ELSEVIER Annals of Pure and Applied Logic 71 (1995) 69–106

ANNALS OF PURE AND APPLIED LOGIC

Applications of Cohomology to Set Theory I: Hausdorff Gaps

Daniel E. Talayco[1]

Department of Mathematics, Boston College, Chestnut Hill, MA 02167, USA

Received 22 August 1993; communicated by T. Jech

Abstract

We explore an application of homological algebra to set theoretic objects by developing a cohomology theory for Hausdorff gaps. This leads to a natural equivalence notion for gaps about which we answer questions by constructing many simultaneous gaps. The first result is proved in ZFC while new combinatorial hypotheses generalizing ♣ are introduced to prove the second result. The cohomology theory is introduced with enough generality to be applicable to other questions in set theory.

Additionally, the notion of an *incollapsible gap* is introduced and the existence of such a gap is shown to be independent of ZFC.

1. Introduction

Hausdorff gaps appear in a wide scope of applications in the literature on set theory and a fitting and voluminous tribute to the importance of these objects has recently been published by Scheepers [16]. Hence, I shall not attempt to provide further incentive for their study at this time, but anticipate an interest in understanding their structure.

In this article we explore a new approach to viewing gaps, adapting tools from cohomology to describe them. In fact, although new information about gaps is gained in this way, this framework for investigating set theoretic structures is of as much interest to the author as the information about gaps. The apparatus of cohomology will be shown to be an appropriate tool for understanding and directing questions about objects in set theory by exploring this particular example. We will see how cohomology provides insight into the pertinent issues underlying gaps and from the direction so obtained will formulate and answer questions about gaps.

[1] Results in this paper constitute a substantial portion of my dissertation completed at The University of Michigan in Ann Arbor under the supervision of Andreas Blass.

SSDI 0168-0072(94)00020-4

In a future paper, we will examine the case of ω_1-trees and develop a cohomology theory for a class of Aronszajn trees. From this cohomological formulation of trees, we will be able to describe a connection between gaps and trees. This will add evidence to the circumstantial similarities between gaps and trees that presently exist because of their constructions and similar behavior under the alchemy of forcing.

This cohomological framework is thus initially unifying. More enticing is the vast amount of technical apparatus available in the various guises of the categorical theory of homological algebra and derived functors.

The modus operandi for this investigation is as follows: 1. Describe the choices and interpretations of chain groups and connecting maps for the selected context. 2. Show that these interpretations capture the given objects. 3. Manipulate the cohomological apparatus to extract information about the associated groups. 4. Interpret this information and use it to direct further investigation of the given objects. One benefit of this approach is the appearance of other venues of generalization not visible from a traditionally set theoretic viewpoint. Further, topological cohomology offers some useful mnemonic and visual associations.

This method of exploration will be both economical and flexible. The set theorist with little familiarity with cohomology should find the objects familiar enough to make the terminology understandable. For the reader familiar with cohomology but unfamiliar with set theory, the approach should permit her or him to be introduced to the pertinent objects within a familiar context.

1.1. Notation and conventions

Notation used is essentially standard for set theory. We use [12] as a reference. Additionally, we use $\triangle$ to denote the symmetric difference of sets. $P(\omega)/fin$ is considered as a boolean algebra, though it should be noted that it maintains a group structure (actually a $\mathbb{Z}/2$-module structure) under the operation induced by $\triangle$. The adverb "almost" will mean "modulo a finite quantity".

The functions max and min take as arguments a list of numbers, while sup, inf and $\sup^+$ (the proper supremum of a set of ordinals) take a set as an argument.

α, β, ξ, η, etc., will always denote ordinals, usually countable. Whenever possible, we will follow the convention that $\beta \leqslant \alpha$ and $\eta \leqslant \xi$. λ will always represent a countable limit ordinal and Λ will be the collection of all countable limit ordinals. κ will usually denote an infinite cardinal. n, m, and i will usually denote elements in ω. If the context is clear, then quantification of these variables will implicitly be over objects of the proper type. Thus in a proof where β is used as an arbitrary countable ordinal, $(\forall\beta)$ will mean $(\forall\beta \in \omega_1)$.

1.2. Introduction to Hausdorff gaps

The usual formulation of a Hausdorff gap is a pair of sequences of length ω_1 of subsets of ω, one almost increasing (increasing modulo finite) and the other almost decreasing, with the decreasing sequence "above" the increasing sequence (i.e., every element of the decreasing sequence almost includes every element of the increasing sequence as a subset). The defining property of a gap is that these sequences have no space in the middle, i.e., there is no set that fits between, continuing both sequences. Hausdorff gaps, then, are a measure of incompleteness (in the sense of Dedekind) of $\mathscr{P}(\omega)$. Note that the order "inclusion modulo finite sets" on $\mathscr{P}(\omega)$ is substantially different from the usual order on the reals. Most notably here, there are uncountable well-ordered chains in the former.

If each element of the top (decreasing) sequence of sets in a gap is complemented relative to ω, the result is a pair of increasing sequences such that any pair of elements, one from each sequence, is almost disjoint. The defining property then becomes: There is no subset of ω continuing one sequence and remaining almost disjoint from each element of the other. This is the formulation of gap used most frequently in this paper. Increasing sequences will be called towers.

Definition 1. Let $\subseteq^*$ be the pre-order on $\mathscr{P}(\omega)$ given by $A \subseteq^* B$ if and only if $A \setminus B$ is finite. If this relationship holds, we say A is almost included in B and B almost includes A. If $A \subseteq^* B$ and additionally $B \setminus A$ is infinite, we say B is strictly above A and write $A \subsetneq^* B$. Denote by $A =^* B$ the statement that $A \triangle B$ is finite and say A and B are almost equal. We use this notation for sets in general.

Definition 2. Let $\mathscr{P}(\omega)/fin$ denote the Boolean algebra obtained by taking the subsets of the natural numbers modulo the ideal of finite sets. The partial order in this algebra is induced by $\subseteq$, or equivalently, by $\subseteq^*$. Hence we use $\subseteq^*$ as the order symbol on the Boolean algebra and may conflate elements of the Boolean algebra with representatives when convenient.

Definition 3. Two elements of $\mathscr{P}(\omega)$ are said to be *almost disjoint* if their intersection is finite. This can also be denoted $A \cap B =^* \emptyset$. Unless explicitly noted otherwise, we use the notion upwards hereditarily: That is, two collections of subsets of ω are said to be almost disjoint if every pair of elements, one from either collection, is almost disjoint.

Definition 4. A *tower*, $T = \langle T_\alpha : \alpha \in \kappa \rangle$, in $\mathscr{P}(\omega)$ is an indexed sequence of elements of $\mathscr{P}(\omega)$ increasingly linearly ordered by $\subseteq^*$: $\beta < \alpha \Rightarrow T_\beta \subseteq^* T_\alpha$. The ordinal κ is said to be the height of the tower. T_α will be called the αth level of T. A *subtower*, S, of T is a tower satisfying for each α, $S_\alpha \subseteq T_\alpha$ (levelwise inclusion) and additionally for each $\beta < \alpha$, $S_\alpha \cap (T_\beta \setminus S_\beta)$ is finite. (Restriction to lower levels of the subtower is "faithful".)

The notion of a tower is more natural in $\mathscr{P}(\omega)/fin$. We introduce towers as objects in $\mathscr{P}(\omega)$ because most of the technical work is done in this setting. However, it will benefit the reader to think about all statements referring to towers in the context of $\mathscr{P}(\omega)/fin$.

Definition 5. A *pregap* (or an ω_1-pregap) is a pair of towers $\langle A, B\rangle$, each of height ω_1 such that for each $\alpha < \omega_1$, $A_\alpha \cap B_\alpha =^* \emptyset$. Note that both A and B are subtowers of the tower $A \cup B$, the level-wise union of A and B.

Definition 6. A *Hausdorff gap* is a pregap, $\langle A, B\rangle$, satisfying: There is no X in $\mathscr{P}(\omega)$ such that for all α, $X \supseteq^* A_\alpha$ and X is almost disjoint from B_α. We say such an X *separates* or *fills* a pregap. The definition is actually symmetric between A and B as can be seen by taking the complement of X relative to ω.

We will usually be interested in gaps that exist inside a given tower. Hence we give the following definition.

Definition 7. Given a tower, T, in $\mathscr{P}(\omega)$ of height ω_1, and a subtower of T, A, let B be the levelwise complement of A in T: $B_\alpha := T_\alpha \setminus A_\alpha$. We say *$A$ is a gap in T* if and only if the pair $\langle A, B\rangle$ is a Hausdorff gap. Assuming only that A is a subtower of T, it is easy to show the tower B is a subtower of T and that the pair $\langle A, B\rangle$ is a pregap.

Notice that a subtower, A, is a gap in a tower T just in case there is no $X \subseteq \omega$ with $X \cap T_\alpha =^* A_\alpha$ for all α. As above, we say such an X fills or separates A. Restricting attention to gaps inside of a tower does not reduce the generality of the considerations since for any Hausdorff gap $\langle A, B\rangle$, A is a gap in the tower $A \cup B$.

The existence of a gap, proven from the axioms of ZFC, was demonstrated by Hausdorff in the first decade of this century. It is this theorem which we shall generalize in later sections. We produce many "different" gaps simultaneously, building them side by side. The number of simultaneous gaps produced in eponymous, hence the $\aleph_0$ and $\aleph_1$ gap theorems. In fact, we do substantially better in each case, at least for our purposes, as indicated by the corollaries following each theorem. The motivation for these theorems and the notion of "different" are developed in the next section. We first state the basic theorem and sketch the methods of generalization.

Theorem 8 (The basic gap theorem Hausdorff [9]). *There is a Hausdorff gap.*

For a proof of this theorem, see [8, p. 36]. It is this proof from which we depart in the generalizations which follow. We take a moment to review the important elements of the proof and indicate the directions in which changes will be made.

Two towers are simultaneously constructed by recursion on the level. There are two conflicting requirements to fulfill during the construction. The first is a "minimize intersection" requirement to ensure that sets in the different towers are almost disjoint. The second is a "maximize intersection" requirement to ensure the result is in fact a gap. If the sets being built are A_α and B_α for $\alpha \in \omega_1$ then one possible maximization requirement is phrased:

$$(\forall\alpha)(\forall r \in \omega)\ \{\beta < \alpha\colon A_\alpha \cap B_\beta \subseteq r\} \text{ is finite.}$$

This (with the minimization requirement) is sufficient to ensure the pair $\langle A, B\rangle$ is a gap and is used in the $\aleph_0$ gap theorem.

There is an alternative to these requirements (attributed to Todorčević) which can be stated as follows:

$$(\forall\alpha)(\forall\beta) \quad A_\alpha \cap B_\beta = \emptyset \iff \alpha = \beta.$$

This clearly implies the previous minimize requirement, but the proof, under this hypothesis, that the pair $\langle A, B\rangle$ is a gap is slightly different. It is a variant of this second condition which is used in the $\aleph_1$ gap theorem.

In the basic gap theorem, most of the work occurs at limit stages. At those stages, a first approximation to one side of the gap is defined satisfying the minimization requirement. This set is carefully enlarged to satisfy the maximization requirement without ruining the previous work. This is the process which is recursively applied in the $\aleph_0$ gap theorem at limit stages. However, in the $\aleph_1$ gap theorem, we depart from this procedure adapting (the term is used loosely) the techniques of forcing to obtain the desired result.

2. The gap cohomology group

We turn now to the motivation for the theorems which are to follow. In fact, though substantial and technical, those theorems only begin to explore the potential generalizations and applications of homological algebra to problems in set theory and other areas in logic. The original observation that gaps are cohomological in nature is due to Blass.

The following is a list of homological ideas used but not defined here: short and long exact sequences; the properties of boundary maps and how they produce cocycles and coboundaries; the definition of a cohomology group and how it is a measure of failed exactness. The unfamiliar reader can find a reasonable introduction to this material in [18] or [10]. More advanced uses of homological algebra are alluded to, but the results do not depend upon them.

Herein, fix a tower, T, of height ω_1 which may be referred to as the ambient tower. Let $\mathscr{D} \subseteq \mathscr{P}(\omega)$ be the family of subsets of ω generated by the closure under "finite upward modification" of sets in T, i.e.,

$$D \supseteq T_\alpha \wedge D \setminus T_\alpha \text{ finite} \Rightarrow D \in \mathscr{D}.$$

Note that if $\alpha, \beta \in \omega_1$ then $T_\alpha \cup T_\beta \in \mathscr{D}$. More generally, $(\mathscr{D}, \subseteq)$ as a partial order is directed upwards. To each element $D \in \mathscr{D}$, associate the following coefficient groups:

- $G_D := {}^D(\mathbb{Z}/2)$, the functions from D into $\mathbb{Z}/2$, which will be associated with subsets of D,
- $F_D := \bigoplus_D(\mathbb{Z}/2)$, the finitely supported functions from D into $\mathbb{Z}/2$, associated with finite subsets of D, and
- $(G/F)_D := G_D/F_D$, associated with the Boolean algebra $\mathscr{P}(D)/\mathit{fin}$.

These function groups have their group operation induced componentwise from $\mathbb{Z}/2$. We interpret such functions as characteristic functions following the convention that f is the characteristic function of the set $f^{-1}``\{1\}$ which will be denoted by $\bar{f}$. (Also, if $s(\xi)$ is a function into $\{0,1\}$ then $\bar{s}(\xi)$ is the pre-image of $\{1\}$). In this case, the sum of two functions is the characteristic function of the symmetric difference of the represented sets.

If $D_1 \supseteq D_2$ then there is a natural restriction map, ρ_2^1 from G_{D_1} to G_{D_2} namely $f \mapsto f \upharpoonright D_2$: similarly for F and G/F. We may now define cochain groups for each collection of coefficients. A 0-cochain is just a choice function fixing for each $D \in \mathscr{D}$ an element of G_D. An n-cochain associates to each linearly ordered $n+1$ element set, $D_0 \supseteq D_1 \supseteq \cdots \supseteq D_n$, an element of the "smallest" coefficient group, G_{D_n}. Hence we define

$$C^n(\mathscr{D}, G) := \prod_{\substack{D_0 \supseteq D_1 \supseteq \cdots \supseteq D_n \\ D_i \in \mathscr{D}}} G_{D_n}.$$

The coboundary operator connects the n- and $n+1$-cochain groups. The elements it sends to 0 can be thought of as locally patching. We employ functional notation to denote the evaluation of a cochain on its arguments. For $x \in C^n(\mathscr{D}, G)$ we have $\delta x \in C^{n+1}(\mathscr{D}, G)$ given by

$$(\delta x)(D_0 \supseteq \cdots \supseteq D_{n+1})$$
$$:= \sum_{i=0}^{n} [(-1)^i x(D_0 \ldots \hat{D}_i \ldots D_{n+1})] + (-1)^{n+1} \rho_{n+1}^n (x(D_0 \ldots D_n)),$$

where $\hat{D}_i$ means D_i is removed from the sequence. This is essentially cohomology on a presheaf. See [11]. The corresponding definitions for F and G/F are similar.

Having made these definitions, we introduce a useful cohomological tool that relates the three cohomologies. From the short exact sequence of coefficient groups,

$$0 \to F \to G \to G/F \to 0$$

we induce the following long exact sequence of cohomology groups:

$$0 \to H^0(\mathscr{D}, F) \xrightarrow{i^*} H^0(\mathscr{D}, G) \xrightarrow{\pi^*} H^0(\mathscr{D}, G/F) \xrightarrow{\delta^*}$$
$$\to H^1(\mathscr{D}, F) \to H^1(\mathscr{D}, G) \to H^1(\mathscr{D}, G/F) \to \cdots \tag{1}$$

It is time to state a few properties of these groups and understand what is represented in the above sequence. Again, assume for the following that T is an ω_1 tower in $\mathscr{P}(\omega)$ and $\mathscr{D}$ is generated from T by closing under finite upward modification. We additionally assume that $\bigcup_{\alpha \in \omega_1} T_\alpha = \omega$, though this is inconsequential.

