Commentationes Mathematicae Universitatis Carolinae

14,4 (1973)

CHANGING COFINALITY OF A MEASURABLE CARDINAL

(An alternative proof)

Lev BUKOVSKÝ, Košice

Abstract: Using the method of iterated ultrapower in Set Theory with a measurable cardinal, it is shown that there are model-classes N_ω and its generic extension N such that for a cardinal $\varkappa_\omega$ the following holds: $\varkappa_\omega$ is measurable in N_ω and $\varkappa_\omega$ is a Rowbottom cardinal in N of cofinality ω .

Key words: Set theory, measurable cardinals, Rowbottom cardinal, model-class, generic extension, iterated ultrapower.

AMS: 02K05, 02K35 Ref. Ž. 2.641.5

In the theory of extensions of models of the set theory, there is an open difficult problem: is it possible to change cofinality of a cardinal number not collapsing it? For a measurable cardinal $\varkappa$, K. Prikry in [6] answers this question affirmatively by constructing a generic extension in which $\varkappa$ is cofinal with ω_0 . Moreover, in this extension, $\varkappa$ remains to be a Rowbottom cardinal and all cardinals are preserved. In this note we prove similar result by using the method of iterated ultrapower introduced by H. Gaifman [3] . Namely, we prove the following (for the notations, see the part 1)):

Theorem. Let $\varkappa$ be a measurable cardinal, $\mathcal{U}$ a normal measure on $\varkappa$. Let N_n be the transitive class isomorphic to the n-th iterated ultrapower of the universe by using the ultrafilter $\mathcal{U}$. Let N_{ω_0} be the Gaifman's direct limit of N_n, $n \in \omega_0$ and $N = \bigcap_{n \in \omega_0} N_n$. Then

a) N is a model of ZFC and $N_{\omega_0} \subseteq N$.

b) Cardinals of N are those of N_{ω_0}.

c) $\varkappa_{\omega_0}$ (the measurable in N_{ω_0}) is cofinal with ω_0 in N.

d) $\varkappa_{\omega_0}$ is a Rowbottom cardinal in N.

e) N is a generic extension of N_{ω_0}.

The proof of a) - d) will use only elementary properties of iterated ultrapowers already known to H. Gaifman. For the proof of e), the theorem A of the author's paper [1] will be used.

The relation of our theorem to Prikry's result is clear. By my opinion, the assertion e) is a little surprising. Unfortunately, we cannot explicitly describe the set of forcing conditions for this generic extension.

1. Preliminaries. We remind some notations and well known facts. We follow K. Kunen [5] with some modifications.

Let $\varkappa$ be a measurable cardinal, $\mathcal{U}$ be a normal measure on $\varkappa$. It is well known that there exists an isomorphism θ of the ultrapower ${}^{\varkappa}V/\mathcal{U}$ onto a transitive class N_1 (V is the universal class). If $x \in N_0 = V$, we denote by $\check{x}$ the function defined as $\check{x}(\xi) = x$ for $\xi \in \varkappa$. The mapping $i_{0,1} : N_0 \longrightarrow N_1$ defined by $i_{0,1}(x) = \Theta(\check{x})$ is an elementary embedding. Thus, $\varkappa_1 = i_{0,1}(\varkappa)$ is a measurable cardinal in N_1 and $\mathcal{U}_1 = i_{0,1}(\mathcal{U})$ is a normal measure on $\varkappa_1$ in N_1. One can construct the ultrapower $({}^{\varkappa_1}N_1) \cap N_1 / \mathcal{U}_1$ and the isomorphic transitive class N_2. Going on, we obtain a system $N_0 \supseteq N_1 \supseteq N_2 \supseteq \dots$ of elementarily equivalent models of ZFC + "there is a measurable cardinal" and a system $i_{n,m}$, $n \leq m \in \omega_0$ of elementary embeddings ($i_{m,m}$ is the identity mapping). As H. Gaifman [3] has shown, the direct limit of the system N_n, $i_{n,m}$ is a well-founded model. We denote by N_{ω_0} the corresponding isomorphic transitive class and i_{n,ω_0} will denote the natural (elementary) embedding of N_n into N_{ω_0}. For $\xi \leq \omega_0$, $\mathcal{U}_\xi = i_{0,\xi}(\mathcal{U})$ is a normal measure on $\varkappa_\xi = i_{0,\xi}(\varkappa)$ in N_ξ.

