THE JOURNAL OF SYMBOLIC LOGIC
Volume 56, Number 3, Sept. 1991

FORCING MINIMAL DEGREE OF CONSTRUCTIBILITY

HAIM JUDAH[1] AND SAHARON SHELAH

Abstract. In this paper we will study four forcing notions, two of them giving a minimal degree of constructibility. These constructions give answers to questions in [Ih].

§0. Introduction. In this work we will study the possible connections between the structure of the degrees of constructibility and certain properties of the set of constructible reals, such as Lebesgue measurability. To be more general, we will use the term "constructibility" to denote constructibility over an arbitrary ground model V, not necessarily L, but often satisfying CH. We will also show that the forcing notions used for "shooting a real through an ultrafilter" produce many degrees of constructibility. In the literature [Sa] Sacks introduces a real number (a Sacks real) which has minimal degree of constructibility, i.e., if S denotes the Sacks forcing, then

$$(*) \qquad 0 \Vdash_S \text{“}(\forall x \in \mathbb{R})(x \in V \text{ or } \underset{\sim}{g} \in V[x])\text{”},$$

where $\underset{\sim}{g}$ is the canonical name for the Sacks real.

This real number, the Sacks real, is not useful when we are interested in Lebesgue measurability or in Baire property of the old set of reals, because

$$\Vdash_S \text{“}\mathbb{R} \cap V \text{ is not Lebesgue measurable”}$$

and

$$\Vdash_S \text{“}\mathbb{R} \cap V \text{ does not have the Baire property”}.$$

From this we can ask if the Lebesgue measurability, or the Baire property, of the constructible reals implies that the number of degrees of constructibility is more than two.

For the Baire property of the constructible reals the answer is no:

Gray [Gr], in his Ph.D. thesis, has shown that a Laver real has minimal degree of constructibility. Because a Laver real is a dominating real, we have that in such

Received September 1, 1988; revised September 15, 1989.
The first author would like to thank the National Science Foundation for its partial support under grant number DMS-8701828, and the second author would like to thank the U.S.-Israel Binational Science Foundation for partial support.

[1] Note new spelling (formerly Jaime Ihoda).

generic extensions the old reals have the Baire property (they are meager; see [Ih]). In [ISh2] we have shown that if we add a Laver real (or a countable support iteration of such reals) we get an extension where the reals of the ground model have outer measure one, i.e. they are not Lebesgue measurable.

Some time ago, J. Truss conjectured that if the constructible reals have measure zero then it is possible to get a Cohen real over L. This conjecture was proved false by A. Miller, who remarked that by adding a Mathias real to L, we produce the negation of the Truss conjecture [Mi]. In [Ma] it was proved that, over L, Mathias reals produce a great number of degrees of constructibility, and therefore it was possible to think that the following holds: if $L \cap \mathbb{R}$ has measure zero then there are more than two degrees of constructibility (clearly this is a weakening of the Truss conjecture (see [ASh])). In §3 we will prove that the Blass-Shelah ([BSh]) is a minimal real. This real has the following property (like a Mathias real): let P be the Blass-Shelah forcing; then

$$0 \Vdash_P (\forall x \in [\omega]^\omega \cap V)(\underset{\sim}{g} \subseteq^* x \vee \underset{\sim}{g} \subseteq^* \omega - x),$$

where $\underset{\sim}{g}$ is the canonical name for the generic subset of ω. Clearly this implies that

$$0 \Vdash_P \text{“}2^\omega \cap V \text{ has measure zero”},$$

and this gives an answer to our question. In §1 we will show that the forcing notion used for “shooting a real through an ultrafilter” produces many degrees of constructibility. This will answer a question that appears in [Ih].

Looking at the forcing notions that have minimal degree of constructibility over the ground model, like (*), we see that they do not satisfy the countable chain condition (e.g. Silver forcing, Laver forcing, rational perfect forcing). The natural question is: if $V \models$ ZFC, does there exist $P \in V$ satisfying

(i) $P \models$ “ccc”,

(ii) $0 \Vdash_P$“$(\forall x \in \mathbb{R})(x \in V$ or $\underset{\sim}{g} \in V[x])$”?

We do not yet have a general answer to this question. We will show, in §2, that under CH (MA) there are such partially ordered sets.

All our notation is standard. We finish this section by giving some definitions.

0.1. Definition. (a) $[\omega]^\omega = \{a \subseteq \omega: |a| = \aleph_0\}$.

(b) $[\omega]^{<\omega} = \{a \subseteq \omega: |a| < \aleph_0\}$.

(c) For a, b in $[\omega]^\omega$ we say

(i) $a \subseteq^* b$ iff $(\exists n \in \omega)(a - n \subset b)$, and

(ii) $a =^* b$ iff $a \subseteq^* b$ and $b \subseteq^* a$.

0.2. Definition. Let D be a filter over ω.

(a) We say that D is *selective* iff $(\forall g \in^\omega \omega)(\forall n)(g^{-1}(n) \in [\omega]^{<\omega} \to (\exists a \in D)(g \restriction a$ is one-to-one)) (also this means “rare”).

(b) We say that D is *Ramsey* iff D is a nonprincipal ultrafilter and for every $\pi: [\omega]^2 \to \{0,1\}$ there is $x \in D$ such that $|\pi''[x]| = 1$. It is well known (and easy to show) that Ramsey ultrafilters are selective.

In this work we assume that all the filters are proper, are nonprincipal, and contain the filter of the cofinite sets.

We thank the referee for simplifying the proof of 1.2 and for many suggestions for improvement of the presentation.

§1. Ramsey reals and Silver reals. In [Ma] it was proved that forcing with P_D, when D is a Ramsey filter, produces a great number of reals which have incomparable degrees of constructibility. In [ISh1] we proved that in such cases, i.e. when D is a Ramsey ultrafilter, $P(D)$ produces the same generic extension as P_D. In this section we will show, without any assumption on D, that $P(D)$ produces a great number of reals which have incompatible degrees of constructibility. We will also show that P_D adds a Cohen real (and therefore many degrees of constructibility), if D is an ultrafilter that is not Ramsey. This was previously proved by many people.

1.0. DEFINITION [ISh1]. (a) $P(D)$ will denote the following partially ordered set:

(i) $p \in P(D)$ iff p is a subtree of $\omega^{<\omega}$ with the property that there exists $s \in p$ (denoted $s(p)$) so that $\forall t \in p$, $t \subseteq s$ or $s \subseteq t$, and if $s \subseteq t \in p$ then

$$p^t = \{n \in \omega: t^\frown\langle n\rangle \in p\} \in D$$

for every $p \in P(D)$ and for every $t \in p$, t is an increasing function;

(ii) $p_1 \leq p_2$ iff $p_1 \supseteq p_2$.

(b) If $s \in p$ then $p^{[s]} = \{t \in p: t \subseteq s \text{ or } s \subseteq t\}$.

(c) $p_1 \leq^\circ p_2$ iff $s(p_1) = s(p_2)$ and $p_1 \leq p_2$.