Proposition 9. (a) *$H^0(\mathcal{D}, G) \cong \mathcal{P}(\omega)$ with $\triangle$ (symmetric difference) as the group operation in $\mathcal{P}(\omega)$.*

(b) *$H^0(\mathcal{D}, F)$ depends on the structure of T. If T is an inextendable tower then $H^0(\mathcal{D}, F) \cong fin$ (the finite subsets of ω) with $\triangle$ as the group operation.*

(c) $H^1(\mathcal{D}, G) = 0$.

This is where the real connection between the set theoretic property of being a tower, for example, and cohomological properties is made. This proposition deals with the "concrete" case where there is no concern about modulo finite.

Proof of Proposition 9. (a) If $x \in H^0(\mathcal{D}, G)$ and $D_0 \supseteq D_1 \in \mathcal{D}$, then the coboundary condition on x indicates $x(D_0) \upharpoonright D_1 = x(D_1)$. Thus we can define without ambiguity a function $\bigcup_{D \in \mathcal{D}} x(D)$. This is the characteristic function of a subset of ω. Conversely, a subset of ω, X, induces an element of $H^0(\mathcal{D}, G)$ via $x(D) = X \cap D$ for each $D \in \mathcal{D}$ (conflating sets and their characteristic functions). Since distinct subsets of ω give rise to distinct cocycles and every cocycle arises in this way, we have a bijective correspondence. The group structure is preserved because the operation in $H^0(\mathcal{D}, G)$ is induced by the operation in $\mathbb{Z}/2$.

(b) If $x \in H^0(\mathcal{D}, F)$, the above applies as well. If T is an inextendable tower, then for any infinite $X \subseteq \omega$, there is an α with $T_\alpha \cap X$ infinite. Thus the function $\bigcup_{D \in \mathcal{D}} x(D)$ must have finite support. Consequently, there is always a T, though not necessarily of height ω_1, with $H^0(\mathcal{D}, F) \cong fin$. If T is extendable, let $X \subseteq \omega$ be an infinite set almost disjoint from all levels of T. Then the cochain given by $x(D) := D \cap X$ is a cocycle (in $H^0(\mathcal{D}, F)$). However, distinct X's may not give distinct cochains.

For one last example here, consider an extendible ω_1 tower, T, still satisfying the weak condition $\bigcup_\alpha T_\alpha = \omega$. If X is an infinite set almost disjoint from each T_α (as T is extendible) and $X' \subsetneq X$ is also infinite, then there is an α with $T_\alpha \cap X' \neq T_\alpha \cap X$ (by the assumption that $\bigcup_\alpha T_\alpha = \omega$). Thus the F cohomology classes associated with X and X' are not equal. As a consequence, $|H^0(\mathcal{D}, F)| = 2^{\aleph_0}$; the previous sentence demonstrates $\geqslant$ while the fact that an element in $H^0(\mathcal{D}, F)$ defines a subset of ω gives the reverse inequality. It is unknown whether under $\neg$CH other alternatives are possible.

(c) This is an example of the facility with which homological algebra can make statements about these structures. The presheaf of interest, G is *flasque* (also called flabby) which implies the higher derived functors of $\varprojlim$ (the cohomology groups of interest) are trivial. See [11]. However, the statement can also be proved directly and doing so reveals how topological visualization can guide our proofs. We wish to show that given a cocycle $x \in C^1(\mathcal{D}, G)$ there is an element $y \in C^0(\mathcal{D}, G)$ such that $\delta y = x$.

First, assume that $T_0 \subseteq T_\alpha$ for all α. (Alternatively, we may assume $T_0 = \emptyset$.) We need to define $y(D)$ for each $D \in \mathcal{D}$ so that

$$(\forall D_1 \supseteq D_2)\ x(D_1, D_2) = y(D_2) - y(D_1) \upharpoonright D_2. \tag{2}$$

For each $n \in \omega$ we will simultaneously define $y(D)(n)$ for each D containing n. We then verify the above equation holds for each $n \in D_2$.

For each $n \in \omega$ we define $y(D)(n)$ for those $D \in \mathscr{D}$ with $n \in D$ by

$$y(D)(n) := -x(D, T_0 \cup \{n\})(n). \tag{3}$$

Applying this definition to the right-hand side of Eq. (2), we have

$$y(D_2)(n) - y(D_1)(n) = x(D_1, T_0 \cup \{n\})(n) - x(D_2, T_0 \cup \{n\})(n),$$

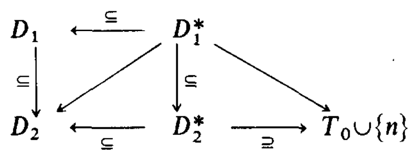
which, by the cochain condition on x applied to the triple $D_1 \supseteq D_2 \supseteq T_0 \cup \{n\}$, is equal to $x(D_1, D_2)(n)$ as desired.

If we do not assume that $T_0 \subseteq T_\alpha$ for all α, then the proof can be completed by defining the operation $D^* := D \cup T_0$ and defining

$$y(D)(n) := (x(D^*, D) - x(D^*, T_0 \cup \{n\}))(n). \tag{4}$$

The motivation for these definitions is in fact geometrical as will be described below.

Suppose D_1 and D_2 are arbitrary elements of $\mathscr{D}$ and that $n \in D_2$. Consider the following diagram



The arrows represent the restriction maps induced on the coefficient groups. The diagram resembles a simplical complex. We can "push off" the information from $x(D_1, D_2)$ to the other edges of the 2-simplex containing that edge because x satisfies the cochain condition. This was the motivation for the definition of y. With this in mind, we present the formal argument, guided by this process of "pushing off" via the cochain condition

$$\begin{aligned}(y(D_2) - y(D_1))(n) &= (x(D_2^*, D_2) - x(D_2^*, T_0 \cup \{n\}))(n) \\ &\quad - (x(D_1^*, D_1) - x(D_1^*, T_0 \cup \{n\}))(n) \\ &= x(D_2^*, D_2)(n) - x(D_1^*, D_1)(n) \\ &\quad + (x(D_1^*, T_0 \cup \{n\}) - x(D_2^*, T_0 \cup \{n\}))(n) \\ &= x(D_2^*, D_2)(n) - (x(D_1^*, D_2) - x(D_1, D_2))(n) + x(D_1^*, D_2^*)(n) \\ &= x(D_1, D_2)(n),\end{aligned}$$

where the first equality is by definition of y, the second by re-arrangement, the third by two applications of the cochain on x to the triples (D_1^*, d_1, D_2) and $(D_1^*, D_2^*, T_0 \cup \{n\})$, and the last equality by rearrangement and another application of the cochain condition on x to the triple (D_1^*, D_2^*, D_2). This gives the desired equality and shows y is sent to x by δ. $\square$

For the following proposition, we shift to the more natural context of $\mathscr{P}(\omega)/fin$. Recall that the partial order in $\mathscr{P}(\omega)/fin$ induced by $\subseteq$ is in fact $\subseteq^*$ and that towers and gaps transfer to this context. Notice that in examining elements of $H^0(\mathscr{D}, G/F)$, we need only consider the original tower as all other elements of $\mathscr{D}$ are finite modifications of elements of the tower and such finite modification is "washed away" by our definitions.

Definition 10. If $x \in H^0(\mathscr{D}, G/F)$, then x is an equivalence class (modulo boundaries) of equivalence classes (modulo finite). By a *representative* of $x(T_\alpha)$ I mean a subset of ω whose characteristic function is a representative (under the modulo finite equivalence relation) of the evaluation at T_α of a cochain representative of x.

Proposition 11. *There is a bijective correspondence between $H^0(\mathscr{D}, G/F)$, and subtowers of T (under the equivalence relation of levelwise almost equality, i.e. in $\mathscr{P}(\omega)/fin$). Further, if x and A are so related, then A is a gap in T if and only if x does not lift under π^* in the long exact sequence,* (1).

Through Proposition 11, we see that the cohomology structure completely captures the notion of Hausdorff gap within a given tower.

Proof of Proposition 11. If $x \in H^0(\mathscr{D}, G/F)$, the cocycle condition on x implies that the choices it makes (as a cochain) for subsets of each element of the tower patch in $\mathscr{P}(\omega)/fin$. That is, if A_α is a representative of $x(T_\alpha)$ for each α, then $\langle A_\alpha : \alpha \in \omega_1 \rangle$ is a subtower of T. Conversely, if A is a subtower of T, define $x(T_\alpha)$ to be (the class containing) A_α. These are inverse operations in $\mathscr{P}(\omega)/fin$.

To demonstrate the second sentence of the proposition, we show the contrapositive of each direction. Recall that by Proposition 9, there is a bijection between $\mathscr{P}(\omega)$ and $H^0(\mathscr{D}, G)$. Suppose $y \in H^0(\mathscr{D}, G)$ and $\pi^* y = x$ where π^* is the map $H^0(\mathscr{D}, G) \to H^0(\mathscr{D}, G/F)$. Let $Y \subseteq \omega$ be the set given for y by Proposition 9 and A the subtower of T associated with x given by the preceding paragraph. We wish to see that Y fills the pregap induced by A, that is, $Y \cap T_\alpha =^* A_\alpha$. But this follows immediately from the fact that $Y \cap T_\alpha = y(T_\alpha)$ and $\pi^* y = x$.

Conversely, suppose A is a subtower of T and $x \in H^0(\mathscr{D}, G/F)$ is the cocycle associated with A. Suppose that $Y \subseteq \omega$ fills A. Then the cocycle $y \in H^0(\mathscr{D}, F)$ associated with Y has $(\pi^* y)(T_\alpha)$ induced by $Y \cap T_\alpha$ which is almost equal to A_α. Thus $\pi^* y = x$ as desired. □

From Proposition 11, we also see a new equivalence relation arising for gaps within a tower. Hence, we give the following definition.

Definition 12. We say that two gaps A and A' in a tower T are *cohomologous* if the levelwise symmetric difference is not a gap in T, i.e., if the subtower given by $\langle A_\alpha \triangle A'_\alpha : \alpha \in \omega_1 \rangle$ is not a gap in T.

By Proposition 11, this is equivalent to saying that the difference of the G/F cohomology classes associated with the gaps does not lift under π^*. As one example of the reasonableness of such a relation, we have the following result.

Proposition 13. *If A and A' are two gaps in T which are cohomologous and $\mathbb{P}$ is a notion of forcing such that it is forced by $\mathbb{P}$ that A is not a gap, then it is similarly forced by $\mathbb{P}$ that A' is not a gap.*

Proof. We argue in the generic extension under the stated assumptions. Let X be a set that fills A. Thus $X \cap T_\alpha = A_\alpha$ for each α. Let Y be a set that fills $A \triangle A'$ by the assumption that A and A' are cohomologous. It is straightforward to check that $X \triangle Y$ fills A'. □

As a corollary to Proposition 13, we have: If A is indestructible under notions of forcing that preserve ω_1 and A is cohomologous to A' then A' is similarly indestructible.

Up to this point, we have gaps associated to the quotient of a cohomology group rather than simply to a cohomology group. We remedy this with the following theorem.

Theorem 14. *The group $H^1(\mathscr{D}, F)$ is isomorphic to the set of gaps in T modulo the equivalence relation "cohomologous", with group operation being levelwise symmetric difference.*

Proof. Examining sequence (1) and Proposition 9(a) and (c) we see

$$H^1(\mathscr{D}, F) \cong H^0(\mathscr{D}, G/F)\ker(\delta^*) \cong H^0(\mathscr{D}, G/F)/\mathrm{im}(\pi^*)$$

where the first $\cong$ is true because δ^* is a surjection (by Proposition 9(c)) while the second is true as sequence (1) is exact. But Proposition 13 gives the desired correspondence between this last group and the gaps in T. □

Thus, there is a cohomology group which represents gaps. Notice that the characteristic properties of gaps are captured by finite sets. As a result of this proposition, we give the following definition.

Definition 15. $H^1(\mathscr{D}, F)$ will be called the *gap cohomology group.*

Next, we state a few more properties about this equivalence relationship on gaps.

Proposition 16. (a) *If A is a gap in T and A' is (almost) obtained from A by symmetric difference by a constant set (i.e., there is an $X \subseteq \omega$ such that for each α, $A'_\alpha =^* A_\alpha \triangle (X \cap T_\alpha)$) then A and A' are cohomologous.*

(b) *If A and A' are two gaps in T such that for all α we have $A_\alpha =^* A'_\alpha$ then A and A' are cohomologous.*

(c) *If A and A' are two gaps in T such that for cofinally many α we have $A_\alpha =^* A'_\alpha$ then A and A' are cohomologous.*

Proof. For (a), we have $A'_\alpha \triangle A_\alpha =^* X \cap T_\alpha$ Thus X fills the pregap $A' \triangle A$.

(b) follows from (a) with $X = \emptyset$. The supposition in (c) for each α implies the stronger condition used in (b): Let β be given and let $\alpha \geqslant \beta$ satisfy $A_\alpha =^* A'_\alpha$. Now for each subtower A and A', we know $A_\alpha \cap T_\beta =^* A_\beta$, etc. Thus $A_\beta =^* A_\alpha \cap T_\beta =^* A'_\alpha \cap T_\beta =^* A_\beta$. Since β was arbitrary, the condition in (b) is satisfied. □

3. The $\aleph_0$ gap theorem

Having shown that cohomology induces an equivalence relation on the gaps within a tower, being cohomologous, we can ask what properties the equivalence classes have. We have seen in Proposition 16 that this equivalence relation smooths out some unimportant differences in gaps. However, it is conceivable that every pair of gaps is cohomologous. The main theorems of this paper, the $\aleph_0$ and $\aleph_1$ gap theorems indicate that this is not the case as is explained in the corollaries following each. The constructions, however, are of interest in their own right, and the additional hypotheses that have arisen in consideration of these questions seem important as well.

We now present the $\aleph_0$ gap theorem, so called because it is based on the construction of $\aleph_0$ simultaneous subtowers in a given tower. It implies that the size of the gap cohomology group is at least $2^{\aleph_0}$.

Theorem 17 (The $\aleph_0$ gap theorem). *Let $T = \langle T_\alpha : \alpha \in \omega_1 \rangle$ be a tower in $\mathscr{P}(\omega)$. Then there is an ω by ω_1 matrix $\langle A(m,\alpha): m \in \omega, \alpha \in \omega_1 \rangle$, with the following properties:*

($\aleph_0$ 1) *For each $m \in \omega$, $\langle A(m,\alpha): \alpha \in \omega_1 \rangle$ is a subtower of T.*
($\aleph_0$ 2) $(\forall m_0)(\forall m_1 > m_0)(\forall \alpha)(\forall r \in \omega)\ \{\beta < \alpha : A(m_0,\alpha) \cap A(m_1,\beta) \subseteq r\}$ *is finite.*
($\aleph_0$ 3) $(\forall \alpha \in \omega_1)(\forall \beta < \alpha)(\exists m_0)(\forall m > m_0)\ A(m,\alpha) \cap T_\beta = A(m,\beta)$.
($\aleph_0$ 4) $(\forall \alpha \in \omega_1)\ T_\alpha = \bigcup \{A(m,\alpha): m \in \omega\}$, *a disjoint union.*

The intuition behind the construction is to see A(column, row) with $(0,0)$ at the lower left. Then each column is a tower growing upwards and each row is a partition of the associated level of the tower T. The first and second conditions ensure that each pair of columns is a Hausdorff gap. See [8, p. 36] for a proof of this. Hence each column is a gap in T.

As previously discussed, to be a gap requires "interaction" (non-empty intersection) between (sets in different) columns, a property which is assured by condition ($\aleph_0$ 2). But to obtain the following corollary where unions of collections of columns are gaps (in particular, are towers) requires the intersections to be controlled as formalized by condition ($\aleph_0$ 3) which can be seen as a more complicated "minimize intersection" requirement. This increases the delicateness and technicality required in the proof making it reminiscent of a priority argument.

Corollary 18. *The cardinality of the gap cohomology group is at least* $2^{\aleph_0}$.

