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MINIMAL COLLAPSING EXTENSIONS OF MODELS OF ZFC

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We present some results concerning extensions of models of ZFC in which cofinalities of cardinals are changed and/or cardinals are collapsed, in particular on minimal such extensions. Our main tools are perfect tree forcing PF(S) and Namba forcing Nm(S). We prove that if $N \supseteq M$ is an extension such that (i) $M \models \kappa = \lambda^+ > \aleph_1$, (ii) ${}^{\omega}\lambda \cap N \subseteq M$ and (iii) $N = M[f]$ for some cofinal $f: \omega_0 \to \kappa$, then $N \supseteq M$ is $\mathrm{cf}(\kappa) = \omega_0$-minimal. On the other hand Namba forcing Nm(S) where S is a normal ultrafilter on a measurable cardinal κ produces an extension satisfying (iii) and (ii) for every $\lambda < \kappa$, which is not $\mathrm{cf}(\kappa) = \omega_0$-minimal.

We show that if S is an $\aleph_2$-complete splitting criterion on κ then Pf(S) collapses κ^+ to $\aleph_1$ (assuming GCH). Moreover, we prove, under some reasonable assumptions, that every extension changing the cofinality of a successor cardinal κ must collapse κ^+.

Using these results and results on trees from Sections 2, 3 we construct, assuming e.g. GCH, for every regular uncountable κ a $|\kappa| = \aleph_0$-minimal extension, a $\mathrm{cf}(\kappa) = \omega_0$-minimal extension and a $|\kappa^+| = \aleph_1$-minimal extension.

0. Introduction

More than 20 years ago the question arose whether it is possible to find a generic extension of a given model of ZFC in which cofinalities of some cardinals are changed but no cardinals are collapsed. The answer is well known. Prikry [20] defined a forcing notion by which the cofinality of a measurable cardinal is changed to ω_0 and no cardinals are collapsed. Later, in 1966, Vopěnka posed the question whether one can change the cofinality of ω_2 to ω_0 without collapsing $\aleph_1$. An affirmative answer was given by the first author in [3], [5]. Independently Namba [19], solving a problem concerning Boolean algebras, constructed a forcing notion with similar properties as the one in [3]. After Jensen proved his important Covering Lemma [8], both forcing notions turned out to be examples showing that the Covering Lemma is best possible. Shelah [23], Gitik [11] and others generalized these forcing notions and used them in various constructions. However, several problems concerning these forcing notions remainded open.

This paper arose from two independent results of the authors. The first author proved Theorem 1.1[1] solving the problem of minimality of Namba forcing. The second author proved Theorem 5.1, which answers the question about collapsing of cardinals for the perfect tree forcing of [3]. The paper is organized as follows: Section 1 contains general results on minimal extensions of models of ZFC. In Section 2 we investigate general properties of families of trees from the forcing point of view. We tried to develop the methods used in [3], [5] and [19]. However, one can see that they are closely related to those of Shelah [23]. Section 3 introduces some generalizations of the forcing notions of [3] and [19] and studies their properties. The goal of Section 4 is the nonminimality of Namba forcing for a measurable cardinal. Section 5 is devoted to the proof of the above mentioned Theorem 5.1. Finally, in Section 6 we present some generic extensions of models of ZFC, where cofinalities of cardinals are changed or cardinals are collapsed and they are minimal with respect to these properties. Some open problems are collected in Section 7.

Our set-theoretical terminology is fairly standard and may be found e.g. in [13]. $\kappa, \lambda, \mu, \ldots$ denote infinite cardinal numbers, $\xi, \zeta, \ldots$ are ordinals. ${}^{x}y$ is the set of all functions defined on x with values in y. $\lambda^{\underline{\kappa}}$ is the weak power $\sum_{\mu<\kappa} \lambda^{\mu}$. An ideal J or a filter F is always proper (i.e. $\bigcup J \notin J$ and $\emptyset \notin F$).

If P is a (separative) partially ordered set, then r.o.(P) is the unique complete Boolean algebra which extends P as a partially ordered set and such that P is a dense subset of r.o.(P).

Let P, Q be partially ordered sets. We say that a function ϕ from P into Q is normal if it is order preserving, its range is dense in Q and for every $p \in P$ the image of $\{p' \in P : p' \leq p\}$ under ϕ is dense below $\phi(p)$ in Q (see e.g. [12]).

The following lemma is well known and we shall state it without proof.

