is non-empty by 9.20 and satisfies conditions (i) and (ii) of Lemma 9.13, by CH, 9.11, 9.12, the classical fact that the union of countably many Σ^1_1 sets is Σ^1_1, and the fact that if I is Σ^1_1, then so is $\mathrm{id}(I,X) = \{x \mid \exists y\,(y \in I\ \&\ x \subseteq y \cup X)\}$.

9.21. Lemma (CH). *Let $I \in \mathcal{A}_M$. Then there is an $I' \in \mathcal{A}_M$ and a $J \in \mathcal{R}$ with $I \cup J \subseteq I'$.*

9.22. Remark. The reader may be interested in a counterexample to the more natural assertion than that of Lemma 9.21, that if $I \in \mathcal{A}_M$ and $\mathrm{id}(I, I'_\pi)$ is proper, where $I'_\pi \in \mathcal{R}$, then $\mathrm{id}(I, I'_\pi)$ is MH: take I to be the I_0 of Remark 9.17, and π such that no Z_i contains more than one element of any one piece of π, $Z_0 \cup Z_1$ is the union of the two-element pieces of π, $Z_2 \cup Z_3 \cup Z_4$ the union of the three-element pieces of π, and so on. Then each $Z_i \in I'_\pi$, so $\mathrm{id}(I, I'_\pi)$ fails to be MH.

Proof of 9.21. Let $\mathscr{A} = \mathscr{A}_M \cap \{I' \mid I \subseteq I'\}$, and let $\mathscr{B} = \mathscr{R} \cup \mathscr{P}$. Conditions (i) (ii) and (iii) of Lemma 9.13 are satisfied. The argument given in the proof of Proposition 9.10 shows that if $I' \in \mathscr{A}$ and $J \in \mathscr{P}$ then $J \not\subseteq I'$. If there is an $I' \in \mathscr{A}$ and a $J \in \mathscr{R}$ with $J \subseteq I'$, then there is nothing left to prove, so if 9.21 is false for I, then $\mathscr{A}$ and $\mathscr{B}$ satisfy condition (iv) of Lemma 9.13. There is thus a free ultrafilter F such that $\tilde{I} \subseteq F$ and for all $J \in \mathscr{B}$, $F \cap J \neq 0$. But then F is both rare (as $\mathscr{R} \subseteq \mathscr{B}$) and a p-point (as $\mathscr{P} \subseteq \mathscr{B}$); that is to say, F is a Ramsey ultrafilter. But the relationship $\tilde{I} \subseteq F$ is impossible by Theorem 2.12, as I is tall, free and Σ^1_1. □

9.23. Lemma. *If I is* MH *and for all $n, f^{-1}{}''\{n\} \in I$, then $f_* I$ is* MH.

Proof. $f_* I$ is a free ideal by 9.7. Let

$$X_i \supseteq X_{i+1},\ X_i \notin f_* I.$$

Set $Y_i = f^{-1}{}''X_i$. Then $Y_i \notin I$, and $Y_i \supseteq Y_{i+1}$. There is thus a $Y \notin I$ with $Y \setminus Y_i$ finite for each i. Let $X = f''Y$. Then $f^{-1}{}''X \supseteq Y$, so $X \notin f_* I$. For each i, $X \setminus X_i = f''(Y \setminus Y_i)$, which is finite: thus $f_* I$ is MH. □

9.24. Definition. If I is an ideal and $f : \omega \to \omega$,

$$f^{-1}I =_{\mathrm{df}} \mathrm{id}\{f^{-1}{}''x \mid x \in I\}.$$

9.25. Proposition. *Let f be onto ω. Then if I is proper, so is $f^{-1}I$; if I is Σ^1_1, so is $f^{-1}I$; if I is* MH, *so is $f^{-1}I$; if I is free and tall, so is $f^{-1}I$.*

Proof. The first two parts may be safely left to the reader. Suppose then that I is MH, $X_i \supseteq X_{i+1}$, for $i < \omega$, and no $X_i \in f^{-1}I$. Let $Y_i = f''X_i$. Then $X_i \subseteq f^{-1}{}''Y_i$, so $Y_i \notin I$. As i is MH, there is a $Y \notin I$, such that for all i, $Y \setminus Y_i$ is finite. An X may now be constructed such that $f''X = Y$, and for each i, $X \setminus X_i$ is finite. Such an X cannot be in $f^{-1}I$. Hence $f^{-1}I$ is MH.