Proof. The third condition ensures that

$$X \subseteq \omega \;\Rightarrow\; \left\langle \bigcup(A(m,\alpha)\colon m \in X\}\colon \alpha \in \omega_1 \right\rangle$$

is a tower by the following argument. Let $A(X,\alpha)$ be the αth level of this sequence. We must check that

$$\beta \leqslant \alpha \;\Rightarrow\; A(X,\beta) \subseteq^* A(X,\alpha).$$

The problem is that the finite differences, $A(m,\beta)\backslash A(m,\alpha)$, that exist between different levels of a given column may accumulate to an infinite quantity under the infinite union. Examining ($\aleph_0$ 3) reveals that for a fixed α and β only finitely many columns can contribute to $A(X,\beta)\backslash A(X,\alpha)$ which hence is finite.

If the set $X \neq \omega, \emptyset$ then it is easy to see that $A(X)$ is a gap in T. Recall that two gaps are not cohomologous iff their levelwise symmetric difference is still a gap. But it is clear that for $X, Y \subseteq \omega$ we have $A(X,\alpha) \triangle A(Y,\alpha) = A(X \triangle Y,\alpha)$ and hence if $X \neq Y$ and $X \neq \omega\backslash Y$ then $A(X)$ is not cohomologous with $A(Y)$. Thus the cardinality of the gap cohomology group is at least that of the continuum. □

Proof of Theorem 17. We construct $A(m,\alpha)$ by induction on α, and then, for limit stages only, induction on m. I shall refer to the restrictions of ($\aleph_0$ 1–4) to an ordinal γ (replacing ω_1 by γ or by λ during a limit stage) as *the induction hypotheses.* The stage $\alpha = 0$ is inconsequential so long as condition ($\aleph_0$ 4) is fulfilled.

Successor stage: Suppose $A(m,\beta)$ have been constructed satisfying the induction hypotheses for $\beta \leqslant \alpha$ and $m \in \omega$. As $T_{\alpha+1}\backslash T_\alpha$ is infinite, it can be partitioned into infinitely many disjoint sets, $S(m)$, $m \in \omega$. Define

$$A(m,\alpha+1) := (A(m,\alpha) \cap T_{\alpha+1}) \cup S(m).$$

We check the induction hypotheses are maintained. ($\aleph_0$ 1) and ($\aleph_0$ 4) are immediate.

For ($\aleph_0$ 2), fix m_0 and m_1. We need to show that the set $\{\beta < \alpha+1\colon A(m_0,\alpha+1) \cap A(m_1,\beta) \subseteq r\}$ is finite. This set is

$$\subseteq \{\beta < \alpha\colon (A(m_0,\alpha) \cap T_{\alpha+1}) \cap A(m_1,\beta) \subseteq r\} \cup \{\alpha\}$$

$$\subseteq \{\beta < \alpha\colon (A(m_0,\alpha) \cap A(m_1,\beta) \subseteq \sup{}^+(r \cup A(m_0,\alpha)\backslash T_{\alpha+1})\} \cup \{\alpha\},$$

which is finite by induction hypothesis.

For ($\aleph_0$ 3) we need to show

$$(\forall\beta \leqslant \alpha)(\exists m_0)(\forall m > m_0)\ A(m,\alpha+1)\cap T_\beta = A(m,\beta). \tag{5}$$

For $\beta = \alpha$, this is true by the definition of $A(m,\alpha+1)$ and:

(a) $(\forall m) S(m)\cap T_\alpha = \emptyset$.

(b) $T_\alpha \subseteq^* T_{\alpha+1} \Rightarrow (\exists m_1)(\forall m > m_1)\, A(m,\alpha) \subseteq T_{\alpha+1}$.

Now fix $\beta < \alpha$. Then we have:

(c) By induction hypothesis ($\aleph_0$ 3)

$$(\exists m_2)(\forall m > m_2)\ A(m,\alpha)\cap T_\beta = A(m,\beta).$$

(d) As $T_\beta \subseteq^* T_\alpha$, $S(m)$ are disjoint, and for all $m, S(m)\cap T_\alpha = \emptyset$, so

$$(\exists m_3)(\forall m > m_3)\ S(m)\cap T_\beta = \emptyset.$$

Fix m_1, m_2 and m_3 as in (b), (c) and (d) respectively. Let $m_0 = \max(m_1, m_2, m_3)$ and express $A(m,\alpha+1)$ as $(A(m,\alpha)\cap T_{\alpha+1})\cup S(m)$ to derive (5).

This marks the end of the successor stage.

Limit stage: Now suppose λ is a limit ordinal and for $m \in \omega$ and $\alpha < \lambda$, sets $A(m,\alpha)$ have been constructed fulfilling the induction hypotheses. Fix a function $f: \omega \to \lambda$ which is increasing and cofinal in λ, with $f(0) = 0$.

We now construct the sets $A(m,\lambda)$ by recursion on $m \in \omega$. Assume that the sets $A(s,\lambda)$ have been constructed for $s < m$. For notation, let $T(m,\lambda)$ denote the set $T_\lambda \setminus \bigcup\{A(s,\lambda): s < m\}$, the space remaining in which to build $A(m,\lambda)$. At stage m, we have as induction hypotheses the restrictions of ($\aleph_0$ 1) and ($\aleph_0$ 2) and additionally:

(IH1) $(\forall m' \geqslant m)\ (\forall \beta < \lambda)\ T(m,\lambda) \supseteq^* A(m',\beta)$.

(IH2) $\langle A(s,\lambda): s < m\rangle$ is a disjoint family.

Fix $m \in \omega$ and assume $A(s,\lambda)$ are defined for $s < m$, satisfying the induction hypotheses.

For $n \in \omega$, define $K(m,n) \in \omega$ to be the minimum number satisfying the following three conditions:

(K1) $K(m,n) \supseteq A(m,f(n))\cap\bigcup\{A(m',f(p)): p, m' < n \wedge m' \neq m\}$;

(K2) $K(m,n) \supseteq A(m,f(n))\setminus T(m,\lambda)$;

(K3) $K(m,n) \supseteq \bigcup\{A(m,f(n))\cap T_{f(p)}\setminus A(m,f(p)): p < n\}$.

To see that $K(m,n)$ is finite, we examine each item individually. For (K1), $A(m,f(n))\cap A(m',f(p))$ is finite as $m' \neq m$ and the argument of $\bigcup$ is finite. For (K2), this is finite by (IH1) (with $m' = m$). For (K3), fix n and $p < n$. Then $A(m,f(n))\cap T_{f(p)} =^* A(m,f(p))$ by induction hypothesis.

$K(m,n)$ is the amount of $A(m,f(n))$ to be "removed" in order to satisfy $A(m,\lambda) \supseteq^* A(m,f(n))$. Further, item (K1) ensures almost disjointness between sets in different columns is maintained. This will ensure (IH1) is maintained.

Let $B(m) := \bigcup\{A(m, f(n)) \setminus K(m, n) \colon n \in \omega\}$. (Though including m in this notation may seem redundant, other values of $B(m)$ will be referred to later in the proof). $B(m)$ is a first approximation to $A(m, \lambda)$. Notice that induction hypotheses ($\aleph_0$ 1) and (IH1) would be satisfied if we define $A(m, \lambda)$ to be $B(m)$ since:

(1) $B(m) \supseteq^* A(m, \beta)$ for all $\beta < \lambda$ because for some n, we have $A(m, \beta) \subseteq^* A(m, f(n)) \subseteq^* B(m)$.

(2) For $m' \neq m$ and any i, we have $B(m) \cap A(m', f(i))$ finite (because once $n > i, m'$, then $K(m, n) \supseteq A(m, f(n)) \cap A(m', f(i))$ by (K3)). This immediately gives for all $\beta < \lambda$, $B(m) \cap A(m', \beta)$ is finite. Then by (IH1) for m, for any $m' \geqslant m + 1$ and any $\beta < \lambda$, we would have $A(m', \beta) \subseteq^* T(m + 1, \lambda)$ as needed for (IH1) for $m + 1$.

For $\beta < \lambda$, let n_β denote the unique value of n satisfying $f(n) \leqslant \beta < f(n + 1)$. Define

$$J(m) := \{(m', \beta) \colon m < m' < n_\beta \wedge A(m', \beta) \cap B(m) \subseteq n_\beta\}.$$

$J(m)$ is the set of (indices of) sets in the matrix with which $B(m)$ does not yet have "large" intersection (in the sense of ($\aleph_0$ 2)).

Lemma 19. $(\forall n)\ \{\beta \colon n_\beta < n \wedge (\exists m')\ (m', \beta) \in J(m)\}$ *is finite.*

Proof of Lemma 19. Fix $n \in \omega$. It is enough to show that for each m' with $m < m' < n$ the set $pr_{m'} J(m) := \{\beta \colon n_\beta < n \wedge (m', \beta) \in J(m)\}$ is finite since for each β there are only finitely many m' with $(m', \beta) \in J(m)$. By the definition of $B(m)$, $A(m, f(n + 1)) \setminus K(m, n + 1) \subseteq B(m)$, and so

$$\begin{aligned} pr_{m'} J(m) &\subseteq \{\beta < f(n + 1) \colon A(m', \beta) \cap A(m, f(n + 1)) \setminus K(m, n + 1) \subseteq n\} \\ &\subseteq \{\beta < f(n + 1) \colon A(m', \beta) \cap A(m, f(n + 1)) \subseteq \max(n, K(m, n + 1))\}, \end{aligned}$$

which is finite by induction hypothesis. □

Note that for any $\beta < \lambda$, if $m' > n_\beta$ then $(m', \beta) \notin J(m)$.This and the Lemma 19 imply the following result.

Lemma 20. $(\forall \beta < \lambda)\ J(m) \cap (\omega \times \beta)$ *is finite.*

Define the function $j^m \colon J(m) \to \omega$ as follows. Suppose that $(m', \beta) \in J(m)$. Then let $j^m(m', \beta)$ be

$$\inf[(A(m', \beta) \cap A(m', f(n_\beta)) \setminus \bigcup\{T_{f(l)} \colon l < n_\beta\}) \cap T(m, \lambda) \setminus n_\beta].$$

Note that, as $m' > m$, $A(m', \beta) \subseteq^* T(m, \lambda)$ by the induction hypotheses. (Also note that $A(m', \beta) \cap A(m', f(n_\beta)) =^* A(m', f(n_\beta))$ and is thus infinite.) It is not difficult to check in addition that the argument of inf is an infinite set. Finally, note that $j^m(m', \beta) \geqslant n_\beta$.

Let $A(m,\lambda) := B(m) \cup ran(j^m)$. We check that the induction hypotheses are maintained.

($\aleph_0$ 1). As $A(m,\lambda) \supseteq B(m) \supseteq^* A(m,\beta)$ for $\beta < \lambda$, so $(\forall\beta < \lambda)\ A(m,\lambda) \supseteq^* A(m,\beta)$ as required to continue the tower.

($\aleph_0$ 2). Let $m' > m$ and $r \in \omega$. We need to show that $\{\beta < \lambda\colon A(m,\lambda) \cap A(m',\beta) \subseteq r\}$ is finite. Defining

$$S_n := \{\beta\colon n_\beta = n \wedge A(m,\lambda) \cap A(m',\beta) \subseteq r\}$$

we have the set of interest equal to $\bigcup\{S_n\colon n \in \omega\}$.

Claim. *For all n, S_n is finite.*

Proof of Claim.

$$\begin{aligned} S_n &\subseteq \{\beta < f(n+1)\colon A(m,\lambda) \cap A(m',\beta) \subseteq r\} \\ &\subseteq \{\beta < f(n+1)\colon B(m) \cap A(m',\beta) \subseteq r\} \\ &\subseteq \{\beta < f(n+1)\colon A(m,f(n+1)) \setminus K(m,n+1) \cap A(m',\beta) \subseteq r\} \\ &\subseteq \{\beta < f(n+1)\colon A(m,f(n+1)) \cap A(m',\beta) \subseteq \max(r, K(m,n+1))\} \end{aligned}$$

which is finite by induction hypothesis. □

Claim. *If $n > \max(r,m')$ then $S_n = \emptyset$.*

Proof of Claim. Suppose $\beta \in S_n$ where $n > \max(r,m')$. Then $n_\beta = n$ and $A(m,\lambda) \cap A(m',\beta) \subseteq r \subseteq n$. Since $B(m) \subseteq A(m,\lambda)$ this gives $B(m) \cap A(m',\beta) \subseteq n$. But $m < m' < n$, and so $(m',\beta) \in J(m)$. Consequently, $j^m(m',\beta) \in A(m,\lambda) \cap A(m',\beta) \setminus n$, which contradicts the deduction that this set is empty. □

Thus ($\aleph_0$ 2) is verified.

(IH1) Fix $m' > m+1$ and $\beta < \lambda$. We wish to show that $T(m+1,\lambda) \supseteq^* A(m',\beta)$. Since by induction hypothesis, $T(m,\lambda) \supseteq^* A(m',\beta)$, it is sufficient to show that $A(m,\lambda) \cap A(m',\beta)$ is finite. Further, we may assume β is of the form $f(n)$ for some $n \in \omega$. (See the proof of ($\aleph_0$ 3) below for the proof of a similar statement.)

Assume $\beta = f(n)$. It was noted that $B(m) \cap A(m',\beta)$ is finite. To establish $ran(j^m) \cap A(m',\beta)$ is finite, we show

$$ran(j^m) \cap A(m',\beta) \subseteq ran(j^m \restriction [J(m) \cap (\omega \times f(n_\beta + 1))])$$

and appeal to Lemma 20. Let $(p,\gamma) \in J(m) \setminus (\omega \times f(n_\beta + 1))$. Then $\gamma \geqslant f(n_\beta + 1)$ and so $n_\gamma > n_\beta$. By the definition of j^m, we have

$$j^m(p,\gamma) \notin T_{f(n_\beta)} = T_\beta \supseteq A(m',\beta)$$

as desired. This completes the proof for (IH1).

(IH2) is immediate since $A(m,\lambda) \subseteq T(m,\lambda)$ which is disjoint from $A(s,\lambda)$ for $s < m$. This marks the end of the limit construction.

It remains to check that ($\aleph_0$ 3) and ($\aleph_0$ 4) are satisfied after the completion of the construction of $A(m,\lambda)$ for $m \in \omega$.

($\aleph_0$ 3). We must show that

$$(\forall\beta < \lambda)(\exists m_0)(\forall m > m_0)\ A(m,\lambda) \cap T_\beta = A(m,\beta).$$

It is sufficient to check this for β of the form $f(n)$. For suppose it holds of such ordinals and $\beta \in \omega_1$ is arbitrary. Fix n such that $\beta < f(n)$. Then there are m_0, m_1 and m_2 such that:

(1) $(\forall m > m_0)\ A(m,\lambda) \cap T_{f(n)} = A(m,f(n))$, i.e., ($\aleph_0$ 3) holds for $f(n)$,
(2) $(\forall m > m_1)\ A(m,f(n)) \cap T_\beta = A(m,\beta)$, which holds by induction hypothesis,
(3) $(\forall m > m_2)\ A(m,\lambda) \cap (T_\beta \setminus T_{f(n)}) = \emptyset$ which is possible as $T_\beta \setminus T_{f(n)}$ is finite and the $A(m,\lambda)$ are disjoint.

Then for $m > \max(m_0, m_1, m_2)$,

$$\begin{aligned} A(m,\lambda) \cap T_\beta &= A(m,\lambda) \cap [(T_\beta \setminus T_{f(n)}) \cup (T_\beta \cap T_{f(n)})] \\ &= \emptyset \cup (A(m,\lambda) \cap T_{f(n)} \cap T_\beta) \text{ as } m > m_2 \\ &= A(m,f(n)) \cap T_\beta \text{ as } m > m_0 \\ &= A(m,\beta) \quad \text{as } m > m_1, \end{aligned}$$

which is the desired result.