Lemma 0.1. *Let P, Q be separative partially ordered sets and let $\phi : P \to Q$ be normal. Then the mapping $\bar{\phi} : \mathrm{r.o.}(Q) \to \mathrm{r.o.}(P)$ defined by $\bar{\phi}(b) = \bigvee \{p \in P : \phi(p) \leq b\}$ is a complete embedding of Boolean algebras and the mapping $\tilde{\phi} : \mathrm{r.o.}(P) \to \mathrm{r.o.}(Q)$ defined by $\tilde{\phi}(a) = \bigwedge \{b \in \mathrm{r.o.}(Q) : \bar{\phi}(b) \geq a\}$ extends ϕ.*

1. Minimality of extensions

We shall always consider transitive models of ZFC. If M, N are models then we say that $N \supseteq M$ is an *extension of models* if $M \subseteq N$ and $\mathrm{Ord} \cap M = \mathrm{Ord} \cap N$.

Let $\phi(x_1, \ldots, x_n)$ be a formula of the language of set theory. Let $N \supseteq M$ be an extension of models of ZFC and let $a_1, \ldots, a_n \in M$. The extension $N \supseteq M$ is called a *$\phi(a_1, \ldots, a_n)$-extension* if

(1.1) $\quad N \models \phi(a_1, \ldots, a_n)$

A $\phi(a_1, \ldots, a_n)$-extension is called a *$\phi(a_1, \ldots, a_n)$-minimal extension* if

[1] This theorem was presented to the Oberwolfach Set Theory Meeting in January 1985.

moreover:

(1.2) for every model K, $M \subseteq K \subseteq N$ and $K \models \phi(a_1, \ldots, a_n)$ imply $K = N$.

A forcing notion P is $\phi(a_1, \ldots, a_n)$-minimal if for every M-generic filter G on P the generic extension $M[G] \supseteq M$ is $\phi(a_1, \ldots, a_n)$-minimal.

Historically the first minimal extension was constructed by Sacks [21]: let P_1 denote the set of all perfect subsets of the unit interval ordered by inclusion; constructed in a model M of ZFC. If G is an M-generic filter on P_1 then $M[G] = M[s]$ where s is a real (Sacks real). Sacks showed that for any real $a \in M[s]$, either $a \in M$ or $M[a] = M[s]$. So the forcing notion P_1 is $\phi(\mathcal{P}(\omega_0) \cap M)$-minimal, where $\phi(x)$ denotes the formula $(\exists y)\ (y \subseteq \omega_0\ \&\ y \notin x)$. In [3] (and [5]) the first author found a forcing notion P_2 such that any generic extension N of a model M obtained by the forcing P_2 is $\phi(\omega_2^M)$-minimal, where $\phi(x)$ denotes the formula "x is an ordinal cofinal with ω_0". We shall simply say that the forcing notion P_2 is $\mathrm{cf}(\omega_2^M) = \omega_0$-minimal. Prikry and Abraham [1] found $|\aleph_1^M| = \aleph_0$-minimal extensions $N \supseteq M$, i.e. the cardinal $\aleph_1$ of M is countable in N and for any model K, $M \subseteq K \subseteq N$, $K \neq N$, $\aleph_1$ of M is uncountable in K. Assuming the axiom of constructibility, Sacks [22] constructed a $|\kappa| = \aleph_0$-minimal extension for every regular cardinal κ.

In our paper we shall investigate mainly $\phi(\kappa)$-minimal extensions where $\phi(x)$ is one of the formulas $|x| = \aleph_0$, $|x| = \aleph_1$, $\mathrm{cf}(x) = \omega_0$. For example, we shall construct an ($|\aleph_3^M| = \aleph_1$-minimal) extension $N \supseteq M$ which collapses $\aleph_3$ of M to $\aleph_1^M = \aleph_1^N$ and it is minimal with this property. Our construction of a $|\kappa| = \aleph_1$-minimal extension is based on a $\mathrm{cf}(\lambda) = \omega_0$-minimal extension. Since by Jensen's Covering Lemma [8] the existence of a $\mathrm{cf}(\lambda) = \omega_1$-, $\aleph_2^M = \aleph_2$-extension for $\lambda > \aleph_2$ implies the existence of $0^\#$ in M, our method cannot be immediately generalized to construct a $|\lambda| = \aleph_\alpha$-minimal extension for $\alpha > 1$.

We start with general results on $|\lambda| = \kappa$-minimal and $\mathrm{cf}(\lambda) = \kappa$-minimal extensions.

Theorem 1.1. *Let $N \supseteq M$ be an extension and let μ, λ, κ be cardinal numbers in M such that*

(i) $M \models \kappa = \lambda^+\ \&\ \mu$ *is regular* $\&\ \mu < \lambda$,

(ii) ${}^\mu\lambda \cap N \subseteq M$,

(iii) *there exists an $f \in {}^\mu\kappa \cap N$, f unbounded in κ such that $N = M[f]$.*

Then $N \supseteq M$ is a $\mathrm{cf}(\kappa) = \mu$*-minimal extension.*

The key element of the proof is the following simple result.