Finally suppose i is free and tall. Evidently $f^{-1}I$ is free. Let X be given. If $f''X$ is finite, then $X \in f^{-1}I$; otherwise there is a $Z \in I$ with $Z \subseteq f''X$. Then $X \cap f^{-1}{}''Z$ is infinite and in $f^{-1}I$. Thus $f^{-1}I$ is tall. □

The gap in the proof of 9.19 lies in the proposed proof of the following statement, which I shall now call

9.26. Conjecture. *Let $I \in \mathscr{A}_M$ and suppose that for all $n \in \omega$, $f^{-1}{}''\{n\} \in I$. Then there is a $J \in \mathscr{R}$* and an *$I' \in \mathscr{A}_M$* with *$I \subseteq I'$* and *$J \subseteq f_* I'$.*

Proposed proof. We may suppose that $f_* I$ is tall, since otherwise we may use 9.25 and 9.20. By 9.23, $f_* I$ is MH, so with this supposition, $f_* I \in \mathscr{A}_M$. Let

$$\mathscr{A} = \{f_* I' \mid I \subseteq I' \in \mathscr{A}_M\},$$

and $\mathscr{B} = \mathscr{R} \cup \mathscr{P}$. If the assertion of the lemma is false, the proof of 9.21 may be repeated to obtain a free ultrafilter $F \supseteq (f_* I)^\sim$ such that for all $J \in \mathscr{R} \cup \mathscr{P}$, $F \cap J \neq 0$. But such an F is Ramsey, contradicting Theorem 2.12 as before.

Note that in fact $\mathscr{A}$ does not quite satisfy condition (i) of 9.13; we have though the weaker property (i'): for $I' \in \mathscr{A}$ and $X \notin \tilde{I}$, there is an $I'' \in \mathscr{A}$, such that $I'' \supseteq \mathrm{id}(I', X)$. Examination of the proof of Lemma 9.13 shows that (i') suffices. □

The error lies in the first sentence: if $f_* I$ is not tall, we may by 9.20 find an $I_1 \in \mathscr{A}_M$ with $f_* I \subseteq I_1$, and by 9.25, $f^{-1}I_1$ will also be in $\mathscr{A}_M$; $I_1 \subseteq f_* f^{-1} I_1$, but we do not know that $I \subseteq f^{-1} f_* I$, and hence we cannot conclude that $I \subseteq f^{-1} I_1$, which is what we need to reduce 9.26 to the case that $f_* I$ is tall. In fact "$I \subseteq f^{-1} f_* I$" may well be false: by the result in [18], if I is Σ^1_1 there is a finite-to-one function f such that $f_* I$ is the ideal K of all finite sets, and so $f^{-1} f_* I$ is in this case K, and so in general not equal to I. If 9.26 is true, the following argument will establish 9.19:

Proposed proof of Theorem 9.19. Assuming CH, let us enumerate all infinite subsets of ω as $\langle X_\nu \mid \nu < \aleph_1 \rangle$; the members of $\mathscr{P}$ as $\langle J_\nu \mid \nu < \aleph_1 \rangle$, and all functions from ω to ω as $\langle f_\nu \mid \nu < \aleph_1 \rangle$. We construct a series $\langle I_\nu \mid \nu < \aleph_1 \rangle$ of members of $\mathscr{A}_M$: I_0 can be chosen as an arbitrary member of $\mathscr{A}_M$; $I_{\nu+1}$ a member of $\mathscr{A}_M$ such that either X_ν or $\omega \backslash X_\nu$ is in $I_{\nu+1}$; such that $\tilde{J}_\nu \cap I_{\nu+1} \neq 0$; and such that either $\exists n$ $f_\nu^{-1}{}''\{n\} \in \tilde{I}_{\nu+1}$ or $f_\nu^* \tilde{I}_{\nu+1}$ contains some $J \in \mathscr{R}$; I_λ at limit λ is the union of the I_ν for $\nu < \lambda$. Then if I is the union of all the I_ν, $\tilde{I}$ is the desired ultrafilter. At the successor steps of the construction we use 9.11, 9.26 and the observation in the proof of 9.10 that if $J \in \mathscr{P}$ and $I' \in \mathscr{A}_M$, $J \nsubseteq I'$. □

The story has one happy outcome: Mlle M. Daguenet has found an elegant topological proof of Theorem 9.18 which is modelled on the above argument, but which proceeds by induction, not on the class of Σ^1_1 ideals but on the class of those which are, in 2^ω, the union of countably many compact sets.