We use induction on n. Fix $n \in \omega$ and assume the claim holds for all $p < n$. We wish to show

$$(\exists m_0)(\forall m > m_0)\ A(m,\lambda) \cap T_{f(n)} = A(m,f(n)). \tag{6}$$

Since there are only finitely many $p < n$, we have by induction hypothesis that

$$(\exists m_1)(\forall p < n)(\forall m > m_1)\ A(m,f(n)) \cap T_{f(p)} = A(m,f(p)). \tag{7}$$

Fixing such an m_1, this gives

$$(\forall p \leqslant n)(\forall m > m_1)(\forall m' \neq m)\ A(m,f(n)) \cap A(m',f(p)) = \emptyset \tag{8}$$

(where we use additionally that $A(m,f(p)) \cap A(m',f(p)) = \emptyset$). Next we have

$$(\exists m_2)(\forall m > m_2)\ A(m,f(n)) \subseteq T_\lambda.$$

Let $m_0 := \max(m_1, m_2, n)$.

Claim. $(\forall m < m_0)\ A(m,\lambda) \cap T_{f(n)} = A(m,f(n))$.

Proof of Claim. This is demonstrated by establishing the following three facts.

(A) $(\forall m > m_0)\ ran(j^m) \cap T_{f(n)} = \emptyset$.
(B) $(\forall m > m_0)\ B(m) \cap T_{f(n)} \subseteq A(m,f(n))$.
(C) $(\forall m > m_0)\ K(m,n) = 0$.

(C) implies $(\forall m > m_0)\ B(m) \supseteq A(m,f(n))$ which combined with (B) gives $(\forall m > m_0)\ B(m) \cap T_{f(n)} = A(m,f(n))$. With (A) we can deduce the claim since $A(m,\lambda) = B(m) \cup ran(j^m)$.

To prove (A), let $(m',\beta) \in J(m)$. We may infer that $n \leqslant m_0 < m < m' < n_\beta$, where the first inequality holds by the definition of m_0 and so $f(n) < f(n_\beta) \leqslant \beta$. Thus $j^m(m',\beta) \notin T_{f(n)}$ by the definition of j^m. This establishes (A).

To prove (B) we show that $(\forall m > m_0)\ B(m) \cap T_{f(n)} \subseteq A(m,f(n))$ or, from the definition of $B(m)$, that

$$\bigcup_{p' \in \omega} (A(m,f(p')) \cap T_{f(n)} \setminus K(m,p')) \subseteq A(m,f(n)).$$

We examine the two cases when $p' \leqslant n$ and when $p' > n$. In the first case we appeal to display (7) which implies $(\forall m > m_0)\ (\forall p \leqslant n)\ A(m,f(p)) \subseteq A(m,f(n))$.

When $p' > n$, $K(m,p') \supseteq [A(m,f(p')) \cap T_{f(n)} \setminus A(m,f(n))]$ by the third part of the definition of $K(m,p')$. Consequently $A(m,f(p')) \cap T_{f(n)} \setminus K(m,p') \subseteq A(m,f(n))$. Hence the displayed union over p' is contained in the desired set, $A(m,f(n))$. This establishes (B).

We now prove (C). There are three parts to the definition of $K(m,n)$. It is sufficient to show that for $m > m_0$ the right-hand side of each part is empty.

For (K1) we must show

$$(\forall m > m_0)\ (A(m,f(n)) \cap \bigcup \{A(m',f(p)) \colon p, m' < n \text{ and } m' \neq m\}) = \emptyset.$$

This is immediate from the definition of m_0 and display (8).

For (K2), show $(\forall m > m_0)\ A(m,f(n)) \setminus T(m,\lambda) = \emptyset$. Using the definition of $T(m,\lambda)$, and since $m > m_0$ implies that $A(m,f(n)) \subseteq T_\lambda$, this reduces to $(\forall m > m_0)\ (\forall s < m)$ $A(m,f(n)) \cap A(s,\lambda) = \emptyset$. As $A(s,\lambda) = B(s) \cup ran(j^s)$, we show
(i) $(\forall m > m_0)\ (\forall s < m)\ A(m,f(n)) \cap ran(j^s) = \emptyset$, and
(ii) $(\forall m > m_0)\ (\forall s < m)\ A(m,f(n)) \cap B(s) = \emptyset$.

For (i), suppose for arbitrary β, m', that $(m',\beta) \in J(s)$. We show that $j^s(m',\beta) \notin A(m,f(n))$ in two cases. First, if $n_\beta \leqslant n$ then $m \geqslant m_0 \geqslant n \geqslant n_\beta > m'$ (where the last inequality follows from the definition of $J(s)$) gives $m \neq m'$ which by display (8) gives $A(m,f(n)) \cap A(m',f(n_\beta)) = \emptyset$. But $j^s(m',\beta) \in A(m',f(n_\beta))$. In the second case, $n_\beta > n$. From the definition of j^s, $j^s(m',\beta) \notin T_{f(n)} \supseteq A(m,f(n))$. This proves (i).

For (ii), note that

$$A(m,f(n)) \cap B(s) = \bigcup_{p \in \omega} [A(m,f(n)) \cap A(s,f(p)) \setminus K(s,p)],$$

and again we have two cases. If $p \leqslant n$ then immediately $A(m,f(n)) \cap A(s,f(p)) = \emptyset$ by display (8) and the fact that $s \neq m$.

In the second case, where $p > n$, we have by the third clause in the definition of $K(s,p)$ that

$$\begin{aligned} K(s,p) &\supseteq A(s,f(p)) \cap T_{f(n)} \setminus A(s,f(n)) \\ &\supseteq A(s,f(p)) \cap A(m,f(n)) \setminus A(s,f(n)) \\ &\supseteq A(s,f(p)) \cap A(m,f(n)), \end{aligned}$$

where the last line is true because $A(m,f(n)) \cap A(s,f(n)) = \emptyset$. Thus, $A(s,f(p)) \cap A(m,f(n)) \setminus K(s,p) = \emptyset$ as desired. This establishes (ii) which completes the proof for (K2).

For (K3), we show that

$$\bigcup_{p<n} [A(m,f(n)) \cap T_{f(p)} \setminus A(m,f(p))] = \emptyset.$$

This follows immediately from display (7) since

$$A(m,f(p)) = A(m,f(n)) \cap T_{f(p)} \Rightarrow A(m,f(n)) \cap T_{f(p)} \setminus A(m,f(p)) = \emptyset.$$

The premise of this display holds when $m > m_0$. This completes the proof for (K3) which finishes (C) that $K(m,n) = \emptyset$ for all but finitely many m. □

By the reasoning presented after the statements of (A)–(C), we conclude the claim that for all but finitely many m, $A(m,\lambda) \cap T_{f(n)} = A(m,f(n))$. This completes the proof that ($\aleph_0$ 3) holds through the limit stage.

($\aleph_0$ 4). The sets $A(m,\lambda)$, $m \in \omega$ are disjoint by construction. Suppose that their union does not exhaust T_λ. Notice that the quantity of T_λ remaining must be almost disjoint from T_β for all $\beta < \lambda$ (and hence almost disjoint from each $A(m,\beta)$ for $m \in \omega$ and $\beta < \lambda$). This follows from ($\aleph_0$ 3). Consequently, $A(0,\lambda)$ can be expanded to contain this set without affecting the other hypotheses. Note in particular that $\langle A(0,\alpha): \alpha \leqslant \lambda \rangle$ will continue to be a subtower of T.

This marks the end of limit stage.

This completes the construction of the desired $\omega \times \omega_1$ matrix. Since the properties of this matrix are all stated with quantifiers over countable ordinals, the proofs of the persistence of the induction hypotheses through the recursion establishes that the matrix has the stated properties. □

Immediate attempts to improve Theorem 17 were resisted by apparently combinatorial complications. These difficulties had the flavor of independence results and indeed the only successful attacks on the problem have relied on combinatorial principles which are known to be consistent with and independent of the axioms of ZFC. At this point, it is necessary that these principles be introduced in their proper context.

4. New combinatorial hypotheses

We turn now to the combinatorial hypothesis that will be used to prove the $\aleph_1$ gap theorem. This and related hypotheses seem to be of interest in their own right, and so I take the opportunity to prove some statements about their relative consistency.

An important type of object for these definitions is the following. The reader can find further information on such objects in [4].

Definition 21. A *ladder system*, $\langle \varphi_\lambda : \lambda \in \Gamma \rangle$, on a set of limit ordinals of countable cofinality, Γ, is a Γ-indexed collection of increasing ω-sequences, φ_λ, each cofinal in its respective λ.

Recall the definition of $\clubsuit$ from [14].

Definition 22. $\clubsuit$ is the statement that there is a ladder system, $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$, such that for every uncountable set $X \subseteq \omega_1$ there is a $\lambda \in \Lambda$ with $\varphi_\lambda \subseteq X$.

Most of the hypotheses involved follow the basic form of $\clubsuit$. That is, they state the existence of a sequence of sets having some property with respect to other sets. In general, we will follow the convention that a sequence satisfying these properties is called a $\clubsuit$-sequence ($\diamondsuit$-sequence, H2-sequence, etc.)

The reader is referred to [12, p. 80] for the statements of the hypotheses in the $\diamondsuit$ family.

Blass has pointed out that H2, defined below, can be phrased as a negative partition relation connecting these ideas to the work of Todorčević and others. This seems to reflect the implicit connection between the properties used to ensure a pair of towers is a gap – an event occurring between sets at different levels – and partition relations on pairs of ordinals. In addition, it has led to weakened forms of the hypotheses, also given below, which are more easily seen to be independent of ZFC.

Definition 23. H0 is the statement that there is a ladder system $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$ such that for each stationary subset S of ω_1 there is a $\lambda \in S$ such that $\varphi_\lambda \subseteq S$.

Compared to $\clubsuit$, we have strengthened the statement in requiring the "self-reference" of S, while weakening the universal quantifier to stationary sets. In any case, this turns out to be inconsistent with ZFC.

Claim. H0 *is not consistent with* ZFC.

Proof of Claim. Suppose $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$ were an H0-sequence. Inductively define a set S such that

$$\lambda \in S \Leftrightarrow \varphi_\lambda \nsubseteq S.$$

We show that S is stationary, immediately contradicting H0. Let C be a club and suppose $C \cap S = \emptyset$. By the assumption of H0, there is a $\lambda \in C$ such that $\varphi_\lambda \subseteq C$. But then $\lambda \in S$, a contradiction. Hence S is stationary. □

Fortunately, the same fate does not befall the following weakenings of H0.

Definition 24. H1 is the statement that there is a ladder system $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$ such that for each stationary subset S of ω_1 there is a $\lambda \in S$ such that $|\varphi_\lambda \cap S| = \aleph_0$.

Definition 25. H2 is the statement that there is a ladder system $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$ such that for each stationary subset S of ω_1 there is a $\lambda \in S$ such that $\varphi_\lambda \cap S \neq \emptyset$.

Proposition 26. $\diamondsuit^* \Rightarrow$ H1 $\Rightarrow$ H2. *In particular,* H1 *and* H2 *are consistent with* ZFC.

Proof. The second implication is immediate. Let $\langle \mathscr{D}_\alpha : \alpha \in \Lambda \rangle$ be a $\diamondsuit^*$-sequence. For each $\lambda \in \Lambda$ define φ_λ a cofinal ω-sequence in λ such that for each $D \in \mathscr{D}_\lambda$ which is cofinal in λ, $\varphi_\lambda \cap D$ is infinite. This is done by enumerating the $D \in \mathscr{D}_\lambda$ which are cofinal in λ and recursively defining φ_λ. If there are no cofinal elements in $\mathscr{D}_\lambda$, then let φ_λ be arbitrary.

I claim that this φ_λ sequence is an H2-sequence. For let S be a stationary subset of ω_1 and let C be a club as in the definition of the $\diamondsuit^*$-sequence, that is, where S is predicted. Let $\lambda \in C \cap S$ such that $C \cap S$ is cofinal in λ. This is possible because $C \cap S$ is stationary. As $\lambda \in C$, we know $S \cap \lambda \in \mathscr{D}_\alpha$ and $S \cap \lambda$ is cofinal in λ. By the definition of φ_λ, we have the desired statement that $\varphi_\lambda \cap S$ is infinite. This shows that H1 and H2 hold in L and are thus consistent with ZFC. □

Of course, $\diamondsuit \Rightarrow$ CH, and we are interested in statements about gaps when CH does not hold, too. Further, the use of $\diamondsuit^*$ in the above proof seems to be more than is necessary. It would be more satisfying to have a better understanding of the power of H2. Towards this end, we will show H2 is consistent with $\neg$CH. In fact, the proof below shows H2 is consistent with the continuum being anything reasonable and can be easily adapted to show the same for H1.

This theorem is proved by showing that an H2-sequence is preserved under notions of forcing that satisfy an apparent strengthening of a previously known condition. We begin by stating this new condition and proving lemmas that will help show familiar notions of forcing satisfy the condition.

Definition 27. We say that a notion of forcing, $\mathbb{P}$, has *property SK* if and only if for any sequence of conditions $\langle p_\alpha : \alpha \in S \rangle$ indexed by a stationary set $S \subseteq \omega_1$ there is a stationary $T \subseteq S$ such that for all $\alpha, \beta \in T$, p_α and p_β are compatible.

SK can be read as strong Knaster or stationary Knaster as this is a strengthening of property K.

Lemma 28. *If $\langle A_\alpha : \alpha \in S\rangle$ is a collection of finite subsets of ω_1 with $S \subseteq \omega_1$ stationary, then there is a stationary $T \subseteq S$ such that $\langle A_\alpha : \alpha \in T\rangle$ is a Δ-system.*

Proof (Blass). Thinning S, we may assume all A_α have the same cardinality, n; for $k < n$ let $a_\alpha(k)$ be the kth element of A_α. If there is a stationary set of α's for which $a_\alpha(n-1)$ is bounded, then for these α's there are only countably many different A_α's and so stationarily many are the same.

Otherwise, let k be the least number such that $a_\alpha(k)$ is unbounded on every stationary set of α's. Note that the same must hold for all i between k and n. As above, we can thin the index set to a stationary set such that $\{a_\alpha(0), \ldots, a_\alpha(k-1)\}$ is independent of α. As $a_\alpha(k)$ cannot be a regressive function of α on any stationary set, we can thin to arrange that $\alpha \leqslant a_\alpha(k)$ for all α. Further, by intersecting the index set with a suitable club (namely, $\{\alpha: (\forall \beta < \alpha)\ a_\beta(n-1) < \alpha\}$) we have $a_\beta(n-1) < \alpha$ whenever $\beta < \alpha$. This collection of A_α's forms a Δ-system with kernel $\{a_\alpha(0), \ldots, a_\alpha(k-1)\}$. □

Now we demonstrate that two of the most familiar notions of forcing have property SK.

Lemma 29. *If $\kappa \geqslant \omega_1$ and $\mathbb{P}$ is the set of finite partial functions from κ into 2, then $\mathbb{P}$ has property SK (i.e., Cohen forcing has property SK).*

Proof. Let $p_\alpha \in \mathbb{P}$, $\alpha \in S$, where $S \subseteq \omega_1$ is stationary be given. Notice that the cardinality of the set of all finite partial functions on the union of the domains of the p_α is $\aleph_1$. As Lemma 28 addresses only properties of extensionality and cardinality, we can apply it to $\langle dom(p_\alpha): \alpha \in S\rangle$ to get a Δ-system on a stationary $T' \subseteq S$. We may then reduce to a set $T \subseteq T'$ such that the restriction of p_α to the root of the Δ-system is independent of α in T. Then the union of any two conditions indexed by T is a common extension of each. □

Lemma 30. *Random forcing has property SK.*

The proof actually shows that any σ-linked forcing has property SK.