Lemma 1.2. *Let M, N, f be as in the theorem. If $g \in {}^\mu\kappa \cap N$ is unbounded in κ then there exists a strictly increasing function $h \in M[g]$ such that $f(\xi) < h(\xi)$ for each $\xi \in \mu$.*

Proof. By induction on $\xi < \mu$, we define a function $d \in {}^{\mu}\mu$ as follows:

$$d(\xi) = \min\{\zeta \in \mu : g(\zeta) > f(\xi) \text{ and } g(\zeta) > g(\eta) \text{ for each } \eta < \xi\}.$$

Since $\mu < \lambda$, by (ii) and (iii) we have $\sup\{g(\eta) : \eta < \xi\} < \kappa$ whenever $\xi < \mu$, and so there exists a $\zeta < \mu$ such that $g(\zeta)$ is greater than $\sup\{g(\eta) : \eta < \xi\}$. Therefore the function d is well defined. Using (ii) again, we have $d \in M$. Now it suffices to set $h(\xi) = g(d(\xi))$ for $\xi \in \mu$. $\square$

Proof of Theorem 1.1. Assume $M \subseteq K \subseteq N$ and $K \models \mathrm{cf}(\kappa) = \mu$. We show that $f \in K$.

Let $g \in {}^{\mu}\kappa \cap K$ be unbounded in κ. By the lemma there is a strictly increasing function $h \in M[g] \subseteq K$ such that $f(\xi) < h(\xi)$ whenever $\xi \in \mu$. Without loss of generality we may assume that $f(\xi) > \lambda$ for each $\xi \in \mu$.

Now, in M, for each $\eta \in \kappa - \lambda$ fix a bijection p_η of λ onto η. In N, define a function k by letting for each $\xi \in \mu$

$$k(\xi) = p_{h(\xi)}^{-1}(f(\xi)).$$

Clearly $k \in {}^{\mu}\lambda$ hence by (ii), $k \in M$. Since $f(\xi) = p_{h(\xi)}(k(\xi))$ we obtain

$$f \in M[h] \subseteq K. \qquad \square$$

Specifying the parameters we obtain

Corollary 1.3. *Let $M \subseteq N$ be an extension such that $\mathcal{P}(\omega_0) \cap N \subseteq M$ and $N = M[f]$, where f is an unbounded function from ω_0 into ω_2^M. Then $N \supseteq M$ is a $\mathrm{cf}(\omega_2^M) = \omega_0$-minimal extension.*

As we have already mentioned the first author showed that the forcing introduced in [3] is $\mathrm{cf}(\omega_2^M) = \omega_0$-minimal. By Corollary 1.3 also the forcing introduced by Namba [19] is $\mathrm{cf}(\omega_2^M) = \omega_0$-minimal.

It was a natural open question whether the forcing notions of [3] and [19] do collapse $\aleph_3$ or not (see [5, p. 48], compare [23, p. 368]). We are able to answer this question affirmatively.

Theorem 1.4. *Let $N \supseteq M$ be an extension, let μ, λ, κ be cardinals in M such that*

(i) $M \models \lambda = 2^{\kappa}$, (ii) $N \models \mathrm{cf}(\kappa) = \mu$.

Then

$$N \models |{}^{\kappa}\kappa \cap M| \leq |\lambda|^{\mu}.$$

Proof. Let B be the set of all nondecreasing functions from κ into κ in M. Then

$$M \models |B| = \kappa^{\kappa}.$$

Let $f \in {}^{\mu}\kappa \cap N$ be a strictly increasing function unbounded in κ. For ξ, $\eta < \mu$

we set

$$B_{\xi,\eta} = \{g \in B : g(f(\xi)) < f(\eta)\}.$$

Clearly

$$B = \bigcap_{\xi\in\mu} \bigcup_{\eta\in\mu} B_{\xi,\eta}.$$

Hence

$$B = \bigcup_{\phi\in{}^{\mu}\mu} \bigcap_{\xi\in\mu} B_{\xi,\phi(\xi)}.$$

For a given $\phi \in {}^{\mu}\mu$ consider the set $A_{\phi,\xi}$ consisting of all functions $h \in M$ with domain $f(\xi)+1$ and with values in $f(\phi(\xi))$. Since $A_{\phi,\xi} \in M$ and $f(\xi)+1$, $f(\phi(\xi)) < \kappa$, by (i) we get

$$M \models |A_{\phi,\xi}| \leq \lambda.$$

On the other hand we can easily see that, in N

$$\left|\bigcap_{\xi\in\mu} B_{\xi,\phi(\xi)}\right| \leq \left|\prod_{\xi\in\mu} A_{\phi,\xi}\right| \leq |\lambda|^{\mu}.$$