Let us remark that all classes N_ξ, $i_{n,\xi}$ are definable from $\mathcal{U}$.

If M is a transitive class which is a model of ZF (i.e. M is closed under Gödel's operations - see e.g. Gödel [4] - and M is almost universal), then the superscript M over a notation indicates that the corresponding notion is considered in this model.

The famous Łoś's theorem may be expressed as

$$(1) \quad {}^{\varkappa}V/\mathcal{U} \models \varphi(f_1,\dots,f_n) \equiv \{\xi \in \varkappa : \varphi(f_1(\xi),\dots,f_n(\xi))\} \in \mathcal{U} .$$

We shall need the following simple facts (see [3],[5]):

(2) $x \subseteq \xi < \varkappa_n , x \in N_n \longrightarrow i_{n,n+1}(x) = i_{n,\omega_0}(x) = x$.

(3) $x \subseteq N_1$, $card(x) \leq \varkappa \longrightarrow x \in N_1$.

(4) $\varkappa_{\omega_0} = \lim_{n \in \omega_0} \varkappa_n$.

$(N_n)^{N_m}$ (i.e. the class N_n constructed in N_m from $\mathcal{U}_m$) is

(5) equal to N_{n+m}, $(i_{n,\xi})^{N_m} = i_{m+n,m+\xi}$ for $\xi \leq \omega_0$, $n \in \omega_0$ and $(N_{\omega_0})^{N_m} = N_{\omega_0}$, $N^{N_m} = N$.

${}^x y$ denotes the set of all functions defined on x with values in y . $\mathcal{P}(x)$ is the set of all subsets of x . If f, g are functions, we define $f \in \in g \equiv (\forall u \in \mathcal{D}(f))(f(u) \in g(u))$ and $f \subseteq \subseteq g \equiv (\forall u \in \mathcal{D}(f))(f(u) \subseteq g(u))$. We denote $Wd(f)$ = the least cardinal α such that $(\forall u \in \mathcal{D}(f))(card(f(u)) < \alpha)$.

If $M_1 \subseteq M_2$ are two transitive models, $Apr_{M_1,M_2}(\alpha)$ means (see Vopěnka-Hájek [8] and also [1]): for every function $f \in M$, there exists a function $f \in M_2 \cap \mathcal{P}(M_1)$ such that $f \in \in g$ and $Wd^{M_1}(g) \leq \alpha$.

It is well known (compare [8]) that

(6) $Apr_{M_1,M_2}(\alpha)$ implies that every cardinal σ of M_1 , $\sigma \geq \alpha$, is a cardinal in M_2 .

In [1], the following has been proved:

(7) $Apr_{M_1, M_2}(\alpha)$ implies that there is a partially ordered set $P \in M_1$ satisfying α -chain condition and a generic set $G \subseteq P$ such that $M_2 = M_1(G)$. Moreover, $M_2 = M_1(\mathcal{P}(\alpha) \cap M_2)$.

2. Some auxiliary results. We remind the definition of the sets $V(\xi), \xi \in On: V(0) = \emptyset, V(\alpha) = \mathcal{P}(\bigcup_{\xi \in \alpha} V(\xi))$. By the axiom of regularity, $V = \bigcup_{\xi \in On} V(\xi)$. For any transitive model class M, $V(\xi)^M = V(\xi) \cap M$. Thus, especially $V(\xi)^{N_K} = V(\xi) \cap N_K$. By (5), we obtain

$$(V(\xi) \cap N)^{N_K} = V(\xi)^{N_K} \cap N^{N_K} = V(\xi) \cap N_K \cap N = V(\xi) \cap N .$$

Therefore $V(\xi) \cap N \in N_K$. By the definition of N, we have

(8) for every ordinal ξ, $V(\xi) \cap N \in N$.

Let $Fix_m = \{\xi \in On: i_{m,\omega_0}(\xi) = \xi\}$. It is easy to see that Fix_0 is a proper class, $Fix_0 \subseteq Fix_1 \subseteq \ldots \ldots \subseteq Fix_m \ldots$. Evidently $\xi \in Fix_0 \longrightarrow \Theta(\check{\xi}) = \xi$.

Fix_m is a class definable in N_m from $\mathcal{U}_m$, thus $(\forall x \in N_m)(x \cap Fix_m \in N_m)$.