Proof of Lemma 30. We use the following fact: If p is a Borel set of positive measure, then for almost all $x \in p$, the density of p in intervals around x goes to 1 as the interval width goes to 0. (This result, known as the Lebesgue density theorem, can be established by showing sets without this property have measure 0.) So for $q \in \mathbb{Q}$, the rationals, and $n \in \omega$, consider the set

$$S_n^q := \left\{p\text{: the density of } p \text{ in } \left(q - \frac{1}{n}, q + \frac{1}{n}\right) \text{ is greater than } \frac{1}{2}\right\}.$$

Notice that any pair of conditions in this set are compatible (have intersection with positive measure). Secondly, for any p, there is a $q \in \mathbb{Q}$ and an $n \in \omega$ with $p \in S_n^q$. If S is stationary and indexes a set of conditions, then there must be a q and an n with $\{\alpha\colon p_\alpha \in S_n^q\}$ stationary. This completes the lemma. □

Lemma 31. *Having property SK is preserved under finite support iteration,* i.e., *if* $\langle Q_i, \pi_i\colon i \in \kappa\rangle$ *is a finite support iteration and* $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{Q}_\gamma$ *has property SK" for all* $\gamma < \kappa$, *then the resulting partial order has property SK.*

Proof. Let $\langle Q_i, \pi_i\colon i \in \kappa\rangle$ be a finite support iteration and $p_\alpha \in \mathbb{P}$ for $\alpha \in S$ where $S \subseteq \omega_1$ is stationary. Since $spt(p_\alpha) \in [\kappa]^{<\aleph_0}$, we may apply Lemma 28 to get a stationary $T_0 \subseteq S$. For each γ in the root of this Δ-system, successively get $T_{n+1} \subseteq T_n$ stationary such that

$$\Vdash_{\mathbb{P}_\gamma} (\forall \alpha, \beta \in T_{n+1}) \text{ "} p_\alpha(\gamma) \text{ is compatible with } p_\beta(\gamma)\text{"},$$

by applying the fact that Q_γ, π_γ has property SK. As the original root was a finite set, there is a single T_N demonstrating the lemma. □

We are finally ready to state and prove the theorem.

Theorem 32. Con(ZFC + H2 + $\neg$CH). *That is, if* ZFC *is consistent, then so is* ZFC + H2 + $\neg$CH.

Proof. Start with a model of H2, for example a model of $\diamond^*$. Add $\aleph_2$ Cohen reals. I intend to show that the H2-sequence in the ground model continues to enjoy this property in the extension.

Let $\dot{S}$ be the name of a stationary subset of ω_1 and fix $p \in \mathbb{P}$. Define $p_\alpha := \|\alpha \in \dot{S}\| \wedge p$. As $\Vdash$ "$\dot{S}$ is stationary", we have $\{\alpha\colon p_\alpha \text{ is not } 0\}$ is stationary in the generic extension (associated with any generic set containing p) as it contains $\dot{S}$. Hence it is stationary in the ground model where it is definable. Since $\mathbb{P}$ has property SK, there is a stationary $T \subseteq \omega_1$ such that $\{p_\alpha\colon \alpha \in T\}$ is a pairwise compatible set. By H2 in the ground model, there is a $\lambda \in T$ such that $\varphi_\lambda \cap T \neq \emptyset$. So for some $\alpha \in \varphi_\lambda$, we have p_α and p_λ are compatible, $p_\alpha \wedge p_\lambda \leqslant p$ and

$$p_\alpha \wedge p_\lambda \Vdash \lambda \in \dot{S} \wedge \dot{S} \cap \varphi_\lambda \neq \emptyset.$$

Since we started with an arbitrary condition and an arbitrary name for a stationary set and we found an extension of the condition which forces that the set has nonempty intersection with some φ_λ and forces that λ is in the stationary set, we see that the H2 sequence in the ground model remains an H2 sequence in the extension. We have thus completed the proof of the theorem. □

Notice $\aleph_2$ Cohen reals in the above proof could be replaced with any number of Cohen or random reals (added in a finite support iteration), or any other notion of forcing known to have property SK.

Since the development of H2 and the above discourse, Blass has noted that a weakening of H2 is all that is really needed in the proof of the $\aleph_1$ gap theorem below. Additionally, this weakening follows from $\diamond$ rather than $\diamond^*$.

Definition 33. Weak-H1 is the statement that there is a ladder system $\langle \varphi_\lambda \colon \lambda \in \Lambda \rangle$ such that for all functions $f \colon \omega_1 \to \omega$ there is a limit λ such that $\{\alpha \in \varphi_\lambda : f(\lambda) = f(\alpha)\}$ is infinite.

Definition 34. Weak-H2 is the statement that there is a ladder system $\langle \varphi_\lambda \colon \lambda \in \Lambda \rangle$ such that for all functions $f \colon \omega_1 \to \omega$ there is a limit λ and an $\alpha \in \varphi_\lambda$ such that $f(\lambda) = f(\alpha)$.

Notice that H1 implies weak-H1 and H2 implies weak-H2. These can be phrased in terms of partitions as well. For example, for a weak-H2 sequence, given any partition of ω_1 into countably many pieces, there is necessarily one piece which contains both some λ and an element of that φ_λ.

Proposition 35 (Blass). $\diamond \Rightarrow$ *weak*-H1 $\Rightarrow$ *weak*-H2.

Proof. Fix a $\diamond$-sequence of functions $f_\alpha \colon \alpha \to \omega$. Let φ_λ be an ω-sequence increasing to λ additionally satisfying

$$(\forall n \in \omega) \text{ if } f_\lambda^{-1}\{n\} \text{ is cofinal in } \lambda \text{ then it meets } \varphi_\lambda \text{ in an infinite set.}$$

Such a φ_λ may be constructed recursively by infinitely often addressing each n for which $f_\lambda^{-1}\{n\}$ is cofinal in λ. We show this sequence satisfies weak-H1.

Let $g \colon \omega_1 \to \omega$. Let $A := \{n \in \omega \colon g^{-1}\{n\} \text{ is uncountable}\}$. Define $\beta := \sup^+ \bigcup \{g^{-1}\{n\} \colon n \in \omega \setminus A\}$. Notice that $\beta < \omega_1$ and that if $\xi > \beta$ then $g(\xi) \in A$.

For each $n \in A$, the set of limit points of $g^{-1}\{n\}$ is a club; let C be the intersection of these countably many clubs. Because the f_α's form a $\diamond$-sequence, there is a limit ordinal $\lambda \in C$ with $\lambda > \beta$ and $g \restriction \lambda = f_\lambda$.

Fix such a λ and let $n = g(\lambda)$. By the above remark, $n \in A$. As $\lambda \in C$, it follows that λ is a limit point of $g^{-1}\{n\} \cap \lambda = f_\lambda^{-1}\{n\}$. So, by construction, φ_λ meets $f_\lambda^{-1}\{n\}$, say at γ. Then $g(\gamma) = f_\lambda(\gamma) = n$ and $g(\lambda) = n$ and $\gamma \in \varphi_\lambda$ as required.

The second implication is immediate from the definitions. □

There is another reason for the introduction of the weak forms of these hypotheses. Given a ladder system, it is relatively easy to describe a c.c.c notion of forcing which adjoins, via $\aleph_1$ many dense sets, a function from ω_1 to ω which demonstrates that the given ladder system is not a weak-H2 sequence. Consequently, we have the following result.

Proposition 36. *Under* MA, *weak*-H2 *does not hold; hence neither do weak*-H1, H2 *nor* H1 *hold.*

Before leaving these hypotheses, there is one more curiousity to point out. It is well known that in the conclusion of $\diamond$, it is equivalent to assume that the $\diamond$-sequence guesses correctly just once or to assume that the set of correct guesses is stationary. Not surprisingly, this turns out to be true of H2 as well. In particular we have the following.

Proposition 37. *If $\langle \varphi_\lambda \colon \lambda \in \Lambda \rangle$ is an* H2*-sequence, then for any stationary set S, $\{\lambda \in S\colon S \cap \varphi_\lambda \neq \emptyset\}$ is stationary.*

There is a similar proposition for H1. What is more surprising than this proposition is the following.

Proposition 38. *If $\langle \varphi_\lambda \colon \lambda \in \Lambda \rangle$ is an* H2*-sequence, then for any set $C \subseteq \omega_1$ which contains a club, the set $\{\lambda \in C\colon C \cap \varphi_\lambda \neq \emptyset\}$ contains a club.*

Proof. Let $\langle \varphi_\lambda \colon \lambda \in \Lambda \rangle$ be an H2-sequence and suppose $C \subseteq \omega_1$ is a set containing a club for which the set in question does not contain a club. Then the complement of this set, namely $\{\lambda\colon \lambda \notin C \vee C \cap \varphi_\lambda = \emptyset\}$, is stationary. We may intersect this set with C to get a stationary set: $A := \{\lambda \in C\colon C \cap \varphi_\lambda = \emptyset\}$. Since the φ_λ form an H2-sequence, there is a $\lambda \in A$ such that $A \cap \varphi_\lambda \neq \emptyset$. But $A \subseteq C$ implies $\varphi_\lambda \cap C \neq \emptyset$ which by the definition of A implies $\lambda \notin A$, a contradiction. □

As an aside, Proposition 37 indicates there is an intermediary between H2 and weak-H2, namely, the following definition.

Definition 39. Not-as-weak-H2 is the statement that there is a ladder system $\langle \varphi_\lambda \colon \lambda \in \Lambda \rangle$ such that for all functions $f\colon \omega_1 \to \omega$, the set $\{\lambda\colon f^{-1}(\lambda) \cap \varphi_\lambda \neq \emptyset\}$ is stationary.

Again, there is a similar statement for H1.

5. The $\aleph_1$ gap theorem

In this section, we prove a strengthening of the $\aleph_0$ gap theorem. This theorem is not stated as a consistency result. It is a construction that occurs in ZFC. However, to show that the constructed object is in fact a large collection of gaps, and in particular to prove the corollary corresponding to that following the $\aleph_0$ gap theorem, we use hypothesis H2. Towards this corollary, we first prove the following lemma which shows that under H2 a weak condition on a pregap makes it a gap.

Lemma 40 (H2 and gaps lemma). *Assume* H2 *holds for the ladder system $\langle \varphi_\lambda \colon \lambda \in \Lambda \rangle$. Let $\langle A, B \rangle$ be a pregap satisfying $A_\alpha \cap B_\alpha = \emptyset$ for all α and*

$$(\exists \eta)\ (\exists \text{ stationary } S \subseteq \Lambda)(\forall \lambda \in S)(\forall \beta \in \varphi_\lambda \setminus (\eta + 1))\ \ A_\lambda \cap B_\beta \neq \emptyset.$$

Then $\langle A, B \rangle$ is a gap.

The appearance of η in this lemma is for technical reasons that will be clear in its application after the proof of the $\aleph_1$ gap theorem. The proof of the lemma is better understood ignoring η.

Proof of Lemma 40. Suppose $D \subseteq \omega$ separated $\langle A, B\rangle$. Then there is an $n \in \omega$ and a stationary set $T \subseteq S\backslash(\eta + 1)$ such that $(\forall\lambda, \lambda' \in T)$

$$A_\lambda \backslash n \subseteq D, \qquad (B_\lambda \backslash n) \cap D = \emptyset,$$

$$A_\lambda \cap n = A_{\lambda'} \cap n, \qquad B_\lambda \cap n = B_{\lambda'} \cap n.$$

By H2, $(\exists\lambda \in T)\ \varphi_\lambda \cap T \neq \emptyset$. Let $\beta \in \varphi_\lambda \cap T$. Since $\beta \in \varphi_\lambda$, $A_\lambda \cap B_\beta \neq \emptyset$, while β and λ being in T implies $A_\lambda \cap B_\beta = \emptyset$, by the previous displayed equations. This is a contradiction, and hence no such D exists. □

We now state and prove the title theorem of this section.

Theorem 41 (The $\aleph_1$ gap theorem). *Let the following objects be given.*
(G1) *A tower* $T = \langle T_\alpha: \alpha \in \omega_1\rangle$ *with* $T_0 = \emptyset$.
(G2) *A ladder system* $\langle \varphi_\lambda: \lambda \in \Lambda\rangle$.
(G3) *A collection of disjoint stationary subsets of* Λ: $\{S(\xi,\eta): \eta < \xi < \omega_1\rangle$. *We shall also assume that each* $\lambda \in \Lambda$ *is in some* $S(\xi,\eta)$ *and that*
$$(\lambda \in S(\xi,\eta)) \Rightarrow (\eta < \xi < \lambda).$$
Then there is a collection $\langle A(\xi,\alpha): \xi < \alpha < \omega_1\rangle$ *of subsets of* ω *satisfying*:
($\aleph_1$ 1) $(\forall\xi \in \omega_1)\ \langle A(\xi,\alpha): \xi < \alpha < \omega_1\rangle$ *is a subtower of* T.
($\aleph_1$ 2) $(\forall\alpha \in \omega_1) T_\alpha = \bigcup\{A(\xi,\alpha): \xi < \alpha\}$, *a disjoint union.*
($\aleph_1$ 3) $(\forall\eta < \xi < \omega_1)(\forall\lambda \in S(\xi,\eta))(\forall\beta \in \varphi_\lambda.\beta > \eta) A(\xi,\lambda) \cap A(\eta,\beta) \neq \emptyset$.
($\aleph_1$ 4) $(\forall\beta < \alpha < \omega_1)$ the set $\{\xi: A(\xi,\beta) \nsubseteq A(\xi,\alpha)\}$ *is finite.*

Thinking of the indexing of $A(\xi,\alpha)$ as A (column, row), we can visualize the result as an $\omega_1 \times \omega_1$ upper triangular matrix with (0, 0) at the lower left corner. This follows the intention of building ω_1-towers (the columns) while keeping each row a countable disjoint collection of sets whose union is T_α.

These conditions on the matrix will be satisfied by a recursive construction of sets at level γ (λ when working with limit stages). Consequently we will frequently refer to the "restrictions" of ($\aleph_1$ 1–4) as the *induction hypotheses*, which are obtained by replacing ω_1 by γ (or λ) and quantifying over the sets constructed to that point in the proof. When no confusion can result, we shall refer to these restrictions as ($\aleph_1$ 1), etc.

There is a corollary to this theorem corresponding to the one after the $\aleph_0$ gap theorem. However, the corollary is stated in terms of an independence result and its proof is more involved, so we delay its consideration until after the construction.

Proof of Theorem 41. Assume the objects in (G1)–(G3) have been fixed. We construct sets $A(\xi,\alpha)$ with the convention that $A(\xi,\alpha) = \emptyset$ for $\xi \geqslant \alpha$. We first establish the following lemma which provides a convenient equivalent formulation of ($\aleph_1$ 4).

Lemma 42. *Suppose that sets $A(\xi,\alpha)$ for $\xi < \alpha < \gamma$ satisfy the restrictions of $(\aleph_1\ 1)$ and $(\aleph_1\ 2)$. Fix $\beta < \alpha < \gamma$. Then the following are equivalent:*
(A) *The set $U_1 := \{\xi\colon A(\xi,\beta) \nsubseteq A(\xi,\alpha)\}$ is finite.*
(B) *The set $U_2 := \{\xi\colon A(\xi,\alpha) \cap T_\beta \neq A(\xi,\beta)\}$ is finite.*

Proof of Lemma 42. (A) $\Rightarrow$ (B): Assume U_1 is finite. If $\xi \in U_2$ then at least one of the following three conditions holds of ξ:
(i) $A(\xi,\beta) \nsubseteq T_\alpha$, or
(ii) the set $(A(\xi,\beta) \setminus A(\xi,\alpha)) \cap T_\alpha$ is non-empty, or
(iii) the set $A(\xi,\alpha) \cap T_\beta \setminus A(\xi,\beta)$ is non-empty.

It is sufficient to show that there are only finitely many ξ satisfying each of (i), (ii) and (iii). This is true for (i) by our assumption that $(\aleph_1\ 1)$ and $(\aleph_1\ 2)$ hold. Next, if ξ satisfies (ii) then $A(\xi,\beta) \nsubseteq A(\xi,\alpha)$, and so $\xi \in U_1$ which is assumed to be finite.