Therefore

$$N \models |M \cap {}^{\kappa}\kappa| = |B| \leq 2^{\mu} \cdot |\lambda|^{\mu} = |\lambda|^{\mu}. \qquad \square$$

Again specifying the parameters we have the following

Corollary 1.5. *Let $N \supseteq M$ be a* $\mathrm{cf}(\omega_2^M) = \omega_0$*-extension such that*

(i) $M \models 2^{\aleph_1} = \aleph_2$, (ii) $N \models 2^{\aleph_0} = \aleph_1$.

Then

$$N \models |\aleph_3^M| \leq \aleph_1.$$

Remark. Komjath called our attention to Lemma 4.9 of Shelah [23, p. 440], which actually solves the problem of collapsing $\aleph_3$ in the extensions considered. However, since our Theorem 1.4 is not covered by Shelah's lemma and our proof is different from the one of Shelah we have included it in the paper.

As a consequence of both theorems we obtain

Theorem 1.6. *Let $N \supseteq M$ be an extension such that*

(i) $M \models \mathrm{GCH}$, (ii) $N \models \mathrm{GCH}$, (iii) $\mathcal{P}(\omega_0) \cap N \subseteq M$,

(iv) *there is an $f \in {}^{\omega_0}\omega_2^M$ unbounded in ω_2^M such that $N = M[f]$.*

Then the extension $N \supseteq M$ is both $|\aleph_3^M| = \aleph_1$-minimal and $|\aleph_2^M| = \aleph_1$-minimal.

Proof. By Corollary 1.5, $N \supseteq M$ is a $|\aleph_3^M| = \aleph_1$-extension.

Now, let $M \subseteq K \subseteq N$, $K \models |\aleph_3^M| = \aleph_1$. Then also $K \models |\aleph_2^M| = \aleph_1$. Therefore there exists a cardinal $\lambda \leq \aleph_1$ such that $K \models \mathrm{cf}(\omega_2^M) = \lambda$. By (ii) and (iv), $\lambda = \omega_0$. Using Corollary 1.3 we get $K = N$. $\square$

The idea of the proof above will be used very often in Section 6 without any comment. Closely related to Theorem 1.4 is the following result which we shall need too in Section 6.

Theorem 1.7. *Let $N \supseteq M$ be an extension, let $\lambda < \kappa$ be cardinals in M, $M \models \mathrm{cf}(\kappa) = \mu$, and assume $N \models |{}^{\mu}\lambda| = |\lambda|$. Then the successor of λ in N is not the successor of κ in M.*

Proof. It is easily seen that $M \models \kappa^{\mu} \geq \kappa^{+}$. If $\kappa^{+M} = |\lambda|^{+N}$ then $N \models |\kappa| \leq |\lambda|$ and therefore $N \models |{}^{\mu}\kappa \cap M| \leq |{}^{\mu}\lambda| = |\lambda| < |\lambda|^{+}$. $\square$

Corollary 1.8. *If $N \supseteq M$ is an extension, $N \models 2^{\aleph_0} \leq \aleph_n$, $n \in \omega$, then $\aleph^M_{\omega+1}$ is not $\aleph^N_{n+1}$.*

Proof. In Theorem 1.7 take $\kappa = \aleph^M_{\omega}$ and $\lambda = \aleph^N_n$. $\square$

Remark. It is easy to construct an extension $M \subseteq N$ such that $N \models \mathrm{GCH}$ and $\aleph^M_{\omega+1} = \aleph^N_1$.

The following result is in a sense a converse to Theorem 1.6 and partially explains why the presented methods (i.e. methods for constructing minimal extensions for changing cofinalities of cardinals) should probably be used in order to obtain minimal collapsing extensions.