(d) We say that $A \subseteq p \in P(D)$ is a *front* of p iff, for every $s, t \in A$, $s \not\subseteq t$, and for every branch X of p there exists $k \in \omega$ such that $X \restriction k \in A$.

This notion of forcing is the same as the $Q(U)$ defined by Blass in [BL, §5].

1.1. DEFINITION. If $\underset{\sim}{\tau}$ is a $P(D)$-name of a subset of ω and $a \in [\omega]^\omega$, we define $\underset{\sim}{\tau}^a \in V^{P(D)}$ by setting

$$0 \Vdash_{P(D)} \text{“}n \in \underset{\sim}{\tau}^a \text{ if } |\underset{\sim}{\tau} \cap n| \in a \wedge n \in \underset{\sim}{\tau}\text{”}.$$

Clearly this defines $\underset{\sim}{\tau}^a$ unequivocally from $\underset{\sim}{\tau}$.

1.2. THEOREM. *If $\underset{\sim}{\tau}$ is the $P(D)$-name of the generic subset of ω, produced by forcing with $P(D)$, and $a \neq^* \omega$, then*

$$0 \Vdash \text{“}\underset{\sim}{\tau} \notin V[\underset{\sim}{\tau}^a]\text{”}.$$

PROOF. Suppose a condition p forced $\tau = \mathrm{Val}(\underset{\sim}{x}, \tau^a)$ for a certain name $\underset{\sim}{x}$ in the ground model. By extending p, we can arrange that p^t depends only on $\max(t)$ and decreases (with respect to $\subseteq$) as $\max(t)$ increases; this ensures that, for any path through the tree p, any infinite subset that includes the stem is also a path through p. Let σ be a generic path through p. Clearly, it has two distinct infinite subsets τ and τ', each containing the stem of p, with $\tau^a = \tau'^a$. By [ISh1, 1.14], both τ and τ' are generic paths through p. So $\tau = \mathrm{Val}(x, \tau^a) = \mathrm{Val}(x, \tau'^a) = \tau'$, a contradiction.

1.3. DEFINITION. Let D be an ultrafilter on ω. Let P_D be the following forcing notion:

(a) $(a, A) \in P_D$ iff $a \in [\omega]^{<\omega}$ and $A \in D$ and $\sup(a) < \inf(A)$;

(b) $(a, A) \leq (b, B)$ iff $a \subseteq b$ and $B \subseteq A$ and $b - a \subseteq A$.

This forcing notion was introduced by J. Silver.

1.4. *Fact. If D is not Ramsey, then forcing with P_D produces Cohen reals.*

Proof. If D is not Ramsey, then there is a function $\pi: [\omega]^2 \to \{0, 1\}$ such that for all $x \in D$, $\pi''[x]^2 = \{0, 1\}$. Hence

$$(*) \qquad \forall x \in D \forall i \in \{0, 1\} \exists a, b \in x: a < b \text{ and } \pi(a, b) = i.$$

For any (finite or infinite) $y \subseteq \omega$ with increasing enumeration $y = \{y_0, y_1, \ldots\}$, define

$$\bar{\pi}y = \langle \pi(y_0, y_1), \pi(y_2, y_3), \ldots \rangle.$$

Let $\underset{\sim}{r}$ be the name for the generic real produced by P_D, i.e., $\Vdash_{P_D}$ "$\underset{\sim}{r} = \bigcup\{s : (s, A) \in \underset{\sim}{G}\}$". We will show that $\Vdash_{P_D}$ "$\bar{\pi}\underset{\sim}{r}$ is Cohen-generic over V".

Let $\langle s_i : i < \omega \rangle$ be a maximal antichain of the Cohen forcing $2^{<\omega}$, and let $(s, A) \in P_D$. It is enough to find $(t, B) \geq (s, A)$ and $i < \omega$ such that $\bar{\pi}t \supseteq s_i$, because clearly $(t, B) \Vdash \bar{\pi}\underset{\sim}{r} \supseteq \bar{\pi}t$.

Find i such that $\bar{\pi}s$ is compatible with s_i, say $(\bar{\pi}s)^\frown\langle c_1, \ldots, c_k \rangle \supseteq s_i$. By (*) we can find $a_1 < b_1$ in A such that $\pi(a_1, b_1) = c_1$. Let $s_1 = s \cup \{a_1, b_1\}$ and $A_1 = A - (b_1 + 1)$. Then $(s_1, A_1) \supseteq (s, A)$, and $\bar{\pi}s_1 = \bar{\pi}s^\frown\langle c_1 \rangle$. Continuing by induction, we can find $(s_1, A_1) \leq (s_2, A_2) \leq \cdots \leq (s_k, A_k) = (t, B)$ such that $\bar{\pi}t = \bar{\pi}s^\frown\langle c_1, \ldots, c_k \rangle \supseteq s_i$. □

(This fact is mentioned without proof in [BL], and was probably first noticed by Baumgartner.)

1.5. Remark. We call $r \in [\omega]^\omega$ a *Ramsey real over* V, if for every π in V, $\pi : [\omega]^2 \to \{0, 1\}$ there is an n such that $|\pi''[r - n]| = 1$. If D is a Ramsey ultrafilter, then P_D or $P(D)$ produces Ramsey reals. Conversely, if r is a Ramsey real over V, we can define $D \subseteq P(\omega) \cap V$ by

$$a \in D \leftrightarrow r \subseteq^* a.$$

D will be a Ramsey ultrafilter over $P(\omega) \cap V$, and if $P(\omega)^{V[D]} = P(\omega)^V$, then r is $P(D)$-generic over $V[D]$ (see [ISh1]). We do not know if Ramsey reals could have minimal degree.

§2. Minimal degrees. In this section we introduce a new forcing notion which is similar to the Laver forcing and Mathias forcing with an ultrafilter. The aim is to force a minimal degree of constructibility and satisfy c.c.c. First we will give these two facts under the assumption of the existence of some family of filters; then in the end we will show that CH implies this assumption. (The same proof works from MA.)

2.1. Definition. (i) We say that $\bar{D} = \langle D_\eta : \eta \in \omega^{<\omega} \rangle$ is a *sequence of filters* (on ω) if for every $\eta \in \omega^{<\omega}$, D_η is an ultrafilter over ω.

(ii) For a sequence $\bar{D}$ of filters, let $P(\bar{D})$ be the following partially ordered set:

(a) $p \in P(\bar{D})$ iff $p \subseteq \omega^{<\omega}$ is a tree and there exists $s \in p$, called the *stem* of p, such that for every $t \in p$, $t \subseteq s$ or $s \subseteq t$, and if $s \subseteq t$ then $\{n \in \omega : t^\frown\langle n \rangle \in p\} \in D_t$;

(b) if $p, q \in P(\bar{D})$ we say that $p \leq q$ iff $q \subseteq p$.

(iii) If $p \in P(\bar{D})$ and $t \in p$, we define

$$p^{[t]} = \{s \in p : s \subseteq t \text{ or } t \subseteq s\} \in P(\bar{D}).$$

(iv) If $p_1, p_2 \in P(\bar{D})$ then we say that $p_1 \leq^\circ p_2$ iff $p_1 \leq p_2$ and $\text{stem}(p_1) = \text{stem}(p_2)$.