A construction similar to that proposed above, but simpler and correct, gives the following:

9.27. Theorem (CH). *There is a free ultrafilter F such that for no F is $f_* F$ rare or a p-point.*

Proof. We construct as before an ascending sequence $\langle I_\nu \mid \nu < \aleph_1 \rangle$ of Σ^1_1 ideals: this time we arrange that either $\exists n f_\nu^{-1}{}''\{n\} \in \tilde{I}_{\nu+1}$ or $f_\nu^* I_{\nu+1}$ contains a $J \in \mathscr{R}$ and a $J' \in \mathscr{P}$. Firstly, it is shown in [16] that if I is Σ^1_1 and free, then for some $J' \in \mathscr{P}$, $\mathrm{id}(I, J')$ is proper. Secondly it follows from Theorem 7.1 that if I is Σ^1_1, then for some $J \in \mathscr{R}$, $\mathrm{id}(I, J)$ is proper. Thus if $f_\nu^{-1}{}''\{n\} \in I_\nu$ for each n, we may take $I_{\nu+1}$ to be $f_\nu^{-1} \mathrm{id}(f_\nu^* I_\nu, J, J')$ for some $J \in \mathscr{R}$ and $J' \in \mathscr{P}$. □

One difference between happy and moderately happy families is that if I is Σ^1_1 and tall, $\mathscr{P}(\omega) \setminus I$ cannot by Proposition 4.6 be happy but may by Example 9.3 be moderately happy. A second but conjectural difference is this: the author is unable to prove that the intersection of a countable descending sequence of happy families is happy, in contrast to Lemma 9.12, which indeed admits the following improvement:

9.28. Proposition (AC). *If Martin's axiom holds, $\lambda < 2^{\aleph_0}$ and $\langle I_\nu \mid \nu < \lambda \rangle$ is a strictly ascending sequence of* MH *ideals, then $\bigcup\{I_\nu \mid \nu < \lambda\}$ is* MH.

Proof. Set $I = \bigcup\{I_\nu \mid \nu < \lambda\}$. I is clearly a free proper ideal. Let $X_i \supseteq X_{i+1}$, $X_i \notin I$. For each ν select $Y_\nu \notin I_\nu$ such that for all i, $Y_\nu \setminus X_i$ is finite. We construct a Z such that $Z \setminus X_i$ is finite for each i, and $Y_\nu \setminus Z$ is finite for each ν, which ensures that Z is in no I_ν, and thus not in I, as required.

Consider the following notion of forcing. A condition is a pair $\langle X, Y \rangle$, where $X \Delta X_i$ is finite for some i, Y is in the ideal J generated by $\{Y_\nu \mid \nu < \lambda\}$, and $Y \subseteq X$. We define the partial ordering of the set of conditions by setting $\langle X, Y \rangle \leq \langle X', Y' \rangle$ if and only if $Y \supseteq Y'$ and $X \subseteq X'$. As the conditions $\langle X, Y_1 \rangle$ and $\langle X, Y_2 \rangle$ have the common refinement $\langle X, Y_1 \cup Y_2 \rangle$, and there are only countably many possibilities for X, the partial ordering satisfies the countable chain condition. For $\nu < \lambda$ let $\Delta_\nu = \{\langle X, Y \rangle \mid Y_\nu \setminus Y$ is finite$\}$; for $i < \omega$ let $\Delta'_i = \{\langle X, Y \rangle \mid X \setminus X_i$ is finite$\}$; and for $n < \omega$ let $\Delta''_n = \{\langle X, Y \rangle \mid n \in Y$ or $n \notin X\}$. Then each Δ_ν, Δ'_i and Δ''_n is a dense closed set of conditions. By Martin's axiom there is a set $\mathscr{G}$ of pairwise compatible conditions that meets each of those dense closed sets. Let

$$Z = \bigcup\{Y \mid \text{for some } X, \langle X, Y \rangle \in \mathscr{G}\}.$$

Then for each $i < \omega$ and each $\nu < \lambda$, $X_i \setminus Z$ is finite and $Z \setminus Y_\nu$ is finite. □

It follows from that and from 9.13 that 9.10 can be proved from AC + MA rather than CH.

We now complete the proof of Theorem 2.13. Part (i) is easily proved by combining the method of proof of Proposition 0.11 for the case $A = H$ with the well-known facts that there are $2^{\aleph_0}$ Σ^1_2 sets and that if MA holds, $\lambda < 2^{\aleph_0}$ and F is a free filter generated by λ elements than F is contained in some countably generated free filter. Readers of Booth [3] will be able to formulate the notion of a super-happy family and a generalisation of 0.11 provable from MA.

To prove Part (ii) we quote two results; the first is Theorem 4.55 (2) in [29]:

9.29. Theorem (Solomon). *Suppose that* MA *holds and that $2^{\aleph_0} > \aleph_1$, and let $\langle X_\nu \mid \nu < \aleph_1 \rangle$ be a sequence such that*

$$\nu < \nu' < \aleph_1 \rightarrow (X_{\nu'} \setminus X_\nu \text{ is finite and } X_\nu \setminus X_{\nu'} \text{ infinite}).$$

Then there is a Ramsey ultrafilter G containing each X_ν and each set X such that for all $\nu < \aleph_1$, $(\omega \setminus X) \setminus X_\nu$ is finite.