Theorem 1.9. *Let $N \supseteq M$ be an extension, let μ, λ, κ be regular cardinals in M and $\omega_0 < \mu < \lambda < \kappa$. Assume that*
(i) *$M \models \lambda^{\underline{\mu}} = \lambda$ & $2^{\lambda} \leq \kappa$,*
(ii) *$N \supseteq M$ is a $|\kappa| = \mu$-extension,*
(iii) *$N \supseteq M$ is a $|\lambda| = \mu$-minimal extension,*
(iv) *${}^{\xi}\mu \cap N \subseteq M$ for each $\xi < \mu$.*
Then there exists a cardinal δ in M, $\mu < \delta \leq \lambda$, such that

$$N \models \mathrm{cf}(\delta) < \mu.$$

Proof. We first claim that there is a $\xi < \mu$ such that ${}^{\xi}\lambda \cap N \nsubseteq M$. For suppose not and consider the forcing notion

$$\mathrm{Col}(\mu, \lambda) = \bigcup_{\xi<\mu} {}^{\xi}\lambda.$$

By our assumption

$$\mathrm{Col}^M(\mu, \lambda) = \mathrm{Col}^N(\mu, \lambda).$$

The set

$$\mathcal{D} = \{D \subseteq \mathrm{Col}^M(\mu, \lambda) : D \in M \text{ \& } D \text{ is dense}\}$$

is of cardinality 2^{λ} in M. Therefore

$$N \models |\mathcal{D}| \leq \mu.$$

Since $\mathrm{Col}^N(\mu, \lambda)$ is μ-closed in N one can easily construct, by induction in N, a $\mathscr{D}$-generic filter G on $\mathrm{Col}^N(\mu, \lambda)$. Hence G is an M-generic filter on $\mathrm{Col}^M(\mu, \lambda)$. Then

$$M \subseteq M[G] \subseteq N,$$

and we have

$$M[G] \models |\lambda| = \mu \ \&\ \kappa \geq |\lambda^{+M}| > \mu,$$

which yields a contradiction with (ii) and (iii).

Now, let ξ be the least ordinal such that ${}^{\xi}\lambda \cap N \not\subseteq M$. Let δ be the least ordinal such that ${}^{\xi}\delta \cap N \not\subseteq M$. Then $N \models \mathrm{cf}(\delta) \leq \xi < \mu$ and by (iv), $\delta > \mu$. But now we are done because δ is a cardinal in M. □

Corollary 1.10. *Let $N \supseteq M$ be an extension such that*

(i) $M \models \mathrm{GCH}$, (ii) $\mathscr{P}(\omega_0) \cap N \subseteq M$,

(iii) $M \subseteq N$ *is* $|\aleph_2^M| = \aleph_1$*-minimal,* (iv) $N \models |\aleph_3^M| = \aleph_1$.

Then

$$N \models \mathrm{cf}(\omega_2^M) = \omega_0.$$

We conclude this section with a lemma which we shall need later. It was essentially proved by Magidor and Shelah (see [23], p. 367]). First, let us introduce a new notion. Let $N \supseteq M$ be an extension and let $\lambda < \kappa$ be regular cardinals in M. A function $f \in {}^{\lambda}\kappa \cap N$ is called *fast growing* if f is unbounded in κ and for each $F \in {}^{\kappa}\kappa \cap M$ there exists a $\xi_0 < \lambda$ such that $f(\xi + 1) > F(f(\xi))$ whenever $\xi > \xi_0$.

Lemma 1.11. *Let $N \supseteq M$ be an extension and let $\lambda < \kappa$ be regular cardinals in M. Then either*

(a) *there is no fast growing function from λ into κ in N, or*

(b) *for each $f \in {}^{\lambda}\kappa \cap N$ unbounded in κ, there exists an increasing $\phi \in {}^{\lambda}\lambda$ such that $f \circ \phi$ is fast growing.*

Proof. Assume (a) is false, i.e. there exists an increasing fast growing function $h \in {}^{\lambda}\kappa \cap N$. Let $f \in {}^{\lambda}\kappa \cap N$ be unbounded. (We may assume that f is increasing.) By induction on $\xi < \lambda$, we define two functions

$$\phi(\xi) = \min\{\eta < \lambda : f(\eta) > h(\psi(\zeta) + 1) \text{ for each } \zeta < \xi\},$$

$$\psi(\xi) = \min\{\eta < \lambda : h(\eta) > f(\phi(\xi)) \ \&\ \eta > \psi(\zeta) \text{ for } \zeta < \xi\}.$$

Given $F \in {}^{\kappa}\kappa \cap M$ increasing, for sufficiently large ξ we have

$$F(f(\phi(\xi))) < F(h(\psi(\xi))) < h(\psi(\xi) + 1) < f(\phi(\xi + 1)). \qquad \square$$

2. Trees

Our notion of a tree is closely related to Shelah's notion of a tagged tree (see [23, p. 359]). The main difference is in keeping the splitting criterion constant for all trees (actually in many applications Shelah did the same).

The letters s, t, u, v will denote finite sequences of ordinals; $s \leqslant t$ means that s is an initial segment of t. If s is a sequence with length(s) $= n$, ξ an ordinal then $s^\frown\xi$ denotes the sequence of length $n+1$ extending s whose $n+1$-th term is ξ.