2.2. Definition. If $I \subseteq P(\bar{D})$ is a dense open subset of $\langle P(\bar{D}), \leq \rangle$ and $p \in P(\bar{D})$, we define $\text{rk}^I : p \to \text{ORD}$ by induction on the ordinals:

(i) $\text{rk}^I(s) = 0$ if and only if there exists $q \in I$ such that $p^{[s]} \leq^\circ q$;

(ii) $\mathrm{rk}^I(s) \leq \alpha > 0$ iff $\{n: s^\wedge\langle n\rangle \in p$ and $\mathrm{rk}^I(s^\wedge\langle n\rangle)$ is well defined and less than $\alpha\} \in D_s$;

(iii) $\mathrm{rk}^I(s) = \infty$ iff there does not exist $\alpha \in \mathrm{ORD}$ such that $\mathrm{rk}^I(s) = \alpha$.

2.3. *Claim. For every* $\mathrm{stem}(p) \subseteq s \subseteq p$ *we have that* $\mathrm{rk}^I(s) < \infty$.

Proof. Let $s \in p$ be such that $\mathrm{rk}^I(s) = \infty$ and $s(p) \subseteq s$. We define $p^* = \{t \in p: t \subseteq s$ or $s \subseteq t$ and for every $k \in [\lg(s), \lg(t)), \mathrm{rk}^I(t \restriction k) = \infty\}$. Clearly $p^* \subseteq p$, and by definition of rk^I, if $s \subseteq t \in p^*$ then $\{n: t^\wedge\langle n\rangle \in p^*\} \in D_t$ (D_t is an ultrafilter). Therefore $p^* \in P(\bar{D})$ and, as I is dense open in $P(\bar{D})$, there exists p^{**} such that $p^* \leq p^{**} \in I$. By hypothesis stem $(p^{**}) \in p^*$, and this implies that $\mathrm{rk}^I(\mathrm{stem}(p^{**})) = \infty$; but clearly $\mathrm{rk}^I(\mathrm{stem}(p^{**})) = 0$. □

2.4. LEMMA. *If $I \subseteq P(\bar{D})$ is a dense open subset of $P(\bar{D})$ and $p \in P(\bar{D})$, then there exists $q \in P(D)$ such that*

(i) *$p \leq^\circ q$, and*

(ii) *$\{s \in q: q^{[s]} \in I\}$ contains a front.*

(Remember that $A \subseteq p \in P(\bar{D})$ is a front of p iff for every $s, t \in A$, $s \nsubseteq t$ and for every branch x of p there exists $k \in \omega$ such that $x \restriction k \in A$.)

PROOF. By induction on $\mathrm{rk}^I(\mathrm{stem}(p))$. □

2.5. THEOREM. *If Φ is a $P(\bar{D})$-sentence and $p \in (\bar{D})$ then there exists $q \in P(\bar{D})$ such that $p \leq^\circ q$ and*

$$q \Vdash \text{“}\Phi\text{”} \quad or \quad q \Vdash \text{“}\neg\Phi\text{”}.$$

PROOF. Let $I = \{q: q \Vdash \text{“}\Phi\text{”}$ or $q \Vdash \text{“}\neg\Phi\text{”}\}$. Clearly I is a dense open subset of $\langle P(\bar{D}), \leq\rangle$. We will prove by induction on $\mathrm{rk}^I(t)$ (for $\mathrm{stem}(p) \subseteq t \in p$) that there exists q, $p^{[t]} \leq^\circ q$, such that $q \Vdash \text{“}\Phi\text{”}$ or $q \Vdash \text{“}\neg\Phi\text{”}$. If $\mathrm{rk}^I(t) = 0$ then this is clear. If $\mathrm{rk}^I(t) = \alpha$ then $\{n: \mathrm{rk}^I(t^\wedge\langle n\rangle) < \alpha\} = a \in D_t$. For each $n \in a$, let q_n, $p^{[t^\wedge\langle n\rangle]} \leq^\circ q_n$, be such that $q_n \Vdash \text{“}\Phi\text{”}$ or $q_n \Vdash \text{“}\neg\Phi\text{”}$. Let $a_\Phi = \{n \in a: q_n \Vdash \text{“}\Phi\text{”}\}$ and $a_{\neg\Phi} = a - a_\Phi$. Therefore, without loss of generality, $a_\Phi \in D_t$, and we define $q \in P(\bar{D})$ by $q = \bigcup_{n \in a_\Phi} q_n$. Clearly $p^{[t]} \leq^\circ q$ and $q \Vdash \text{“}\Phi\text{”}$. □

(This theorem was also proved in [Bl].)

2.6. DEFINITION. We say that $\bar{D} = \langle D_\eta: \eta \in \omega^{<\omega}\rangle$ is *good* if $\bar{D}$ is a sequence of filters and

(i) for each $\eta \in \omega^{<\omega}$, D_η is a Ramsey filter,

(ii) $\bigcap D_\eta = \{a: (\forall\eta \in \omega^{<\omega})(a \in D_\eta)\}$ is a selective filter, and

(iii) there exists $\langle A_\eta: \eta \in \omega^{<\omega}\rangle$ such that

$$A_\eta \in D_\eta \quad \text{for every } \eta \in \omega^{<\omega},$$
$$A_\eta \cap A_\xi = \varnothing \quad \text{for every } \eta \neq \xi,$$
$$\bigcup A_\eta = \omega.$$

For the rest of this section we will assume that $\bar{D}$ is a good sequence of filters and that $\underset{\sim}{r}$ is a $P(\bar{D})$-name for a member of $2^\omega = \mathbb{R}$, and $p \in P(\bar{D})$ is such that

$$p \Vdash_{P(\bar{D})} \text{“}\underset{\sim}{r} \notin V\text{”}.$$

2.7. *Fact. There exists $p_1 \in P(D)$ such that*

(i) *$p \leq^\circ p_1$, and*

(ii) *for every $n \in \omega$, $\{\eta \in p_1: p_1^{[\eta]} \Vert \underset{\sim}{r} \restriction \bar{n}\}$ contains a front.*

Proof. Use induction and 2.4. □

For every $\eta \in p_1$ and $k \in \omega$, we define v^k_η to be the unique member of 2^k such that there exists $q \in P(\bar{D})$ satisfying

$$p_1^{[\eta]} \leq^\circ q \Vdash_{P(\bar{D})} \text{“}\underset{\sim}{r} \upharpoonright k = \hat{v}^k_\eta\text{”}$$

(use 2.5 2^k times).

Clearly $v^k_\eta \subseteq v^{k+1}_\eta$. Then we define v_η to be such that $(\forall k \in \omega)(v^k_\eta \subseteq v_\eta)$. Let $\rho_0 =$ stem(p_1), and let $J_0 = \{\tau: v_\tau = v_{\rho_0}\}$. We say that "$\underset{\sim}{G}$ passes outside J_0" if there exist $\tau \notin J_0$ and $q \in G$ extending p_1 such that stem$(q) = \tau$.