If ξ satisfies (iii) then by $(\aleph_1\ 2)$ for level β, $(\exists\eta) A(\xi,\alpha) \cap A(\eta,\beta) \neq \emptyset$. Let

$$U_2^+(\eta) := \{\xi\colon A(\xi,\alpha) \cap A(\eta,\beta) \neq \emptyset \wedge \eta \neq \xi\}.$$

Then the set of ξ satisfying (iii) is equal to $\bigcup_{\eta<\gamma} U_2^+(\eta)$. By $(\aleph_1\ 2)$ for level α, if $U_2^+(\eta) \neq \emptyset$ then $\eta \in U_1$. Since we are assuming that U_1 is finite, it must be the case that for only finitely many η is the set $U_2^+(\eta)$ non-empty. It remains to show that each $U_2^+(\eta)$ is finite. By $(\aleph_1\ 2)$, the $A(\xi,\alpha)$ are disjoint. So, if for some η the set $U_2^+(\eta)$ were infinite, then $A(\eta,\beta) \nsubseteq^* A(\eta,\alpha)$, contradicting the restriction of $(\aleph_1\ 1)$ for column η. Hence only finitely many ξ satisfy (iii).

(B) $\Rightarrow$ (A): This is immediate since $A(\xi,\beta) \nsubseteq A(\xi,\alpha)$ implies that $A(\xi,\alpha) \cap T_\beta \neq A(\xi,\beta)$. □

Consequently, we assume (B) holds, but need only prove that (A) is maintained.

The construction of the matrix

The stage $\gamma = 0$ is trivial.

Successor stage: $\gamma = \alpha + 1$. Assume that for $\xi < \beta < \gamma$, sets $A(\xi,\beta)$ have been constructed satisfying the induction hypotheses. Let $\langle B(\xi)\colon \xi < \gamma\rangle$ be a partition of $T_\gamma \setminus T_\alpha$ into infinite disjoint sets. Define

$$A(\xi,\gamma) := (A(\xi,\alpha) \cap T_\gamma) \cup B(\xi) \quad \text{for each } \xi < \gamma.$$

(Recall that by convention $A(\alpha,\alpha) = \emptyset$.)

Clearly $(\aleph_1\ 1)$ and $(\aleph_1\ 2)$ now hold. The condition in $(\aleph_1\ 3)$ will be satisfied at limit stages. So it remains to check $(\aleph_1\ 4)$, in particular that

$$(\forall\beta < \gamma)\ \{\xi\colon A(\xi,\beta) \nsubseteq A(\xi,\gamma)\} \text{ is finite.}$$

Case 1: $\beta = \alpha$. Since $T_\alpha \setminus T_\gamma$ is finite and since the rows are disjoint families, it follows from the definition of $A(\xi,\gamma)$ that the desired set is finite. In particular, it is contained in the set $\{\xi\colon A(\xi,\alpha) \cap (T_\alpha \setminus T_\gamma) \neq \emptyset\}$.

Case 2: $\beta < \alpha$. Taking the contrapositive of

$$[A(\xi,\beta) \subseteq A(\xi,\alpha) \wedge A(\xi,\alpha) \subseteq A(\xi,\gamma)] \;\Rightarrow\; A(\xi,\beta) \subseteq A(\xi,\gamma),$$

we have

$$\{\xi\colon A(\xi,\beta) \nsubseteq^* A(\xi,\gamma)\} \subseteq \{\xi\colon A(\xi,\beta) \nsubseteq^* A(\xi,\alpha)\} \cup \{\xi\colon A(\xi,\alpha) \nsubseteq^* A(\xi,\gamma)\}.$$

Both of the sets on the right are finite, the first by induction hypothesis ($\aleph_1$ 4) and the second by Case 1.

This establishes the preservation of ($\aleph_1$ 4) through the successor stage and so completes this part of the construction. We now address the arduous.

Limit stage: λ. Assume that $\lambda \in S(\xi_0, \eta_0)$ for some $\eta_0 < \xi_0 < \lambda$, and that $A(\xi,\alpha)$ are constructed for $\xi < \alpha < \lambda$ satisfying the induction hypotheses. Fix bijections, $f\colon \omega \to \lambda$, and $g\colon \omega \to (\lambda \times 2) \cup T_\lambda$.

The following is a brief description of the construction which is to follow. The method is expressed in the notation and ideas of forcing. However, all objects involved are countable so no new generic objects are needed to obtain the result. Nonetheless, future extensions of this theorem may take advantage of this methodology.

We describe six different properties obtainable by objects of the form $\langle r, s, n\rangle$ where $n \in \omega$, and r and s are finite partial functions on λ with codomains ω and ${}^n2$, respectively. Those tuples satisfying these properties will be called *conditions*.

The function $s(\xi)$ is a finite approximation of (the characteristic function of) the set $A(\xi,\lambda)$. If $\xi \in dom(s)$ and $\beta \in dom(r)$, we consider this a promise to satisfy $A(\xi,\beta) \backslash n \subseteq A(\xi,\lambda)$. The value of $r(\beta) = m \in \omega$ will be a promise to satisfy ($\aleph_1$ 4) between rows λ and β "above" m (really, to satisfy the condition for ξ whose f-preimage is greater than m). Thus if $f^{-1}(\xi) > r(\beta)$ then we promise to have $A(\xi,\beta) \subseteq A(\xi,\lambda)$. n is redundant, but convenient to have explicit as it is frequently referenced.

The idea of forcing is implicit in this description. Here is an informal list of what a condition forces.

(F1) A number, $i \in T_\lambda$, is explicitly forced into some $A(\xi,\lambda)$ by $\langle r,s,n\rangle$ if $\xi \in dom(s)$ and $s(\xi)(i) = 1$.

(F2) A number $i \in T_\lambda$ may be implicitly forced into some $A(\xi,\lambda)$ by $\langle r,s,n\rangle$ due to the "almost containment" for the tower: $i \geqslant n$, $\xi \in dom(s)$ and there is a $\beta \in dom(r)$ with $i \in A(\xi,\beta)$.

(F3) A number $i \in T_\lambda$ can also be implicitly forced into $A(\xi,\lambda)$ due to ($\aleph_1$ 4): there is a $\beta \in dom(r)$ with $f^{-1}(\xi) > r(\beta)$ and $i \in A(\xi,\beta)$.

(F4) A number $i \in T_\lambda$ is forced out of an $A(\xi,\lambda)$ by $\langle r,s,n\rangle$ explicitly if $s(\xi)(i) = 0$ or implicitly just in case it is forced into $A(\eta,\lambda)$ for some $\eta \neq \xi$. Of course, i is forced out of all sets if $i \notin T_\lambda$.

(F5) Hence, $\langle r,s,n\rangle$ forces $A(\xi,\beta)\backslash n \subseteq A(\xi,\lambda)$ whenever $\xi \in dom(s)$ and $\beta \in dom(r)$.

(F6) To ensure ($\aleph_1$ 3) is satisfied, $\langle r,s,n\rangle$ will explicitly force that $A(\eta_0,\delta) \cap A(\xi_0,\lambda)$ is non-empty for all δ satisfying $\delta \in \varphi_\lambda \cap \sup^+ dom(r) \wedge \delta > \eta_0$.

Most of the definition of $\mathbb{P}$ below can be seen as consistency requirements for this "forcing".

A partial order called *extension* will be defined on conditions with the intuition being that an extension contains more information about the sets being constructed. A four part extension lemma is proved with the following implications: (E1) permits n to be incremented and is essentially a service lemma for the following parts. (E2) and (E3) permit the extension of the domains of s and r, respectively, by an element. (E4) permits the addition of an arbitrary element of T_λ into some $A(\xi, \lambda)$.

A recursive definition is given starting with the condition $\langle \emptyset, \{\langle \xi_0, \emptyset \rangle\}, 0 \rangle$, to which is applied the appropriate extension lemma which is dictated by the type of $g(m)$ where m is the stage of the definition.

This generates a chain of objects from which the $A(\xi, \lambda)$ are derived. g is a bookkeeping function that ensures all the desired properties are obtained. This ends the description of the proof mentioned above.

For a function $s(\xi)$ mapping into 2, recall the notation that

$$\bar{s}(\xi) := (s(\xi))^{-1}\{1\}.$$

Definition 43. Define the set

$$\mathbb{P} \subseteq \{\langle r, s, n \rangle : dom(r), dom(s) \in [\lambda]^{<\aleph_0} \wedge n \in \omega \wedge ran(r) \subseteq \omega \wedge ran(s) \subseteq {}^n 2\}$$

as follows:

$\langle r, s, n \rangle \in \mathbb{P}$ if and only if the following requirements are satisfied. Note that the parenthetical statements are meant as explanation, not as part of the definition.

($\mathbb{P}$1) $\langle \bar{s}(\xi) : \xi \in dom(s) \rangle$ is a collection of pairwise disjoint subsets of T_λ. (s builds the final sets. See (F1)).

($\mathbb{P}$2) $(\forall \alpha \in dom(r))(\forall \xi \in dom(s))(\forall m > r(\alpha))$

- ($\mathbb{P}$2A) $\xi = f(m) \Rightarrow A(\xi, \alpha) \cap n \subseteq \bar{s}(\xi)$, (($\aleph_1$ 4) satisfied with row α. See (F3).)
- ($\mathbb{P}$2B) $\xi \neq f(m) \Rightarrow A(f(m), \alpha) \cap \bar{s}(\xi) = \emptyset$. (Allows $f(m)$ to be added to the domain of s.)
- ($\mathbb{P}$2C) $A(f(m), \alpha) \subseteq T_\lambda$.

($\mathbb{P}$3) $(\forall \alpha, \beta \in dom(r))(\forall \xi \in dom(s))(\forall m > r(\alpha))$
$\xi \neq f(m) \Rightarrow A(\xi, \beta) \cap A(f(m), \alpha) \setminus n = \emptyset$. (See (F5).)

($\mathbb{P}$4) $(\forall \alpha, \beta \in dom(r))(\forall m_0 > r(\alpha))(\forall m_1 > r(\beta))$
$m_0 \neq m_1 \Rightarrow A(f(m_0), \alpha) \cap A(f(m_1), \beta) = \emptyset$.
(When $f(m_0)$ and $f(m_1)$ are added to the domain of s, we need to meet ($\aleph_1$ 4) at α, β, respectively, while maintaining disjointness of $\bar{s}(\xi)$.)

($\mathbb{P}$5) $(\forall \xi \neq \xi' \in dom(s))(\forall \beta, \beta' \in dom(r))\ A(\xi, \beta) \cap A(\xi', \beta') \setminus n = \emptyset$. (Needed to allow almost containment to be satisfied. See (F5).)

($\mathbb{P}$6) $\xi_0 \in dom(s)$ and $(\forall \delta \in \varphi_\lambda \cap \sup^+ dom(r).\delta > \eta_0) A(\eta_0, \delta) \cap \bar{s}(\xi_0) \neq \emptyset$. (This will ensure condition ($\aleph_1$ 3) is satisfied. See (F6).)

Definition 44. If $p = \langle r, s, n \rangle$ and $p' = \langle r', s', n' \rangle$ are elements of $\mathbb{P}$, say p' *extends* p and write $p' \leqslant p$ if and only if the following five conditions are satisfied:

(e1) $r' \supseteq r$;
(e2) $n' \geqslant n$;
(e3) $dom(s') \supseteq dom(s)$;
(e4) $(\forall \xi \in dom(s))\ s'(\xi) \supseteq s(\xi)$;
(e5) $(\forall \xi \in dom(s))(\forall \beta \in dom(r))\ \bar{s}'(\xi) \supseteq (A(\xi, \beta) \cap (n' \backslash n))$. (See (F5).)

We now state the extension lemmas. For all these statements, let $p = \langle r, s, n \rangle \in \mathbb{P}$.

Lemma 45 (E1) (The simple extension lemma). *There is an s' such that*
(a) $\langle r, s', n + 1 \rangle \in \mathbb{P}$,
(b) $\langle r, s', n + 1 \rangle \leqslant p$,
(c) $dom(s') = dom(s)$.

Lemma 46 (E2) (Extending the domain of s). *For any $\xi \in \lambda$ there are an s' and an n' such that*
(a) $\langle r, s', n' \rangle \in \mathbb{P}$,
(b) $\langle r, s', n' \rangle \leqslant p$,
(c) $dom(s') = dom(s) \cup \{\xi\}$.

Lemma 47 (E3) (Extending the domain of r). *For any $\beta \in \lambda$ there are an s', an n' and an m' such that*
(a) $\langle r \cup \{\langle \beta, m' \rangle\}, s', n' \rangle \in \mathbb{P}$,
(b) $\langle r \cup \{\langle \beta, m' \rangle\}, s', n' \rangle \leqslant p$,
(c) $dom(s') = dom(s)$.

Lemma 48 (E4) (Adding an element of T_λ). *For any $i \in T_\lambda$ there are an s' and an n' such that*
(a) $\langle r, s', n' \rangle \in \mathbb{P}$,
(b) $\langle r, s', n' \rangle \leqslant p$,
(c) $(\exists \xi \in dom(s'))\ i \in \bar{s}'(\xi)$,
(d) $|dom(s') \backslash dom(s)| \leqslant 1$.

Proof of Lemma 45. Let $\langle r, s, n \rangle \in \mathbb{P}$ be given. To show the claim, we must extend each function $s(\xi)$ by one place such that the resulting triple $\langle r, s', n + 1 \rangle$ satisfies ($\mathbb{P}$1)–($\mathbb{P}$6). Notice that since r and $dom(s)$ do not change, ($\mathbb{P}$2C) and ($\mathbb{P}$3)–($\mathbb{P}$6) will necessarily be satisfied. By ($\mathbb{P}$1), n should belong to at most one $\bar{s}(\xi)$. We perform a minimal extension to satisfy ($\mathbb{P}$2A) and the definition of extension, (e5). That is, define $s'(\xi)(n) := 0$ unless there is an $\alpha \in dom(r)$ with $n \in A(\xi, \alpha)$ in which case $s'(\xi)(n) := 1$. If this new triple is in $\mathbb{P}$, it immediately satisfies the conditions to extend $\langle r, s, n \rangle$. So, we show the following:

Claim. $\langle r, s', n+1 \rangle \in \mathbb{P}$.

Proof of Claim. We first show that only one such ξ can satisfy this second requirement. That is if $\langle r, s, n \rangle \in \mathbb{P}$, then for at most one ξ is there an $\alpha \in dom(r)$ with $n \in A(\xi, \alpha)$. But this is exactly what ($\mathbb{P}$5) states for $\langle r, s, n \rangle$. Hence $\langle r, s', n+1 \rangle$ satisfies ($\mathbb{P}$1).

The condition defining $\langle r, s', n+1 \rangle$ gives ($\mathbb{P}$2A) immediately. If ($\mathbb{P}$2B) failed, there would be an $\alpha \in dom(r)$, a $\xi \in dom(s)$, and an $m > r(\alpha)$ such that $\xi \neq f(m)$ and $n \in A(\xi, \alpha) \cap A(f(m), \alpha)$. But this contradicts the fact that ($\mathbb{P}$3) held for $\langle r, s, n \rangle$. This completes the proof that $\langle r, s', n+1 \rangle \in \mathbb{P}$. □

This completes the proof of Lemma 45.

Before continuing, we state and prove a lemma necessary for the remaining proofs. While (E1) stated that n could be increased, the following lemma shows there is an n' to which n can be increased to meet the other conditions in the definition of condition.

Lemma 49. *Given $\langle r, s, n \rangle \in \mathbb{P}$, and $\xi \notin dom(s)$ there is an $n' > n$ satisfying the following two conditions:*

(a) $(\forall \alpha \neq \beta \in dom(r))(\forall m > r(\alpha))\ A(\xi, \alpha) \cap A(f(m), \beta) \backslash n' = \emptyset$.

(b) $(\forall \eta \in dom(s))(\forall \alpha \neq \beta \in dom(r))\ A(\xi, \alpha) \cap A(\eta, \beta) \backslash n' = \emptyset$.

Proof. Since (a) and (b) are preserved as n' grows and since there are only finitely many triples $\eta \in dom(s)$, $\alpha \neq \beta \in dom(r)$, it suffices to show such an n' exists for an arbitrary such triple. For (b), notice that as $\xi \notin dom(s)$ we have $\xi \neq \eta$. Since $A(\xi, \alpha)$ and $A(\eta, \beta)$ are almost disjoint, (b) follows.

For (a) there are two cases depending on the order of α and β.