Let $\boldsymbol{T}$ be a nonempty tree of finite sequences of ordinal numbers partially ordered by $\leqslant$. Let $\boldsymbol{S}$ and $\boldsymbol{R}$ be functions defined on $\boldsymbol{T}$ such that

(2.1) for each $v \in \boldsymbol{T}$, $\boldsymbol{R}(v)$ is a cardinal,

(2.2) for each $v \in \boldsymbol{T}$, $\boldsymbol{S}(v) \subseteq \mathcal{P}(\boldsymbol{R}(v))$,

(2.3) for each $v \in \boldsymbol{T}$, for every $x \in \boldsymbol{S}(v)$, $|x| \geqslant 2$.

If moreover

(2.4) $\boldsymbol{T} = \bigcup_{n \in \omega} \boldsymbol{T}_n$, where

$$\boldsymbol{T}_0 = \{\emptyset\} \quad \text{and} \quad \boldsymbol{T}_{n+1} = \{v^\frown\xi : v \in \boldsymbol{T}_n,\ \xi \in \boldsymbol{R}(v)\},$$

then the ordered triple $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ is called a *basic tree*. We call $\boldsymbol{R}$ the *ramification* and $\boldsymbol{S}$ the *splitting criterion* of the basic tree.

Note that if e.g. $\boldsymbol{R}(v) = \kappa$ for all $v \in \boldsymbol{T}$ then

$$\boldsymbol{T} = \bigcup_{n \in \omega} {}^n\kappa = {}^{<\omega}\kappa.$$

In the sequel we shall always consider one fixed basic tree $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ at a time. To simplify the notation, we shall use the symbol $\boldsymbol{T}$ instead of $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ for a basic tree.

For a set $T \subseteq \boldsymbol{T}$, $v \in T$ we put $\mathrm{Succ}(v, T) = \{\xi \in \boldsymbol{R}(v) : v^\frown\xi \in T\}$ and $T(v) = \{u \in T : u \geqslant v \vee u \leqslant v\}$. An element $v \in T$ is called a *splitting point* of T if $\mathrm{Succ}(v, T) \in \boldsymbol{S}(v)$.

A tree T is a subset of the basic tree $\boldsymbol{T}$ satisfying

(i) $(\forall v \in T)(\forall n \in \omega)\, v \restriction n \in T$,

(ii) $(\forall v \in T)(\exists u \in T)\, u > v$.

A tree T is said to be *perfect* if

(2.5) for each $u \in T$ there exists a splitting point $v \in T$ with $v > u$.

The *nth splitting level* of T is defined as follows:

$$\mathrm{SL}_n(T) = \{v \in T : v \text{ is a splitting point of } T \text{ and } |\{u < v : u \text{ is a splitting point of } T\}| = n\}.$$

Now, let us introduce some orders among trees as follows:

(a) $T_1 \leq T_2$ iff $T_1 \subseteq T_2$;

(b) $T_1 \leq^* T_2$ iff $T_1 \leq T_2$ and each $v \in T_1$ which is a splitting point of T_2 is also a splitting point of T_1;

and for each $n \in \omega$,

(c) $T_1 \leq^n T_2$ iff $T_1 \leq T_2$ and $\mathrm{SL}_k(T_1) = \mathrm{SL}_\kappa(T_2)$ for each $k \leq n$.

Let us remark that for every tree T there exists a subtree $S \leq^* T$ satisfying the following condition

(2.6) for every $v \in S$, either $\mathrm{Succ}(v, S) \in \boldsymbol{S}(v)$ or $|\mathrm{Succ}(v, S)| = 1$.

If a tree T satisfies condition (2.6) then the unique element (if any) s_T of $\mathrm{SL}_0(T)$ is called the *trunk* of T.

A subset A of ω is called a *splitting set* of a tree T, denoted by $A = \mathrm{SS}(T)$, if for each $v \in T$, v is a splitting point of T if and only if $\mathrm{length}(v) \in A$. A tree having an infinite splitting set is called *regular*.

Usually the splitting criterion $\boldsymbol{S}$ is defined in such a way that $\boldsymbol{S}(v) = \mathcal{P}(\boldsymbol{R}(v)) - J$ for an ideal J over $\boldsymbol{R}(v)$. This explains the following definition. The set $\boldsymbol{S}(v)$ is *λ-complete* if for each $A \subseteq \mathcal{P}(\boldsymbol{R}(v)) - \boldsymbol{S}(v)$ of size less than λ we have $\bigcup A \notin \boldsymbol{S}(v)$. The splitting criterion $\boldsymbol{S}$ is λ-complete if $\boldsymbol{S}(v)$ is λ-complete for each $v \in \boldsymbol{T}$.