2.8. *Fact. There exists p_2 such that*

$$p_1 \leq^\circ p_2 \Vdash_{P(\bar{D})} \text{“}\underset{\sim}{G} \textit{ passes outside } \hat{J}_0\text{”}.$$

Proof. Because "$\underset{\sim}{G}$ passes outside $\hat{J}_0$" is a sentence in the $P(\bar{D})$-language, if the conclusion of the fact is false, there exists p_2 such that

$$p_1 \leq^\circ p_2 \Vdash_{P(\bar{D})} \text{“}\underset{\sim}{G} \text{ does not pass outside } \hat{J}_0\text{”}.$$

Then there is no $q \geq p_2$ such that $q \Vdash_{P(\bar{D})} \text{“}\underset{\sim}{r} \neq v_{\rho_0}\text{”}$. Therefore $p_2 \Vdash_{P(\bar{D})} \text{“}\underset{\sim}{r} = v_{\rho_0}\text{”}$, a contradiction □

Now fix $p_2^\circ \geq p_1^\circ \geq p$ satisfying 2.8.

Let $J_2 = \{\rho \in p_2: \rho_0 \subseteq \rho$ and $\rho \notin J_0$ and for every $\rho_0 \subseteq \eta \subseteq \rho$ if $\eta \neq \rho$ then $\eta \in J_0\}$. Also, by using 2.4, we can assume that J_2 is a front.

Set $J_1 = \{\rho: (\exists k)(\rho\hat{}\langle k\rangle \in J_2 \cap p_2)\}$.

Clearly for every $\rho \in J_1$ we have that $v_\rho = v_{\rho_0}$, and for every $\rho \in J_2$ we have that $v_\rho \neq v_{\rho_0}$.

We define, for every $\rho \in J_2$,

$$k(\rho) = \min\{k: v_\rho \upharpoonright k \neq v_{\rho_0} \upharpoonright k\}.$$

2.9. *Fact. We may assume that for every $\rho \in J_1$*

$$\{l: \rho\hat{}\langle l\rangle \in J_2\} \in D_\rho.$$

Proof. We define $\bar{J}_1 = \{\eta: (\exists \rho \in J_1)(\eta \subseteq \rho)\}$. For every $\eta \in \bar{J}_1$, we define

$$\begin{aligned} \text{rk}(\eta) = 0 \quad &\text{iff} \quad \{l: \eta\hat{}\langle l\rangle \in J_2\} \in D_\eta, \\ \text{rk}(\eta) \geq \alpha \quad &\text{iff} \quad \{l: \text{rk}(\eta\hat{}\langle l\rangle) \geq \beta\} \in D_\eta \text{ for every } \beta < \alpha, \\ \text{rk}(\eta) = \infty \quad &\text{iff} \quad \text{there is not } \alpha \text{ such that rk}(\eta) = \alpha. \end{aligned}$$

Clearly for every $\rho \in \bar{J}_1$, rk$(\rho) \neq \infty$ (J_2 is a front).

Let $J'_1 = \{\rho: \text{rk}(\rho) = 0\} \subseteq J_1$, and let $p'_2 = \{\eta: (\exists \rho \in J'_1)(\eta \subseteq \rho \text{ or } \rho \subseteq \eta)\}$.

It is easy to show that $p_2 \leq^\circ p'_2 \in P(\bar{D})$ and p'_2 satisfies the requirements, and we can work with p'_2 instead of p_2. So let $p_2 = p'_2$. □

Therefore $J_1 \cap p_2$ is a front of p_2. Let $J_1 = J_1 \cap p_2$.

2.10. *Fact.* $\{l: k(\rho\hat{}\langle l\rangle) > m\} \in D_\rho$ *for each $m \in \omega$ and for every $\rho \in J_1$.*

Proof. If not, then $\{l: k(\rho\hat{}\langle l\rangle) = m\} \in D_\rho$, and this implies that

$$\{l: v_{\rho\hat{}\langle l\rangle} \upharpoonright m \neq v_{\rho_0} \upharpoonright m\} \in D_\rho.$$

Hence $\{l: v_{\rho\hat{}\langle l\rangle} \upharpoonright m = t \neq v_{\rho_0} \upharpoonright m\} \in D_\rho$ for some $t \in \omega^m$, and thus $v_\rho \upharpoonright m \neq v_{\rho_0} \upharpoonright m$, contradicting $\rho \in J_0$. □

2.11. *Fact. We can assume that* $\langle k\langle \rho^\frown\langle l\rangle\rangle : \rho^\frown\langle l\rangle \in J_2\rangle$ *is one-to-one, as a function of* l*, for each* ρ.

Proof. Use the fact that D_ρ is a Ramsey filter. □

2.12. *Fact. There exists* $p_3 \in P(\bar{D})$ *such that* $p_2 \leq^\circ p_3$ *and, for every* $\rho \in J_2$, $p_3^{[\rho]} \Vdash$ "$\underset{\sim}{r} \restriction k(\rho) = v_\rho \restriction k(\rho)$".

Proof. Use 2.5. □

Therefore we have $p \leq^\circ p_1 \leq^\circ p_2 \leq^\circ p_3$ satisfying all the above facts.

2.13. *Fact. We can assume that if* $l_1 < l_2$ *and* $\rho^\frown\langle l_1\rangle \in J_2$ *and* $\rho^\frown\langle l_2\rangle \in J_2$*, then*

$$(*) \qquad v_{\rho^\frown\langle l_1\rangle} \restriction k(\rho^\frown\langle l_1\rangle) \text{ and } v_{\rho^\frown\langle l_2\rangle} \restriction k(\rho^\frown\langle l_2\rangle) \text{ are incomparable.}$$

Proof. Let $A = \{l : \rho^\frown\langle l\rangle \in J_2\}$; we define $\pi : [A]^2 \to 2$ by

$$\pi(\langle l_1, l_2\rangle) = \begin{cases} 0 & \text{if } (*) \text{ holds,} \\ 1 & \text{if } (*) \text{ fails.} \end{cases}$$

Then there exists $B \subseteq A$, $B \in D_\rho$, such that $\pi \restriction [B]^2$ is constant. Then if $\pi \restriction [B]^2 = \{0\}$ we finish. Otherwise we have that $\rho \in J_2$, because the sequence defined by B will be v_ρ, a contradiction. □

Now we assume p_3 satisfies the condition in 2.13. We define

$$J_2^k = \{\rho^\frown\langle l\rangle \in J_2 : k(\rho^\frown\langle l\rangle) = k\}.$$

Let $J_1 = \langle \rho_i : i < \omega\rangle$. Then also we may assume that

$$\rho_i{}^\frown\langle l\rangle \in p_3 \Rightarrow k(\rho_i{}^\frown\langle l\rangle) > i.$$

Therefore J_2^k is finite. We define the partial function $h : \omega \to \omega^{<\omega}$. Suppose $l \in A_\rho$ (see 2.6(iii)) and $\rho \in J_2 \cap p_3$. Then

$$h(l) = v_{\rho^\frown\langle l\rangle} \restriction k(\rho^\frown\langle l\rangle) + 1.$$

Clearly h is well defined on $\bigcup_\rho A_\rho$, $\rho \in J_1$.