Case 1: $\beta > \alpha$. By induction hypothesis and Lemma 42, $\{\mu \colon A(\mu, \beta) \cap T_\alpha \neq A(\mu, \alpha)\}$ is finite. So there is an m_0 such that for all m

$$m > m_0 \wedge \xi \neq f(m) \Rightarrow A(f(m), \beta) \cap T_\alpha = A(f(m), \alpha)$$
$$\Rightarrow A(f(m), \beta) \cap A(\xi, \alpha) = \emptyset.$$

But there are only finitely many m with $r(\beta) < m \leqslant m_0$ while $\xi \neq f(m)$ implies $A(f(m), \beta)$ and $A(\xi, \alpha)$ are almost disjoint.

Case 2: $\beta < \alpha$. Similarly to the previous case, the set $\{\mu \colon A(\mu, \beta) \nsubseteq A(\mu, \alpha)\}$ is finite. So there is an m_0 such that for all m

$$m > m_0 \wedge \xi \neq f(m) \Rightarrow A(f(m), \beta) \subseteq A(f(m), \alpha)$$
$$\Rightarrow A(f(m), \beta) \cap A(\xi, \alpha) = \emptyset.$$

The case is completed as above. □

Proof of Lemma 46. Fix $\langle r,s,n\rangle \in \mathbb{P}$ and $\xi \in \lambda \backslash dom(s)$. We wish to add ξ to the domain of s. This is done in two steps, first by extending the functions $s(\eta)$ for the $\eta \in dom(s)$ (i.e., increasing n) and then by defining $s'(\xi)$.

Step 1: Fix n' as in Lemma 49 for $\langle r,s,n\rangle$. By iterated application of (E1), we may extend $\langle r,s,n\rangle$ to $\langle r,s'',n'\rangle \in \mathbb{P}$. This ensures that ($\mathbb{P}$3) and ($\mathbb{P}$5) will be satisfied by the new condition.

Step 2: We now add ξ to $dom(s)$ conforming to condition ($\mathbb{P}$2A). For $\eta \in dom(s)$, let $s'(\eta) := s''(\eta)$. Define $s'(\xi)$ by

$$\bar{s}'(\xi) := \{ i < n' \colon (\exists \alpha \in dom(r))\, i \in A(\xi,\alpha) \wedge f^{-1}(\xi) > r(\alpha) \}.$$

Claim. $\langle r,s',n'\rangle \in \mathbb{P}$.

Proof of Claim (*sketch*). The construction assures that ($\mathbb{P}$2A), ($\mathbb{P}$3) and ($\mathbb{P}$5) are satisfied. ($\mathbb{P}$2C), ($\mathbb{P}$4) and ($\mathbb{P}$6) persist from $\langle r,s'',n'\rangle$. If ($\mathbb{P}$1) fails for $\langle r,s',n'\rangle$, then ($\mathbb{P}$2B) would not hold for $\langle r,s'',n'\rangle$. If ($\mathbb{P}$2B) fails for $\langle r,s',n'\rangle$, then ($\mathbb{P}$4) would not hold for $\langle r,s'',n'\rangle$. □

Hence $\langle r,s',n'\rangle \in \mathbb{P}$. The other desired conditions for (E2) follow from the construction. This completes the proof of Lemma 46. □

Proof of Lemma 47.

Part 1: Let $\beta \in \lambda \backslash dom(r)$. We wish to add β to the domain of r. The initial part of the construction is more complicated if $\beta > \max(\sup(dom(r)), \eta_0)$. If this is *not* the case, we may let $\langle r,s'',n''\rangle := \langle r,s,n\rangle$ and skip to Part 2 of the construction.

But suppose $\beta > \max(\sup(dom(r)), \eta_0)$. Recall that $\lambda \in S(\xi_0, \eta_0)$ and by ($\mathbb{P}$6), $\xi_0 \in dom(s)$. We must ensure that ($\mathbb{P}$6) is satisfied by the extension, in particular for each $\delta \in (\varphi_\lambda \cap (\beta+1) \backslash (\eta_0+1))$, $A(\eta_0,\delta) \cap \bar{s}'(\xi_0) \neq \emptyset$.

Consider an arbitrary $\delta \in (\varphi_\lambda \cap (\beta+1)) \backslash \max(\sup^+(dom(r), \eta_0+1))$, a finite set. We wish to find an "unrestricted" number in $A(\eta_0,\delta)$ to force into $\bar{s}'(\xi_0)$. Say that $i \in A(\eta_0,\delta)\backslash n$ is *restricted* (i.e., is already forced into another set) if either of the following conditions are met:

(1) $(\exists \alpha \in dom(r))(\exists m > r(\alpha))\ i \in A(f(m),\alpha)$, or
(2) $(\exists \alpha \in dom(r))(\exists \xi \in dom(s))\ i \in A(\xi,\alpha)$.

Since $\delta > \sup^+(dom(r))$, the set $A(\eta_0,\delta) \backslash \bigcup \{ T_\alpha \colon \alpha \in dom(r) \}$ is infinite. (Actually, we also use the following facts: ($\aleph_1$ 4) is satisfied as an induction hypothesis, sets in different columns are almost disjoint and $A(\eta_0,\delta) \backslash A(\eta_0,\alpha)$ is infinite for each $\alpha \in dom(r)$.) This means there are arbitrarily large i which are not restricted.

Perform the following extension process for each relevant δ in turn. Let i_δ be the least $i \geqslant n$ such that

$$i_\delta \in \left(A(\eta_0,\delta) \backslash \bigcup \{ T_\alpha \colon \alpha \in dom(r) \} \right) \cap T_\lambda.$$

(That is, i_δ is not restricted.) Repeatedly apply (E1) to get $\langle r, s'', i_\delta \rangle \leqslant \langle r, s, n \rangle$ with $dom(s'') = dom(s)$. Extend this in the obvious minimal way to satisfy $i_\delta \in \bar{s}''(\xi_0)$. Notice that as i_δ is not restricted, this is a condition extending $\langle r, s, n \rangle$. Let the result of this finite iteration be called $\langle r, s'', n'' \rangle$.

Part 2: Assume we have a condition $\langle r, s'', n'' \rangle$ satisfying ($\mathbb{P}$6) for $\delta \in \varphi_\lambda \cap (\beta + 1) \backslash (\eta_0 + 1)$. We now extend s'' and n'' to satisfy ($\mathbb{P}$3) and ($\mathbb{P}$5). By a slight variation of the Lemma 49, there is an $n' \geqslant n''$ such that

(a) If $\alpha \in dom(r)$, $m > r(\alpha)$ and $\xi \in dom(s)$ satisfy $\xi \neq f(m)$ then $A(\xi, \beta) \cap A(f(m), \alpha) \backslash n' = \emptyset$.

(b) If $\xi \neq \xi' \in dom(s'')$ and $\beta' \in dom(r)$ then $A(\xi, \beta) \cap A(\xi', \beta') \backslash n' = \emptyset$. Iteratively applying (E1), obtain $\langle r, s', n' \rangle \leqslant \langle r, s'', n'' \rangle$.

Part 3: Defining $r(\beta)$.

Claim. *There is an $m' \in \omega$ satisfying the following three conditions*:

(1) $(\forall m > m') A(f(m), \beta) \cap n' = \emptyset$.

(2) $(\forall \alpha \in dom(r))(\forall m > m')(\forall l > r(\alpha))$

$$l \neq m \;\Rightarrow\; A(f(l), \alpha) \cap A(f(m), \beta) = \emptyset.$$

(3) $(\forall m > m') A(f(m), \beta) \subseteq T_\lambda$.

Proof of Claim. Since there are only finitely many requirements on m', and since the properties of interest are preserved upward (for larger m'), we may consider each one separately. (1) follows from the fact that $\{A(\eta, \beta): \eta < \beta\}$ is a disjoint family while (3) follows because $T_\beta \backslash T_\lambda$ is finite.

For (2) fix $\alpha \in dom(r)$. If $\alpha < \beta$, take $m' > \max(f^{-1}\{\xi: A(\xi, \beta) \cap T_\alpha \neq A(\xi, \alpha)\})$, a finite set by hypothesis. If $\alpha > \beta$, let $m' > \max(f^{-1}\{\xi: A(\xi, \beta) \nsubseteq A(\xi, \alpha)\}$, again finite by hypothesis. In either case, if $\xi \neq f(m)$ and $m > m'$ then $A(f(m), \beta) \cap A(\xi, \alpha) = \emptyset$. □

Fix m' as in the previous claim.

Claim. $\langle r \cup \{(\beta, m')\}, s', n' \rangle \in \mathbb{P}$.

Proof of Claim. All the work has been done in the previous lemmas and claims. For example, ($\mathbb{P}$1) holds for $\langle r, s', n' \rangle$ and hence holds for $\langle r \cup \{(m', \beta)\}, s', n' \rangle$. ($\mathbb{P}$2) holds by choice of m', condition (3), as does ($\mathbb{P}$3) when β is α. When β instantiates β in ($\mathbb{P}$3), the choice of n', part (2), ensures the condition is met. ($\mathbb{P}$4) holds by choice of m', part (3), ($\mathbb{P}$5) by choice of n' part (2), and finally ($\mathbb{P}$6) holds because of the construction of $\langle r, s'', n'' \rangle$. □

The other properties of this condition are clear from the construction and so this completes the proof of Lemma 47. □

Proof of Lemma 48. Fix $i \in T_\lambda$. We wish to find an s' and an n' such that there is a $\xi \in dom(s')$ with $i \in \bar{s}'(\xi)$ and with $dom(s')$ having at most 1 new element. The conclusion is immediate if i is in $\bar{s}(\xi)$ for some ξ in $dom(s)$, so assume otherwise. We examine three cases.

Case 1: $\langle r, s, n\rangle$ implicitly "forces" $i \in A(\xi, \lambda)$ for some ξ. See (F2) and (F3). Either:
(1) there is an $\alpha \in dom(r)$ and an $m > r(\alpha)$ such that $i \in A(f(m), \alpha)$, or
(2) $i > n$ and there is an $\alpha \in dom(r)$ and $\xi \in dom(s)$ such that $i \in A(\xi, \alpha)$.

Either way, apply (E1) and (E2) to ensure that $\langle r, s', n'\rangle \leqslant \langle r, s, n\rangle$, $n' > i$ and $\xi \in dom(s')$. Then the conclusion holds by ($\mathbb{P}$2A) applied to $\langle r, s', n'\rangle$ or by (e5) according to whether (1) or (2) holds, respectively.

Case 2: Case 1 fails and $i < n$. Then we may fix $\xi \notin dom(s)$ and define a new condition that satisfies $i \in \bar{s}'(\xi)$. This case uses ($\mathbb{P}$4) on $\langle r, s, n\rangle$ to ensure the result holds.

Case 3: Case 1 fails and $i \geqslant n$. Extend by (E1) to $\langle r, s', i\rangle$ and put i in any $\bar{s}'(\xi)$.

These exhaust the cases need to establish (E4) and complete the proof of the extension lemmas. □

Completion of the construction

We recursively define a sequence of conditions p_m for $m \in \omega$. Let $p_0 := \langle \emptyset, \{\langle \xi_0, 0\rangle\}, 0\rangle$ and note that $p_0 \in \mathbb{P}$. Recall that $g : \omega \to (\lambda \times 2) \cup T_\lambda$ is a bijection. Suppose that $p_m = \langle r_m, s_m, n_m\rangle$ is defined. We define $p_{m+1} \leqslant p_m$ according to the value of $g(m)$ as follows:

Case 1: $g(m) = \langle \xi, 0\rangle$ for some $\xi < \lambda$. Put ξ into the domain of s. Apply (E2) to p_m and ξ to get $p_{m+1} = \langle r_{m+1}, s_{m+1}, n_{m+1}\rangle \leqslant p_m$ with $r_{m+1} = r_m$ and $\xi \in dom(s_{m+1})$.

Case 2: $g(m) = \langle \beta, 1\rangle$ for some $\beta < \lambda$. Put β into the domain of r. Apply (E3) to get $p_{m+1} = \langle r_{m+1}, s_{m+1}, n_{m+1}\rangle \leqslant p_m$ with $\beta \in dom(r_{m+1})$.

Case 3: $g(m) \in T_\lambda$. Ensure $g(m)$ is in some $\bar{s}(\xi)$. Apply (E4) to obtain $p_{m+1} = \langle r_{m+1}, s_{m+1}, n_{m+1}\rangle \leqslant p_m$ with $g(m) \in \bar{s}_{m+1}(\xi)$ for some $\xi \in dom(s_{m+1})$.

This completes the definition of p_m and we are ready to define the $A(\xi, \lambda)$. Fix $\xi < \lambda$. Since g is a bijection, there is an $l \in \omega$ with $g(l) = \langle \xi, 0\rangle$. By the definition of p_m, if $m > l$ then $\xi \in dom(s_m)$. Let

$$A(\xi, \lambda) := \bigcup \{\bar{s}_m(\xi) : m > l\}.$$

This completes the construction of $A(\xi, \lambda)$.

It remains to show that the induction hypotheses ($\aleph_1$ 1)–($\aleph_1$ 4) hold for these $A(\xi, \lambda)$. Unless otherwise noted, $p_m = \langle r_m, s_m, n_m\rangle$.

Claim ($\aleph_1$ 1). *For each $\xi < \lambda$, $\langle A(\xi, \alpha) : \xi < \alpha \leqslant \lambda\rangle$ is a tower.*

Proof of Claim. It suffices to show for α with $\xi < \alpha < \lambda$ that $A(\xi, \alpha) \subseteq^* A(\xi, \lambda)$. So fix such an α. Since $A(\xi, \alpha) \setminus T_\lambda$ is finite, there is an $i_0 \in T_\lambda$ with $i_0 > \sup(A(\xi, \alpha) \setminus T_\lambda)$. Let $m > \sup(g^{-1}\{i_0, \langle \xi, 0\rangle, \langle \alpha, 1\rangle\})$. This choice ensures that $\xi \in dom(s_m)$ and $\alpha \in dom(r_m)$. We show that $A(\xi, \alpha) \setminus n_m \subseteq A(\xi, \lambda)$. Suppose $i \in A(\xi, \alpha) \setminus n_m$. By the definition of i_0 and since $i \in A(\xi,\alpha)\backslash i_0$ we know $i \in T_\lambda$. Next, $m' := g^{-1}(i) > m$, since if $m' \leqslant m$ then i would be in some $\bar{s}_m(\xi)$ because of the construction at stage m' and this implies $i \leqslant n_m$ contrary to the choice of i. Hence $p_{m'} \leqslant p_m$ and by the definition of extension, in particular by (e5), we may conclude $i \in \bar{s}_{m'}(\xi)$. □

Claim ($\aleph_1$ 2). *T_λ is the disjoint union of the $A(\xi,\lambda)$ with $\xi < \lambda$.*

Proof of Claim. Only elements of T_λ can be put in any $A(\xi,\lambda)$. That T_λ is contained in the union follows from the third case in the definition of p_m and the fact that T_λ is contained in the image of g. The property of disjointness follows from condition ($\mathbb{P}$1) in the definition of $\mathbb{P}$. □

Claim ($\aleph_1$ 3). *For each β in $\varphi_\lambda\backslash(\eta_0+1)$, $A(\xi_0,\lambda)\cap A(\eta_0,\beta) \neq \emptyset$.*

Proof of Claim. Fix $\beta \in \varphi_\lambda\backslash(\eta_0+1)$ and let $m > g^{-1}(\langle\beta,1\rangle)$. Then $\beta \in dom(r_m)$ and by ($\mathbb{P}$6) we conclude $\bar{s}(\xi_0)\cap A(\eta_0,\beta) \neq \emptyset$. But $\bar{s}(\xi_0) \subseteq A(\xi_0,\lambda)$ and so $A(\xi_0,\lambda)\cap A(\eta_0,\beta) \neq \emptyset$ as desired. □

Claim ($\aleph_1$ 4). *If $\alpha < \lambda$ then $\{\xi\colon A(\xi,\alpha) \nsubseteq A(\xi,\lambda)\}$ is finite.*

Proof of Claim. Temporarily denote $\{\xi\colon A(\xi,\alpha) \nsubseteq A(\xi,\lambda)\}$ by $X(\alpha)$. Fix $\alpha < \lambda$ and let $m := g^{-1}(\langle\alpha,1\rangle)+1$. We show $X(\alpha) \subseteq f``(r_m(\alpha)+1)$.