The set of branches $\mathrm{Br}(T)$ of a tree T is the set of all infinite sequences of ordinals f for which $f \restriction n \in T$ for each $n \in \omega$. Consider the space

$$X = \prod_{n \in \omega} \bigcup_{v \in \boldsymbol{T}_n} \boldsymbol{R}(v)$$

equipped with the product topology, where each $\bigcup_{v \in \boldsymbol{T}_n} \boldsymbol{R}(v)$ carries the discrete topology. The set of all branches of the basic tree $\mathrm{Br}(\boldsymbol{T})$ is a closed subset of X; it is obvious that a nonempty set $A \subseteq \mathrm{Br}(\boldsymbol{T})$ is closed if and only if $A = \mathrm{Br}(T)$ for some tree T.

We shall need the following.

Theorem 2.1. *Suppose that the splitting criterion is μ-complete and let $\lambda < \mu$. Let T be a tree and let*

$$\mathrm{Br}(T) = \bigcup_{\xi < \lambda} C_\xi$$

with every C_ξ a Borel subset of $\mathrm{Br}(T)$. *Then there exist a tree $S \leq^* T$ and an ordinal $\xi_0 < \lambda$ such that*

$$\mathrm{Br}(S) \leq C_{\xi_0}.$$

The theorem was proved by Shelah (see [23, pp. 362–363]) and is based on the determinacy of Borel games, which was proved by Martin [17]. For some special types of trees and when all C_ξ's are closed, the results was proved by the first author [5, p. 43] and Namba [19]. Let us remark that we shall only need the case when all C_ξ's are closed.

From this theorem we can obtain Balcar's theorem [5, p. 47].

Theorem 2.2. *If the splitting criterion is $(2^{\aleph_0})^+$-complete, then every perfect tree T contains a regular subtree $S \leqslant^* T$.*

Proof. Without loss of generality we may assume that T satisfies (2.6). For each $A \in [\omega]^\omega$ let

$$E(A) = \{f \in \mathrm{Br}(T) : A = \{n \in \omega : f \restriction n \text{ is a splitting point in } T\}\}.$$

The set $E(A)$ is closed in $\mathrm{Br}(T)$ and $\mathrm{Br}(T) = \bigcup \{E(A) : A \in [\omega]^\omega\}$. As there are only $2^{\aleph_0}$ subsets of ω and the splitting criterion is $(2^{\aleph_0})^+$-complete, by Theorem 2.1 there exist an $A \in [\omega]^\omega$ and a tree $S \leqslant^* T$ such that $\mathrm{Br}(S) \subseteq E(A)$. Clearly S is regular. □

Since we shall use sets of subtrees of a given basic tree as forcing notions, it is useful to consider first some properties of such sets. A set $\mathscr{K}$ of trees is called *hereditary* if $T \in \mathscr{K}$, $S \leqslant^* T$ implies $S \in \mathscr{K}$. A set $\mathscr{K}$ is said to be *fusion closed* if the following condition is satisfied:

(2.7) if for each $n \in \omega$ there is a tree $S \in \mathscr{K}$ such that $T \leqslant^n S$ then $T \in \mathscr{K}$.

A set of trees $\mathscr{K}$ is said to be a *family of trees* if

(2.8) there exist a set $\mathscr{K}_0 \subseteq \mathscr{K}$ and an increasing function $r \in {}^\omega\omega$ such that

(a) $\mathscr{K}_0$ is hereditary and dense in $(\mathscr{K}, \leqslant)$,
and for each $n \in \omega$, each tree $T \in \mathscr{K}$,

(b) if $v \in \mathrm{SL}_{r(n)}(T)$ then there exists a tree $S \in \mathscr{K}$ such that $S \leqslant^0 T(v)$,

(c) if $W_v \leqslant^0 T(v)$, $W_v \in \mathscr{K}$ for each $v \in \mathrm{SL}_{r(n)}(T)$ then there exist trees $Z_v \leqslant^0 W_v$ such that

$$\bigcup \{Z_v : v \in \mathrm{SL}_{r(n)}(T)\} \in \mathscr{K}.$$

Let us give some examples. The set $A(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ of all perfect subtrees of a given basic tree $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ is a family of trees (with $\mathscr{K}_0 = \mathscr{K} = A(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ and $r(n) = n$).

We shall also consider a special type of perfect trees: a tree $T \subseteq \boldsymbol{T}$ is said to be an $\boldsymbol{S}$-*Namba* tree if

(2.9) there exists a trunk $s_T \in T$ and each $v \in T$, $v \geqslant s_T$ is a splitting point of T.