2.14. *Fact.* h *is a finite-to-one function.*

Proof. By the remark on J_2^k. □

Therefore there exists $A \in \bigcap_{\eta \in \omega^{<\omega}} D_\eta$ such that $h \restriction A$ is one-to-one.

2.15. *Fact. For every* l_1, l_2 *in* A

$$h(l_1) \text{ is incomparable with } h(l_2).$$

Proof. If not, we assume without loss of generality that $h(l_1) \subseteq h(l_2)$. This implies that $k(\rho_1{}^\frown\langle l_1\rangle) = k(\rho_2{}^\frown\langle l_2\rangle)$, where $l_1 \in A_{\rho_1}$ and $l_2 \in A_{\rho_2}$, contradicting the fact that $h \restriction A$ is one-to-one. □

For every $\rho \in J_1$ let

$$J_\rho = \{\rho^\frown\langle l\rangle \in J_2 : l \in A\},$$

and let

$$p_4 = \{\eta : (\exists \rho)(\exists \delta)(\rho \in J_1 \wedge \delta \in J_\rho \wedge (\eta \subseteq \delta \vee \delta \subseteq \eta) \wedge \eta \in p_3)\}.$$

2.16. *Fact.* (a) $p_4 \in P(\bar{D})$.

(b) $p_0 \leq^\circ p_1 \leq^\circ p_2 \leq^\circ p_3 \leq^\circ p_4$.

(c) $I = \bigcup\{J_\rho : \rho \in J_1\}$ *is a front of* p_4.

(d) *If* ρ_1 *and* ρ_2 *are in* J, *then*

$$p_4^{[\rho_i]} \Vdash \text{“}\underset{\sim}{r} \restriction k(\rho_i) = v_{\rho_i} \restriction k(\rho_i)\text{”}, \qquad i = 1, 2,$$

and $v_{\rho_1} \restriction k(\rho_1) + 1$ *is incomparable with* $v_{\rho_2} \restriction k(\rho_2) + 1$.

(e) *Without loss of generality*, $\inf\{k(\rho): \rho \in J\} \geq m$ *for some fixed* m.

Proof. Clear. □

2.17. LEMMA. *For every* $p \in P$ *such that* $p \Vdash \text{“}\underset{\sim}{r} \notin V\text{”}$ *there exist* $p_\omega \in P$, *and* $\langle I_i : i < \omega\rangle$ *and* $k: \omega^{<\omega} \to \omega$ *and* $\langle v_\rho \restriction k(\rho) + 1: \rho \in \omega^{<\omega}\rangle$ *such that the following conditions hold*:

(a) $p \leq^\circ p_\omega$.

(b) I_i *is a front of* p_ω *for every* $i \in \omega$.

(c) *If* $\rho \in I_{i+1}$ *then* $(\exists m \in \omega)(\rho \restriction m \in I_i)$.

(d) $\inf\{k(\rho): \rho \in I_i\} \geq i$ *for every* $i \in \omega$.

(e) *If* $\rho_1, \rho_2 \in I_i$, *then*

$$p_\omega^{[\rho_j]} \Vdash \text{“}\underset{\sim}{r} \restriction k(\rho_j) + 1 = v_{\rho_j} \restriction k(\rho_j) + 1\text{”}, \qquad j = 1, 2,$$

and

$$v_{\rho_1} \restriction k(\rho_1) + 1 \textit{ is incomparable with } v_{\rho_2} \restriction k(\rho_2) + 1.$$

PROOF. Build p_ω, $\langle I_i : i < \omega\rangle$, $k: \omega^{<\omega} \to \omega$ and $\langle v_\rho \restriction k(\rho): \rho \in \omega^{<\omega}\rangle$ by induction on ω using 2.16. □

2.18. LEMMA. *For every* $\underset{\sim}{r} \in V^P$, if $0 \Vdash \underset{\sim}{r} \in 2^\omega$ *then*

$$0 \Vdash \text{“}\underset{\sim}{r} \in V \text{ or } \underset{\sim}{G} \in V[\underset{\sim}{r}]\text{”}.$$

PROOF. By a density argument, we may assume that $p \Vdash \text{“}\underset{\sim}{r} \notin V\text{”}$. Then there exist p_ω, $\langle I_i : i < \omega\rangle$, $k: \omega^{<\omega} \to \omega$ and $\langle v_\rho \restriction k(\rho) + 1: \rho \in \omega^{<\omega}\rangle$ satisfying 2.17(a)–(e).

2.19. *Claim*. $p_\omega \Vdash \text{“}\underset{\sim}{G} \in V[\underset{\sim}{r}]\text{”}$.

Proof of the Claim. Let G be generic over V such that $p_\omega \in G$. Then from $\underset{\sim}{r}[G]$, we can choose only one member ρ_i of each I_i such that $v_{\rho_i} \restriction k(\rho_i) + 1 \subseteq \underset{\sim}{r}[G]$. Then the sequence $\langle \rho_i : i < \omega\rangle$ belongs to $V[\underset{\sim}{r}]$ and this sequence generates G. This proves the claim.

Clearly this finishes the proof of the lemma also. □

Now we will produce good sequences of filters.

2.20. LEMMA. *CH implies that there are good sequences of filters.*

PROOF. Fix $\langle A_\eta : \eta \in \omega^{<\omega}\rangle$, a partition of ω into infinite sets. Fix $\langle g_\alpha : \alpha < \omega_1\rangle$, an enumeration of the one-to-one functions from ω to ω, and fix $\langle B_\alpha : \alpha \in \omega_1\rangle$, an enumeration of the infinite subsets of ω. Assume we have $\langle\langle A_\eta^\alpha : \alpha < \beta\rangle : \eta \in \omega^{<\omega}\rangle$ satisfying the following conditions:

(i) $A_\eta^\alpha \subseteq^* A_\eta$.

(ii) $A_\eta^{\alpha_2} \subseteq^* A_\eta^{\alpha_1}$ for every $\alpha_2 > \alpha_1$ and $\eta \in \omega^{<\omega}$.

(iii) $g_\alpha \restriction \bigcup\{A_\eta^\alpha : \eta \in \omega^{<\omega}\}$ is one-to-one.

Induction. The case $\alpha + 1 = \beta$. We choose $\{i_n : n < \omega\}$ such that $g_\beta \restriction \{i_n : n < \omega\}$ is one-to-one and if $\eta \in \omega^{<\omega}$ then $\{i_n : n < \omega\} \cap A_\eta^\alpha$ is infinite. Then we set $A_\eta^\beta = A_\eta^\alpha \cap \{i_n : n < \omega\}$.

The case $\beta = \bigcup \beta$. Let $\langle \alpha_j : j < \omega \rangle$ be an increasing sequence of ordinals with limit β. Then let D^η_ω be such that $D^\eta_\omega \subseteq^* A^\eta_{\alpha_j}$, $j < \omega$. Then pick $\{i_n : n < \omega\}$ such that $g_\beta \restriction \{i_n : n < \omega\}$ is one-to-one and, for each $\eta \in \omega^{<\omega}$, $D^\beta_\eta \cap \{i_n : n < \omega\}$ is infinite.
Then set

$$A^\beta_\eta = D^\beta_\eta \cap \langle i_n : n < \omega \rangle.$$

This concludes the induction.