Let us consider a function f such that $\mathcal{D}(f) \subseteq Fix_0$ and $x = W(f) \in N_1$. For $y \in x$, let $h_y \in {}^{\varkappa}V$ be such that $\Theta(h_y) = y$. For $\xi \in \varkappa$, we set

$$g(\xi) = \{\langle \alpha, u \rangle : (\exists y \in x)(u = h_y(\xi) \,\&\, f(\alpha) = y)\} .$$

By (1), one can easily show that $\Theta(g) \supseteq f$, $\Theta(g)$ is a function and $\mathcal{D}(\Theta(g)) \subseteq Fix_1$. If we denote $Ext(f) = \Theta(g) \cap (On \times x)$, we have

(9) for every function f such that $\mathcal{D}(f) \subseteq Fix_0$, $W(f) \in N_1$, there exists a function $Ext(f) \in N_1$ such that $f \subseteq Ext(f)$, $\mathcal{D}(Ext(f)) \subseteq Fix_1$ and $W(f) = W(Ext(f))$.

If $x \in N_1$, $card^{N_1}(x) \leq \varkappa_1$, then there exists a set $y \in N_0$ such that $x \subseteq i_{0,1}(y)$ and $card^{N_0}(y) \leq \varkappa_0$. In fact, by (1), there is a function $h \in {}^{\varkappa_0}V$ such that $(\forall \xi \in \varkappa_0)(card(h(\xi)) \leq \varkappa_0)$ and $\Theta(h) = x$. We set $y = \bigcup_{\xi \in \varkappa_0} h(\xi)$.

This observation may be generalized as follows:

Let $f \in N_1$ be a function, $Wd^{N_1}(f) \leq \varkappa_1^+$. Then there is

(10) a function $g \in N_0$ such that $Wd^{N_0}(g) \leq \varkappa_0^+$ and $f \subseteq\subseteq i_{0,1}(g)$.

Since $f \in N_1$, there is a function $h \in {}^{\varkappa_0}N_0$ such that $\Theta(h) = f$. We may suppose (by (1)) that for every $\xi \in \varkappa_0$, $h(\xi)$ is a function and $Wd^{N_0}(h(\xi)) \leq \varkappa_0^+$. We set

$$g(\eta) = u \equiv (\exists \xi \in \varkappa_0)(\eta \in \mathcal{D}(h(\xi))) \,\&\, u = \bigcup_{\xi \in \varkappa_0} h(\xi)(\eta) .$$

Evidently $Wd^{\aleph_0}(q) \leq \aleph_0^+$. Using (1), one can easily show that $f \subseteq \subseteq i_{0,1}(q)$.

A cardinal δ is said to be Rowbottom cardinal if, for any $\lambda < \delta$ and $f : [\delta]^{<\aleph_0} \longrightarrow \lambda$ there exists a subset $x \subseteq \subseteq \delta$ having power δ such that $f''[x]^{<\aleph_0}$ is countable (compare e.g. Silver [7]). The notion of an M -ultrafilter has been introduced by K. Kunen (see [5], p. 181).

Using intelligently a classical idea of Erdős-Hajnal (see [2], p. 126), it is easy to prove:

(11) Let M be a transitive model of ZFC, $x \subseteq M$, $card(x) \leq \leq \aleph_0 \longrightarrow x \in M$. Let $\alpha = \lim_{n \in \omega_0} \alpha_n$, $\alpha_0 < \alpha_1 < \ldots$.

If there exists an M -ultrafilter on every α_n , then α is a Rowbottom cardinal in M .

This assertion is a trivial generalization of the theorem 1.29 in [6]. Replacing the measures " μ_ξ " in Prikry's proof (see [6], pp.14-15) by " M -ultrafilter on α_n ", we obtain a proof of (11).

3. Proof of the theorem. Since an intersection of transitive and closed (under Gödel's operations) classes is such a class, by (8) N is also almost universal, we have that N is a transitive model of ZF.

For to prove $N \models AC$, it suffices, for any $x \in N$, to find a function $f \in N$ such that $\mathcal{D}(f) \subseteq On$ and $W(f) = x$.

Thus, let $x \in N$ and let $f \in N_0$ be a function,

$\mathcal{D}(f) \subseteq Fix_0$ and $W(f) = x$. Let $F(f) = \{f_n ; n \in \omega_0\}$, where $f_0 = f$ and $f_{n+1} = Ext^{N_n}(f_n)$ (see (9)). We set $f_{\omega_0} = \bigcup_{n \in \omega_0} f_n$. By (9), $f_n \in N_n$, $\mathcal{D}(f_n) \subseteq On$ and $W(f_n) = x$. Since $F^{N_n}(f_n) = \{f_k : n \leq k \in \omega_0\}$ and $f_{\omega_0} = \bigcup_{n \leq k \in \omega_0} f_k$, we have $f_{\omega_0} \in N_n$ and therefore, $f_{\omega_0} \in N$. Thus the axiom of choice AC holds true in N.