Recall that $f\colon\omega\to\lambda$ is a bijection. It is sufficient to show that $\lambda\backslash f``(r_m(\alpha)+1)\cap X(\alpha) = \emptyset$. Fix ξ such that $f^{-1}(\xi) > r_m(\alpha)$ and fix $i \in A(\xi,\alpha)$. Note by ($\mathbb{P}$2C) that $i \in T_\lambda$. Let $l > \max(g^{-1}(i), g^{-1}\langle\xi,0\rangle, m)$.

Since $p_l \leqslant p_m$, we have $r_l(\alpha) = r_m(\alpha)$. Further, $\xi \in dom(s_l)$ and $i < n_m$. By ($\mathbb{P}$2A) we can conclude that $i \in \bar{s}_m(\xi)$, hence $i \in A(\xi,\lambda)$. Since i was an arbitrary element of $A(\xi,\alpha)$, we have $A(\xi,\alpha) \subseteq A(\xi,\lambda)$, so $\xi \notin X(\alpha)$. This gives the desired result. □

This completes the proof that the induction hypotheses are satisfied through the limit stage. Continuing the construction through the countable ordinals gives the desired $\omega_1 \times \omega_1$ matrix. □

We are now in a position to apply the theorem to obtain information about the gap cohomology. According to Lemma 40, if H2 holds of the ladder system $\langle\varphi_\lambda\colon \lambda \in \Lambda\rangle$ and we apply the $\aleph_1$ gap construction with this system, then for each pair $\eta < \xi < \omega_1$, the pair of towers $\langle A(\eta,\alpha), A(\xi,\alpha)\colon \xi < \alpha < \omega_1\rangle$ is a Hausdorff gap. This is where the appearance of η in Lemma 40 is used and is necessary because of the form of the gaps constructed.

Suppose that $X \subseteq \omega_1$ is non-empty and not equal to ω_1. Define

$$A(X,\alpha) := \bigcup\{A(\xi,\alpha)\colon \xi \in X\cap\alpha\}.$$

Claim. *The tower* $A(X) := \langle A(X,\alpha)\colon \alpha \in \omega_1 \rangle$ *is a gap in* T.

Proof of Claim. Assume that $\mu_0 \in X$ and $\mu_1 \notin X$. In fact, if $A(X)$ is truly a subtower, it is easy to see that it is a gap in T. This is simply because the tower $A(\mu_0)$ is a subtower of $A(X)$ while $A(\mu_1)$ is a subtower of its levelwise complement. Any separation of $A(X)$ from its levelwise complement would provide a separation of $A(\mu_0)$ from $A(\mu_1)$.

So it remains to show that $A(X)$ is a tower. Fix $\beta < \alpha$. There are three things to check:

(a) $A(X,\beta) \subseteq^* A(X,\alpha)$,
(b) $A(X,\beta) \neq^* A(X,\alpha)$, (i.e. $A(X,\alpha) \backslash A(X,\beta)$ is infinite) and
(c) the set $A(X,\alpha) \cap T_\beta \backslash A(X,\beta)$ is finite.

(a) and (b) show $A(X)$ is a tower while (c) ensures $A(X)$ satisfies the "faithful restriction" clause in the definition of subtower.

For (a) we need to check that $A(X,\beta) \backslash A(X,\alpha)$ is finite. This set is equal to

$$\bigcup_{\eta \in X \cap \beta} \left(A(\eta,\beta) \backslash \bigcup_{\xi \in X \cap \alpha} A(\xi,\alpha) \right).$$

Note that this displayed set is contained in $\bigcup \{A(\xi,\beta) \backslash A(\xi,\alpha)\colon \xi \in X \cap \beta\}$ since we have made the set being subtracted smaller. For each $\xi \in X$, we have $A(\xi,\beta) \subseteq^* A(\xi,\alpha)$. However, if $X \cap \beta$ is infinite, there is the possibility that the finite sets $A(\xi,\beta) \backslash A(\xi,\alpha)$ might accumulate to an infinite set. This is the reason the construction insists that $A(\xi,\beta) \subseteq A(\xi,\alpha)$ (not almost containment) "most of the time". In particular, for this fixed α and β, for all but finitely many ξ we have $A(\xi,\beta) \backslash A(\xi,\alpha) = \emptyset$, and hence the displayed set is finite. This gives (a).

For (b), we need to show $A(X,\alpha) \backslash A(X,\beta)$ is infinite. Since $A(\mu_0,\alpha) \subseteq A(X,\alpha)$, it suffices to show that $A(\mu_0,\alpha) \backslash T_\beta$ is infinite because $A(X,\beta)$ is contained in T_β. Recall that the set $\{\eta\colon A(\eta,\beta) \not\subseteq A(\eta,\alpha)\}$ is finite and that the levels of the matrix are disjoint families. These, together with the fact that $T_\beta \subseteq^* T_\alpha$ holds, imply that $A(\mu_0,\alpha) \cap T_\beta =^* A(\mu_0,\beta)$. But $A(\mu_0,\alpha) \backslash A(\mu_0,\beta)$ is an infinite set.

For (c), the set of interest is contained in $\bigcup \{A(\xi,\alpha) \cap T_\beta \backslash A(\xi,\beta)\}$. Recall that for all but finitely many ξ we have $A(\xi,\alpha) \cap T_\beta = A(\xi,\beta)$, and, as noted in (b), immediately above, for all ξ we have $A(\xi,\alpha) \cap T_\beta =^* A(\xi,\beta)$. Hence only finitely many sets contribute to the noted union, and each only a finite amount. Thus $A(X)$ satisfies the "faithful restriction" requirement. □

This claim leads to the following result.

Corollary 50. *It is consistent with the statement* $2^{\aleph_1} > 2^{\aleph_0}$ *that the gap cohomology group have cardinality* $2^{\aleph_1}$.

Proof. We have shown that H2 is consistent in the preceding section. Build the $\omega_1 \times \omega_1$ matrix in the $\aleph_1$ gap theorem with the ladder system in (G2) satisfying H2. As noted in the corollary to the $\aleph_0$ gap theorem, if X and Y are non-trivial subsets of ω_1 then $A(X) \triangle A(Y) = A(X \triangle Y)$. In the discussion of the gap cohomology, recall that two gaps represent different cohomology classes just in case their levelwise symmetric difference is again a gap. By the immediately preceding claim, this is the case whenever $X \neq Y$ and $X \neq \omega_1 \setminus Y$. □

This ends the discussion of cohomology and gaps for this article. In attempting to settle the issue of, for example, the possible size of the gap cohomology group, it may be useful to look at definable properties of gaps. One example of this is tight gaps which are described in [15, 16]. The next section introduces another example, incollapsible gaps.

6. Incollapsible gaps

Let $\langle A, B \rangle$ be a Hausdorff gap. We ask: On what subsets of ω does the restriction of $\langle A, B \rangle$ remain a gap? Note that the question only makes sense for subsets on which $\langle A, B \rangle$ remains a pregap. For a tower, A, in $\mathscr{P}(\omega)$ let $A \restriction Z$ for $Z \subseteq \omega$ be the tower $\langle A_\alpha \cap Z \colon \alpha \in \omega_1 \rangle$.

Definition 51. Say that a gap, $\langle A, B \rangle$, *collapses* on $Z \subseteq \omega$ if and only if $\langle A \restriction Z, B \restriction Z \rangle$ is a pregap but not a gap. That is, there is some $Y \subseteq Z$ that fills $\langle A \restriction Z, B \restriction Z \rangle$.

Definition 52. Say a gap, $\langle A, B \rangle$, is *incollapsible* if

$$(\forall Z \subseteq \omega) \langle A, B \rangle \text{ does not collapse on } Z.$$

Definition 53. Let IG be the statement: There is an incollapsible gap.

Theorem 54 (Incollapsible gaps). IG *is independent of* ZFC.

Proof. We show that MA + $\neg$CH $\vdash \neg$IG and CH $\vdash$ IG. Similar investigations have been undertaken in [13] who show under MA + $\neg$CH that for any gap there is an infinite proper subset of ω on which the gap remains a gap when restricted.

We now sketch the proof of MA + $\neg$CH $\vdash \neg$IG. The "obvious" partial order works. Fix $\langle A, B \rangle$, a gap. Let

$$\mathbb{P} := \{\langle z, y, s, t \rangle \colon (\exists n \in \omega) z, y \in {}^n\{0,1\} \wedge s, t \in [\omega_1]^{<\omega}\}.$$

z will build a set Z, while y will build a set $Y \subseteq Z$. The set s is a list of ordinals, α, for which we promise to keep $A_\alpha \cap Z$ inside Y "from now on" and t is a list of ordinals keeping $B_\alpha \cap Z$ out of Y "from now on".

The goal is to have $\langle A \restriction Z, B \restriction Z \rangle$ a pregap filled by Y. Recall the notation from previous chapters that $\bar{z}$ is the set $z^{-1}\{1\}$, etc. Let $p = \langle z, y, s, t \rangle$ and $p' = \langle z', y', s', t' \rangle$ and define $\leqslant$ on $\mathbb{P}$ by $p \leqslant p'$ if and only if

(1) $z \supseteq z'$, $y \supseteq y'$, $s \supseteq s'$ and $t \supseteq t'$.
(2) (With the obvious notational conventions) $(\forall i \in n \setminus n')$

(i) $(\forall \alpha \in s') i \in A_\alpha \wedge i \in \bar{z} \Rightarrow i \in \bar{y}$,
(ii) $(\forall \alpha \in t') i \in B_\alpha \wedge i \in \bar{z} \Rightarrow i \notin \bar{y}$

Claim. *$\mathbb{P}$ has the c.c.c. (In fact, it is σ-centered.)*

Proof of Claim. Fix an uncountable subcollection of $\mathbb{P}$, S. By thinning, we may assume all the z and y components are the same, independent of the condition chosen from S. The componentwise union of any pair of these is a common extension. □

We now describe the dense sets that ensure $\langle A, B \rangle$ will collapse as desired. Let $\beta < \alpha < \omega_1$ and $m \in \omega$ be fixed. Define

$$D^m_{\alpha\beta} := \{p = \langle z, y, s, t \rangle : |\bar{z} \cap A_\alpha \setminus A_\beta| \geqslant m\}.$$

Claim. *$D^m_{\alpha\beta}$ is dense for all α, β and m.*

Proof of Claim. By induction on m. The claim is clear for $m = 0$. Suppose it is true for m. Fix $p_0 \in \mathbb{P}$ and $p_1 \leqslant p_0$ (with the obvious notational extensions) so that $|\bar{z}_1 \cap A_\alpha \setminus A_\beta| \geqslant m$. Let $r \in \omega$ be so large that

$$\left(\bigcup_{\alpha' \in s_1 \cup t_1} A_{\alpha'} \setminus r \right) \cap \left(\bigcup_{\alpha' \in s_1 \cup t_1} B_{\alpha'} \setminus r \right) = \emptyset,$$

which is possible since $s_1 \cup t_1$ is a finite set. Fix $n \in A_\alpha \setminus A_\beta$, an infinite set, such that $n > r, n_1$. Let $z \supseteq z_1$ and $n \in \bar{z}$, and let $y \supseteq y_1$ with $n \in \bar{y}$. Then $\langle z, y, s, t \rangle \leqslant p_0$ and $|\bar{z} \cap A_\alpha \setminus A_\beta| \geqslant m + 1$. □

The point of this claim is that $A \restriction Z$ will still be an ω_1 tower. A similar argument shows the same for $B \restriction Z$ and so $\langle A \restriction Z, B \restriction Z \rangle$ remains a pregap. To show that the set Y separates the restricted pregap, note that if $\alpha \in s$ and $\langle z, y, s, t \rangle \in G$ for G generic then $Y \supseteq^* (Z \cap A_\alpha)$. So it suffices to show the set $E_\alpha := \{\langle z, y, s, t \rangle \in \mathbb{P} : \alpha \in s\}$ is dense which is left to the reader.

We now sketch the proof of CH $\vdash$IG. This is a standard diagonalization argument on $[\omega]^{\aleph_0}$ (the infinite subset of ω) which under CH has cardinality $\aleph_1$. Fix an enumeration of $[\omega]^{\aleph_0}$, Y_α for $\alpha < \omega_1$. Recursively define A_α and B_α so that Y_α will not fill $\langle A \restriction Z, B \restriction Z \rangle$ for any $Z \supseteq Y_\alpha$. Suppose A_β and B_β are defined for $\beta < \alpha$ and form a pregap. If for some $\beta < \alpha$ we have $Y_\alpha \subseteq^* A_\beta$ then there is nothing to worry about. Otherwise $Y_\alpha \setminus A_\beta$ is infinite for each $\beta < \alpha$. As the set of A_β's is an increasing chain, the collection of $Y_\alpha \backslash A_\beta$ has the strong finite intersection property. So there is an infinite set $X \subseteq^* Y_\alpha \backslash A_\beta$ and thus X is almost disjoint from each A_β.

Since α is countable, there is a set X' such that for all $\beta < \alpha$, $X' \supsetneq^* B_\beta$ and $X' \cap A_\beta =^* \emptyset$. Let $B_\alpha := X \cup X'$ and notice now that Y_α cannot separate A from B since $B_\alpha \cap Y_\alpha$ is infinite. Let A_α be any set such that $A_\alpha \supsetneq^* A_\beta$ for $\beta < \alpha$ and A_α is almost disjoint from B_α. □

References

[1] J. Baumgartner, Applications of the proper forcing axiom, in: K. Kunen and J. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, New York, 1984).

[2] A. Blass, Cohomology detects failures of the axiom of choice, Trans. Amer. Math. Soc. 279 (1) (1983) 257–269.

[3] K. Devlin, Variations on ◇, J. Symbolic Logic 44 (1) (1979) 51–58.

[4] K. Devlin and S. Shelah, A weak version of ◇ which follows from $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math. 29 (1978) 239–247.

[5] A. Dow, P. Simon and J. Vaughan, Strong homology and the proper forcing axiom, Proc. Amer. Math. Soc. 106 (1989) 821–828.

[6] F. Drake, Set Theory: An Introduction to Large Cardinals (North-Holland, Amsterdam, 1974).

[7] P. Eklof and A. Mekler, Almost Free Modules: Set Theoretic Methods (North-Holland, Amsterdam, 1990).

[8] D. Fremlin, Consequences of Martin's Axiom (Cambridge University Press, Cambridge, 1984).

[9] F. Hausdorff, Summen von $\aleph_1$ Mengen, Fund. Math. 26 (1936) 241–255.

[10] J. Hocking and G. Young, Topology (Dover, New York, 1961).

[11] C. Jensen, Les foncteurs dérivés de $\varprojlim$ et leurs applications en théorie des modules, Lecture Notes in Mathematics, Vol. 254 (Springer, New York, 1972).

[12] K. Kunen, Set Theory: An Introduction to Independence Proofs (Elsevier, Amsterdam, 1980).

[13] K. Kunen, E. van Douwen and J. van Mill, There can be C^*-embedded dense proper subspaces in $\beta\omega - \omega$, Proc. Amer. Math. Soc. 105 (1989) 462–470.

[14] A. Ostaszewski, On countably compact, perfectly normal spaces, J. London Math. Soc. (2), 14 (1976) 505–516.

[15] M. Rabus, Tight gaps in $\mathscr{P}(\omega)$, Preprint.

[16] M. Scheepers, Gaps in $({}^\omega\omega, \prec)$, Israel Math. Conf. Proc. 6 (1993) 439–561.

[17] S. Todorčević, Partition problems in topology, Contemporary Mathematics, Vol. 84 (Amer. Math. Soc., Providence, RI, 1989).

[18] S. Todorčević, Remarks on MA and CH, Canad. J. Math. 43 (1991) 832–841.

[19] J. Vick, Homology Theory, (Academic Press, New York, 1973).