Clearly (a) $\bigcap \{D_\eta : \eta \in \omega^{<\omega}\}$ is a selective filter, and (b) each D_η generates an ultrafilter $\bar{D}_\eta$ which is selective, and we can assume that $\bar{D}_\eta$ is Ramsey.

This shows that $\langle \bar{D}_\eta : \eta < \omega^\omega \rangle$ is a good sequence of filters.

As the referee remarked, a sequence $\langle D_\eta : \eta \in \omega^{<\omega} \rangle$ of Ramsey ultrafilters is good iff the D_η's are pairwise nonisomorphic. This makes the existence of good sequences under CH trivial.

2.21. THEOREM. *If* $V = L$ *(or* $V \models CH$*), then there exists a forcing notion* P *such that* $P \models$ *"ccc" and* P *produces a minimal degree of constructibility.*

PROOF. Clearly $P(\bar{D}) \models$ "ccc".

§3. Minimal degree and splitting families. In this section we will show that the forcing notion introduced in [BSh] produces a minimal degree of constructibility. This forcing notion also produces a subset of ω, say a, with the following property:

$$(\forall b \in [\omega]^\omega)(a \subseteq^* b \text{ or } a \subseteq^* \omega - b)$$

(see [BSh]). The following definitions and lemmas are from [BSh].

3.1. DEFINITION. For natural numbers $n < m$, let $K_{n,m}$ be the set of all binary relations $t \subseteq \mathscr{P}(n) \times \mathscr{P}(m)$ such that, for each $a \subseteq n$, $(a, a) \in t$ and if $(a, b) \in t$ then $b \cap n = a$. (We think of each $t \in K_{n,m}$ as specifying, for each $a \subseteq n$, some permissible extensions of a to subsets of m, each extension being obtained by adjoining to a some elements of $[n, m)$.) If $t \in K_{n,m}$ and $s \in K_{m,l}$, then we write ts for the ordinary composition of these binary relations, so $ts \in K_{n,l}$.

3.2. DEFINITION. (a) For $t \in K_{n,m}$ and $Y \subseteq [n, m)$ we define $t_Y \in K_{n,m}$ by

$$t_Y = \{(a, b) \in t : b \backslash a \subseteq Y\}.$$

(b) We define the *depth* $D_P(t)$ of each $t \in K_{n,m}$ by the following induction

$D_P(t) \geq 0$	always,
$D_P(t) \geq 1$	if, for every $a \subseteq n$, there is $b \subseteq m$ such that $(a, b) \in t$ and $b \neq a$,
$D_P(t) \geq d + 1$	for $d \geq 1$, if, for every partition $\{Y, Z\}$ of $[n, m)$, at least one of $D_P(t_Y)$ and $D_P(t_Z)$ is $\geq d$.

3.3. LEMMA [BSh, 2.1]. *For any* $t \in K_{n,m}$ *and* $s \in K_{m,l}$,

$$\max\{D_P(t), D_P(s)\} \leq D_P(ts) \leq 1 + \max\{D_P(t), D_P(s)\}. \qquad \square$$

Now we will give the definition of the forcing Q.

3.4. DEFINITION. A *condition in* Q is a pair $\langle w, T \rangle$ consisting of a finite subset w of ω and a sequence $T = \langle t_l : l \in \omega \rangle$ such that for some increasing function $n : \omega \to \omega$

(a) $w \subseteq n(0)$,

(b) $t_l \in K_{n(l), n(l+1)}$ for each l, and

(c) $D_P(t_l) \to \infty$ as $l \to \infty$.

(Notice that $\langle w, T\rangle$ determines the function n uniquely, since $\mathscr{P}(n(l))$ is the domain of t_l.) Another such condition $\langle w', T'\rangle$ is an *extension* of $\langle w, T\rangle$ if and only if there is an increasing function $k: \omega \to \omega$ such that, writing t_l^* for $t_{k(l)} t_{k(l)+1} \cdots t_{k(l+1)-1}$,

(a) $(w, w') \in t_0 t_1 \cdots t_{k(0)-1}$, by which we mean $w = w'$ if $k(0) = 0$,

(b) $t'_l \in K_{n(k(l)), n(k(l+1))}$ for all $l \in \omega$, and

(c) $t'_l \subseteq t_l^*$.

Thus, any extension of $\langle w, T\rangle$ is obtained by a succession of operations of the following three sorts.

Compose relations. Partition the sequence T into finite blocks of consecutive t_i's; and compose the t_i's within each block, leave w unchanged. (In the description of extensions above, this is the special case where $k(0) = 0$ and $t'_l = t_l^*$.)

Shrink relations. Replace each t_l by a subset t'_l in $K_{n(l), n(l+1)}$, and leave w unchanged. Of course the t'_l's must be big enough so that their depths tend to ∞ with l. (This is the special case where $k(l) = l$ for all l.)

Fix values. Replace w by some w' such that $(w, w') \in t_0 \cdots t_{m-1}$ for some m, and delete the initial segment $t_0 \cdots t_{m-1}$ from T, so $t'_l = t_{k+l}$. (This is the special case where $k(l) = m + l$ and $t'_l = t_l^*$.)

We think of a condition (w, T) as providing the following information about the generic $W \subseteq \omega$ being produced: $W \cap n(0) = w$, and, for each l, $(W \cap n(l), W \cap n(l+1)) \in t_l$. It will be useful to view a condition (w, T) as a labeled tree in which the root (at level 0) is labeled w and, if a node at level l is labeled with a set $a \subseteq n(l)$, then its immediate successors are labeled with the sets $b \subseteq n(l+1)$ such that $(a, b) \in t_l$. Thus, the set of labels at level m is

$$\mathrm{Lev}_{(w,T)}(m) = \{a \mid (w, a) \in t_0 \cdots t_{m-1}\}.$$

We also write Tree(w, T) for the set $\bigcup_m \mathrm{Lev}(m)$ of all the labels occurring in the tree.

3.5. LEMMA [BSh, 2.6]. *Let $(w, T) \in Q$ and let C map the finite subsets of ω into $\{0, 1\}$. Then either there is an extension (w', T') of (w, T) such that C maps* Tree(w', T') *to 0, or there is an extension (w, T') of (w, T) such that C maps* Tree$(w, T') - \{w\}$ *to 1.* □

3.6. LEMMA [BSh, 2.9]. *Let $\underset{\sim}{A}$ be a Q-name for a subset of ω. Then every condition (w, T) has an extension (w, S) with the following property. If $l \in \omega$, if $n = n(l)$ is the number such that s_l has domain $\mathscr{P}(n)$, if $(w, w^*) \in s_0 \cdots s_{l-1}$ and if $i < n$, then $(w^*, S - l)$ decides whether $i \in \underset{\sim}{A}$ (where $S - l = \langle s_{k+l}: k < \omega\rangle$).* □

All the above material was taken, almost without changes, from [BSh]. Now we will prove new propositions about Q.