Now, we show that

(12) $\quad Apr_{N_{\omega_0}, N}(\varkappa^+_{\omega_0})$ holds true.

Let $f \in N$ be a function. We denote $f_n = \{\langle x, y\rangle : : f(i_{n,\omega_0}(x)) = i_{n,\omega_0}(y)\}$. By the definition of the direct limit N_{ω_0}, we have $f = \bigcup_{n \in \omega_0} i_{n,\omega_0}(f_n)$ and $f_n \in N_n$. We set $h_n(x) = \{f_n(x)\}$ for $x \in \mathcal{D}(f_n)$. For every $n \in \omega_0$, by repeated applications of (10), there exists a function $g_n \in \in N_0$ such that $Wd^{N_0}(g_n) \leq \varkappa^+_0$ and $h_n \subseteq \subseteq i_{0,n}(g_n)$. Thus $f_n \in \in i_{0,n}(g_n)$.

We set

$$h(x) = u \equiv (\exists n)(x \in \mathcal{D}(g_n)) \,\&\, u = \bigcup_{n \in \omega_0} g_n(x) .$$

Evidently $Wd^{N_0}(h) \leq \varkappa^+_0$ and $g_n \subseteq \subseteq h$. Since i_{0,ω_0} is an elementary embedding, we have $Wd^{N_{\omega_0}}(i_{0,\omega_0}(h)) \leq \varkappa^+_{\omega_0}$. By the construction of the function h, one easily obtains $f \in \in i_{0,\omega_0}(h)$. Since

$i_{0,\omega_0}(k) \in N_{\omega_0}$, the assertion (12) follows.

Now, the part b) of the theorem follows by (3),(6) and (12). The part e) follows by (7) and (12).

Let $a = \{ \varkappa_n ; n \in \omega_0 \}$. Evidently $a^{N_k} = \{ a_n ; k \leq n \in \omega_0 \} \in N_k$. Since $a = \{ \varkappa_n ; n < k \} \cup a^{N_k}$, we have $a \in N_k$. Thus also

(13) $$\{ \varkappa_n ; n \in \omega_0 \} \in N .$$

The part c) of the theorem follows by (4) and (13). Since $\mathcal{U}_n$ is an N_n-ultrafilter on $\varkappa_n$, d) follows by (11), (3) and (13).

Finally, let us remark that by (7) and (12), $N = N_{\omega_0}(x)$, where $x = \mathcal{P}(\varkappa^+_{\omega_0}) \cap N$. The author was not able to prove or to disprove the following conjectures:

(14) $N = N_{\omega_0}(x)$, where $x = N \cap \mathcal{P}(\varkappa_{\omega_0})$,

(15) $N = N_{\omega_0}(x)$, where $x = N \cap {}^{\omega_0}\varkappa_{\omega_0}$.

Neither we know the relation of the generic extension N of N_{ω_0} to that constructed in [6], p. 24.

R e f e r e n c e s

[1] L. BUKOVSKÝ: Characterization of generic extensions of models of set theory, to appear.

[2] P. ERDÖS, A. HAJNAL: On the structure of set-mappings, Acta Math.Acad.Sci.Hungar.IX(1958),111-131.

[3] H. GAIFMAN: Self-extending models,measurable cardinals

and the constructible universe, mimeographed.

[4] K. GÖDEL: The consistency of the axiom of choice and of the generalized continuum hypothesis, Princeton Univ.Press, 1966.

[5] K. KUNEN: Some applications of iterated ultrapowers in set theory, Ann.Math.Logic 1(1970),179-227,

[6] K. PRIKRY: Changing measurable into accessible cardinals, Dissertationes Math.,Warszawa 1970.

[7] J. SILVER: Some applications of model theory in set theory, Ann.Math.Logic 3(1971),45-110.

[8] P. VOPĚNKA, P. HÁJEK: The theory of semisets,Academia, Praha 1972.

Prírodovedecká fakulta UPJŠ

Košice

Československo

(Oblatum 12.10.1972)