The second is Theorem 3.2 of [15]:

9.30. Theorem (Solovay). *Suppose that* MA *holds, that $2^{\aleph_0} > \aleph_1$ and that for some X, $\aleph_1^{L[X]} = \aleph_1$. Then every subset of $\mathscr{P}(\omega)$ of power $\aleph_1$ is Π^1_1.*

Now let $B = \{X_\nu \mid \nu < \aleph_1\}$, as in 9.29, and let

$$C = \{x \mid \forall Z :\in B\, x \setminus Z \text{ is finite}\}.$$

Let G be the Ramsey ultrafilter of Theorem 9.29, which contains all of B and no member of C. We assert that C is not in $\mathscr{C}_G$: for let $X \in G$. Then $X \notin C$; but put $Y_\nu = X \cap X_\nu$. Then by MA + $\neg$CH, there is a Z such that for all ν, $Z \setminus Y_\nu$ is finite: for such a $Z, Z \cap X$ is infinite and is in C. But if in addition the hypotheses of 9.30 hold, then B is Π^1_1, and so C is Π^1_2. Then $A = \mathscr{P}(\omega) \setminus C$ is a Σ^1_2 set not in $\mathscr{C}_G$.

Part (iii) was proved in Section 5. For part (iv), let F be a Ramsey ultrafilter, A a set Σ^1_2 in X and κ a strongly inaccessible Rowbottom cardinal. Consider the structure $\langle V_\kappa, F, \mathscr{E} \restriction V_\kappa, \{X\}\rangle$, where $\mathscr{E}$ is the epsilon relation: that is of type $(\kappa, 2^{\aleph_0})$. Let N be an elementary submodel of type $(\kappa, \aleph_0)$ and M be the transitive collapse of N. Then $X \in M$, and $F \cap M$ is in M a Ramsey ultrafilter: however $F \cap M$ is countable, and so as in Section 5 we see that F contains reals $P_{F\cap M}$ generic over M and thus each set Σ^1_2 in X will be $\mathscr{C}_F$. The argument from Chang's conjecture $+ 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$ is similar, and proceeds by considering the structure $\langle H_2, F, \mathscr{E} \restriction H_2, \{X\}\rangle$, where F is a Ramsey ultrafilter, and H_2 is the set of sets hereditarily of power less than $\aleph_2$. □

Finally we list a number of characterisations of Ramsey ultrafilters, the equivalence of which follows from 0.10, 0.13, 2.0, 2.10, 2.12, 9.4, 9.9(vi), and 9.16.

9.31. Theorem (DCR). *The following are equivalent properties for a free ultrafilter F:*

(i) *F is a happy family,*

(ii) *for each $\pi : [\omega]^2 \to 2$ there is an $X \in F$ with π constant on $[X]^2$,*

(iii) *$[\![\forall X\, \forall Y \mathrel{\dot{\subseteq}} X$ (if X is $\hat{\boldsymbol{P}}_F$ generic over $\hat{V}$ so is $Y)]\!]^{\boldsymbol{B}} = 1$, where $\boldsymbol{B}$ is the algebra over $\boldsymbol{P}_F$,*

(iv) *every Σ^1_1 subset of $\mathscr{P}(\omega)$ is in $\mathscr{C}_F$,*

(v) *$F \cap I \neq 0$ for each tall Σ^1_1 free ideal I.*

A word about the history of the work in this paper is now in order. The author's interest in the problem of refuting the relation $\omega \to (\omega)^\omega$ without the axiom of choice was aroused by Friedman during Scott's seminar on partition theorems conducted at Stanford in 1967. The arguments used in the present paper are in part a development of ideas learned by the author from Cohen, in whose paper [5] the seeds of Theorem 8.2 may be found, and from Jensen. Metatheorem 5.1 and a form of Theorem 8.2 were proved in the author's dissertation submitted for a Research Fellowship at Peterhouse in 1968. The proof of Theorem 0.13 given in Section 2 was found in 1969, and that given in Section 1 in 1971. Topological proofs of Silver's theorem 0.12 have been found by Ellentuck [6] and Taylor and of Theorem 0.13 by Louveau [13] and Milliken [19] who has proved a form of Theorem 4.4 as well. Theorem 9.27 was proved first by Pitt [23].

As this is the first time the material in the author's dissertation has been published, he takes the opportunity of recording his gratitude to those who have encouraged and taught him, by word or by example; and in particular to the Master and Fellows of Peterhouse, for admitting him to their Society; to Friedman, Kunen and Silver for many conversations in 1967 and 1968; and to Ronald Jensen, who supervised his dissertation and to whom he dedicates this work.

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