3.7. LEMMA. *If $n, d < \omega$ are given, then there are $m > n$ and $r \in K_{n,m}$ such that $D_P(r) \geq d$ and if*

$$w_a \neq w_b \quad \text{and} \quad w_a r w_c \quad \text{and} \quad w_b r w_d,$$

then $w_c \neq w_d$.

PROOF. We will show that this is possible by induction on d. In order to fix notation we will assume that m was given. Let $a: \mathscr{P}([n, m)) \to \mathscr{P}([0, n))$ be a function. We will give a probability measure to the space of such functions.

First take the equidistributed probability on the space $\mathscr{P}([0,n))$. This says that every member of $\mathscr{P}([0,n))$ has measure 2^{-n}. Then take the product measure for $\{a: \mathscr{P}([n,m)) \to \mathscr{P}([0,n))\}$; that is, the measure of a is $(2^{-n})^{2^{m-n}}$, for each such function.

For each such function a, define a member $r^a \subseteq K_{n,m}$ by

$$wr^a w' \cup w \quad \text{iff} \quad a(w') = w \text{ or } w = w'.$$

We will show, by induction on d, that if $Y \subseteq [n,m)$ and $|Y| \geq f(n,d)$, then there is $1 \geq c(n,d) > 0$ such that the probability (Pr) of the a's such that

$$D_P(r^a_Y) \geq d$$

is greater or equal than $1 - c(n,d)^{2^{(|Y|/2^d)}}$.

Case $d = 0$. Put $f(n,0) = 0$ and $c(n,d) = 0$.

Case $d = 1$. For each $w \in \mathscr{P}([0,n))$ we have

$$\begin{aligned} &\Pr(\text{there is no } w' \neq \varnothing \text{ such that } wr^a w' \cup w) \\ &\quad = \Pr((\forall w' \in \mathscr{P}(Y) - \{0\})(a(w') \neq w)) \\ &\quad = \prod_{w' \in \mathscr{P}(Y) - \{0\}} \Pr(a(w') \neq w) = (1 - 2^{-n})^{2^{|Y|} - 1}. \end{aligned}$$

Therefore

$$\begin{aligned} \Pr\{a: D_P(r^a_Y) < 1\} &\leq 2^n \cdot (1 - 2^{-n})^{2^{|Y|} - 1} \\ &= [(2^{n \cdot 2^{-|Y|}})(1 - 2^{-n})^{1 - 2^{-|Y|}}]^{2^{|Y|}}. \end{aligned}$$

Then fix $f(n,1)$ such that if $|Y| \geq f(n,1)$ we obtain that

$$(2^{n \cdot 2^{-|Y|}})(1 - 2^{-n})^{1 - 2^{-|Y|}} < 1;$$

take this to be $c(n,1)$.

Case $d' = d + 1$. We have

$$\begin{aligned} \Pr\{a: D_P(r^a_Y) \ngeq d+1\} &= \Pr\{a: D_P(r^a_Y) \leq d\} \\ &\leq \sum_{\substack{Z \subseteq Y \\ |Z| \geq |Y|/2}} \Pr\{a: D_P(r^a_Z) \ngeq d\} \leq \sum_{\substack{Z \subseteq Y \\ |Z| \geq |Y|/2}} c(n,d)^{2^{|Z|/2^d}} \\ &\leq 2^{|Y|} \times c(n,d)^{2^{|Y|/2^{d+1}}} = [2^{|Y|/2^{|Y|/2^{d+1}}} \cdot c(n,d)]^{2^{|Y|/2^{d+1}}}. \end{aligned}$$

Then fix $f(n, d+1) > f(n,d)$ such that if $|Y| \geq f(n, d+1)$ we obtain that

$$2^{|Y|/2^{|Y|/2^{d+1}}} \cdot c(n,d) < 1;$$

take this to be $c(n, d+1)$.

Now we finish the proof of the lemma by taking $m > f(n,d)$. □

From now on we will assume that $\underset{\sim}{A}$ is a Q-name for a real, i.e., a subset of ω, and that

$$(w, S) \Vdash \text{“}\underset{\sim}{A} \notin V\text{”}.$$

The reader may check that without loss of generality we may assume that $w = \varnothing$, and by 3.6 we may assume that, over $(\varnothing, S)$, $\underset{\sim}{W} \cap [0, n(l))$ decides $\underset{\sim}{A} \cap [0, n(l))$, where $\underset{\sim}{W}$ is the canonical name for Q and $n(l)$ is the domain of s_l. Also we write $C(w')$

for $\underset{\sim}{A} \cap [0, n(l))$ when $w' = \underset{\sim}{W} \cap [0, n(l))$, and we write $C(w \cup \varphi_{[n(l), n(l+1))})$ for $\underset{\sim}{A} \cap [0, n(l+1))$ when $\underset{\sim}{W} \cap [0, n(l+1)) = w$.

3.8. LEMMA. *For every $n(l)$, if $w \in \mathscr{P}([0, n(l)))$ then there exists S' such that $(w, S - l) \leq (w, S')$, and for every k, if $ws'_k w'$ and $w' - w \neq \varnothing$, then $C(w') \neq C(w \cup \varphi_{[n(l), n'(k+1))})$.*

PROOF. For fixed l, for fixed $w \in P([0, n(l)))$, for every $k > l$ we have that $C(w \cup \varphi_{[n(l), n(k))})$ is an increasing sequence of partial characteristic functions, such that the union defines a real $r(w)$. Clearly $(\varnothing, S) \Vdash$ "$\underset{\sim}{A} \neq r(w)$". Therefore there is w' and $l' > l$ such that $ws_{l'}w'$ and $C(w') \nsubseteq r(w)$. This will induce the following coloring function on Tree$(w, S - l)$: "μ is 1" if there is $k < \omega$ such that $\mu \in \mathscr{P}(n(k))$, $\mu \in$ Tree$(w, s - l)$ and $C(\mu) \nsubseteq r(w)$; "μ is 0" otherwise.

By Lemma 3.5 we obtain an extension (w, S') of $(w, s - l)$ such that the coloring function is constant and equal to 1 on Tree$(w, S') - \{w\}$. □

3.9. LEMMA. *There exists $(\varnothing, S') \in Q$ such that $(\varnothing, S) \leq (\varnothing, S')$ and for every $w, w' \in$ Tree$(\varnothing, S')$ we have that*

$$C(w \cup \varphi_{[n'(l), n'(k))}) \neq C(w')$$

(where $(w, w') \in s'_k$).

PROOF. Use 3.8 and a fusion argument as in [BSh]. □

From now on we will assume that $S = S'$, satisfying Lemma 3.9, and also that the function $D_P(s_l)$ grows very fast. Now by induction on $l \in \omega$ we will define r_l and $k(l)$ such that $\langle \varnothing, \langle r_l \colon l < \omega \rangle \rangle$ will be an extension of $\langle \varnothing, S \rangle$.

Let the power set of $n'(l)$ be dom(r_l).

Suppose we are at the stage $l + 1$. Then we proceed as follows:

By Lemma 3.7 for $n = n'(l)$ and $d = n'(l)$ there are m and $r \in K_{n,m}$ satisfying the requirements of 3.7.

Let $k(l+1) = k(l) + m$.

Let $S \restriction [k(l), k(l) + m) = s_{k(l)}, \ldots, s_{k(l)+m-1}$. Then we define $r_l = \{(w, u) \colon w \in \mathscr{P}(n'(l))$ and $w = u \cap [0, n'(l))$, and there is $v \in \mathscr{P}(m)$, wrv, such that (a) if $n \leq i < m$ and $i \notin v$, then $u \cap [n(k(l) + i), n(k(l) + i + 1)) = \varnothing$; (b) if $n \leq i < m$ and $i \in v$, then

$$u \cap [n(k(l) + i), n(k(l) + i + 1)) \neq \varnothing$$

and $(u \cap [0, n(k(l) + i)))s_{k(l)+i}(u \cap [0, n(k(l) + i + 1)))\}$.

Clearly $r_l = r_l(s_{k(l)}, \ldots, s_{k(l)+m}, r)$ (this notation will be used in the proof of 3.10). It is not hard to see that $r_l \in K_{n'(l), n'(l+1)}$, where $n'(l+1) = n(k(l+1))$.

3.10. *Claim.* $D_P(r_l) \geq d$.

Proof. By induction on $d' \leq d$. The case $D_R(r_l) \geq 1$ follows easily from the definition if we take $v - w \neq \varnothing$.

Case $D_P(r_l) \geq d' + 1$. Suppose $Y \subseteq [n'(l), n'(l+1))$.

For every $i = 0, 1, \ldots, m - 1$, there is

$$Z_i \in \{Y \cap [n(k(l) + i), n(k(l) + i + 1)), [n(k(l) + i), n(k(l) + i + 1)) - Y\}$$

such that $D_P(s_{k(l)+i}) \geq d' + 1$ (here we use the fast growth of D_P). Set

$$Z^* = \{i \colon Z_i = Y \cap [n(k(l) + i), n(k(l) + i + 1))\}.$$

Then $D_P(r_{Z^*}) \geq d'$ or $D_P(r_{m-Z^*}) \geq d'$. Without loss of generality we assume that $D_P(r_{Z^*}) \geq d'$.

By the induction hypothesis

$$D_P(r_l((s_{k(l)})_{Z_0}, \ldots, (s_{k(l)+m-1})_{Z_{m-1}}, r_{Z^*})) \geq d';$$

but

$$r_l((s_{k(l)})_{Z_0}, \ldots, (s_{k(l)+m-1})_{Z_{m-1}}, r_{Z^*}) \subseteq (r_l)_{Y^*},$$

where $Y^* = \bigcup\{Y \cap [n(k(l)+i), n(k(l)+i+1)) \colon i \in Z^*\}$. □

This shows that $(\varnothing, \langle r_l \colon l < \omega\rangle)$ is a condition extending $(\varnothing, S)$; we call this condition $(\varnothing, R)$, or simply $(*)$.

3.11. LEMMA. *Fix $l < \omega$. Suppose that the following conditions hold:*

(a) *$w_1 r_l w'_1$ and v_1 witnesses this, i.e. $w_1 r v_1$.*

(b) *$w_2 r_l w'_2$ and v_2 witnesses this, i.e. $w_2 r v_2$.*

(c) *$w_1 \neq w_2$.*

Then $C(w'_1) \neq C(w'_2)$.

PROOF. There is $n'(l) \leq i < n'(l+1)$ such that $i \in v_1 \Leftrightarrow i \notin v_2$. Suppose that $i \in v_1$ and $i \notin v_2$.

Now in the interval $[n(k(l)+i), n(k(l)+i+1) = I$, w'_1 is not empty and w'_2 is empty. Then $C(w'_1 \cap n(k(l)+i+1)) \neq C(w'_1 \cap n(k(l+1)) \cup \varphi_I)$ (by 3.9), and this says that $C(w'_1) \neq C(w'_2)$. □

3.12. THEOREM. $0 \Vdash (\forall x \in \mathbb{R})(x \in V$ *or* $\underset{\sim}{W} \in V[x])$.

PROOF. Let $\underset{\sim}{A}$ be a Q-name for a subset of ω; without loss of generality we may assume that $0 \Vdash_Q$ "$\underset{\sim}{A} \notin V$". Let (s, S) be a member of Q. Without loss of generality we can take $s = \varnothing$. We may also assume that $(\varnothing, S)$ satisfies the condition of 3.9 and that $(\varnothing, R)$ is an extension of $(\varnothing, S)$ satisfying $(*)$. Then if $\underset{\sim}{W}[G]$ is the realization of $\underset{\sim}{W}$ using G and $(\varnothing, R) \in G$, then by 3.11 we may compute $\underset{\sim}{W}[G]$ from $\underset{\sim}{A}[G]$. This shows that $(\varnothing, R) \Vdash_Q$ "$\underset{\sim}{W} \in V[\underset{\sim}{A}]$". This ends the proof of the theorem. □

REFERENCES

[ASh] U. ABRAHAM and R. SHORE, *The degrees of constructibility of Cohen reals*, ***Proceedings of the London Mathematical Society***, ser. 3, vol. 53 (1986), pp. 193–208.

[Bl] A. BLASS, *Selective ultrafilters and homogeneity*, ***Annals of Pure and Applied Logic***, vol. 38 (1988), pp. 215–255.

[BSh] A. BLASS and S. SHELAH, *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, ***Annals of Pure and Applied Logic***, vol. 33 (1987), pp. 213–243.

[Gr] C. W. GRAY ***Iterated forcing from the strategic point of view***, Ph.D. thesis, University of California, Berkeley, California, 1980.

[Ih] J. IHODA, *Σ^1_2-sets of reals*, this JOURNAL, vol. 53 (1988), pp. 636–642.

[ISh1] J. IHODA and S. SHELAH, *Δ^1_2-sets of reals*, ***Annals of Pure and Applied Logic***, vol. 42 (1989), pp. 207–223.

[ISh2] ———, *The Kunen-Miller chart*, this JOURNAL, vol. 55 (1990), pp. 909–927.

[Ma] A. MATHIAS, *Happy families*, ***Annals of Mathematical Logic***, vol. 12 (1977), pp. 59–111.

[Mi] A. MILLER, *Some properties of measure and category*, ***Transactions of the American Mathematical Society***, vol. 266 (1981), pp. 93–114.

[Sa] G. SACKS, *Forcing with perfect closed sets*, ***Axiomatic set theory*** (D. Scott, editor), Proceedings of Symposia in Pure Mathematics, vol. 13, part 1, American Mathematical Society, Providence, Rhode Island, 1971, pp. 331–355.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720

MATHEMATICAL SCIENCES RESEARCH INSTITUTE
BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICS
BAR ILAN UNIVERSITY
52-100 RAMAT GAN, ISRAEL

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL

DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903

Sorting: The first three addresses are Professor Judah's; the last two are Professor Shelah's. For both, the Israel address is the current one.