The set $\mathrm{Nm}(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ (or simply $\mathrm{Nm}(\boldsymbol{S})$) of all $\boldsymbol{S}$-Namba trees is a family of trees (again $\mathscr{K}_0 = \mathscr{K} = \mathrm{Nm}(\boldsymbol{S})$, $r(n) = n$).

Let $T = {}^{<\omega}\omega$, $R(v) = \omega$, $S(v) = [\omega]^{\omega}$ for each $v \in {}^{<\omega}\omega$. Then $A(T, R, S)$ is the rational perfect set forcing of Miller [18] and $\mathrm{Nm}(T, R, S)$ is Laver's forcing [15].

If $T = {}^{<\omega}\omega_2$ and for each $v \in T$, $R(v) = \omega_2$ and $S(v) = [\omega_2]^{\omega_2}$, then $A(T, R, S)$ is the forcing of [3] and $\mathrm{Nm}(T, R, S)$ is Namba forcing [19].

If $T = {}^{<\omega}\omega_1$ and for each $v \in T$, $R(v) = \omega_1$, $S(v) = [\omega_1]^{\omega_1}$, then $A(T, R, S)$ is the $|\aleph_1| = \aleph_0$-minimal forcing of Prikry and $\mathrm{Nm}(T, R, S)$ is the one of Abraham [1].

Finally, if $T = {}^{<\omega}2$, $R(v) = \{0, 1\}$, $S(v) = \{\{0, 1\}\}$ for each $v \in {}^{<\omega}2$, then $A(T, R, S)$ is Sacks forcing [21] and $\mathrm{Nm}(T, R, S)$ is Cohen forcing.

We begin with introducing a condition on the family of trees which guarantees that certain countable sets will not be added in the corresponding generic extension.

A family of trees $\mathcal{K}$ is said to be *λ-indecomposable* if

(2.10) for each system $\{a_\xi \in \mathrm{r.o.}(\mathcal{K}, \leqslant) : \xi < \lambda\}$ such that $\bigvee_{\xi<\lambda} a_\xi = \mathbf{1}$, for each $T \in \mathcal{K}$ such that $S(v)$ is λ^+-complete for at least one $v \in \mathrm{SL}_0(T)$, there exist a $\zeta < \lambda$ and a tree $S \leqslant^0 T$, $S \in \mathcal{K}$ such that $S \leqslant a_\zeta$.

Now, following the idea of [3] we shall prove

Theorem 2.3. *Let $\mathcal{K}$ be a λ-indecomposable fusion closed family of trees and let the splitting criterion be $(\lambda^{\aleph_0})^+$-complete. Then the complete Boolean algebra $\mathrm{r.o.}(\mathcal{K}, \leqslant)$ is $(\aleph_0, \lambda)$-distributive, i.e. the forcing $(\mathcal{K}, \leqslant)$ does not add a new function from ω into λ.*

Proof. Let $a_{n,\zeta} \in \mathrm{r.o.}(\mathcal{K}, \leqslant)$, $n \in \omega$, $\zeta \in \lambda$, and

$$\bigwedge_{n\in\omega} \bigvee_{\zeta\in\lambda} a_{n,\zeta} = \mathbf{1}.$$

In order to show that

$$\bigvee_{\phi\in{}^{\omega}\lambda} \bigwedge_{n\in\omega} a_{n,\phi(n)} = \mathbf{1},$$

it is sufficient to prove that

$$(\forall T \in \mathcal{K})(\exists \phi \in {}^{\omega}\lambda)(\exists T' \in \mathcal{K})\ T' \leqslant T \,\&\, T' \leqslant \bigwedge_{n\in\omega} a_{n,\phi(n)}.$$

Given $T \in \mathcal{K}$ we shall find a tree $T_\infty \in \mathcal{K}$, $T_\infty \leqslant T$, and a function $H : \bigcup_{n\in\omega} \mathrm{SL}_{r(n)}(T_\infty) \to \lambda$ such that

$$(\forall n \in \omega)(\forall v \in \mathrm{SL}_{r(n)}(T_\infty))(\exists V \in \mathcal{K})\ T_\infty(v) \leqslant V \leqslant a_{n,H(v)}.$$

We proceed by induction. Let $T_0 = T$. Assume that $T_n \in \mathcal{K}$ and $H \restriction \bigcup_{k<n} \mathrm{SL}_{r(k)}(T_n)$ have already been constructed. By (2.8)(b), for each $u \in \mathrm{SL}_{r(n)}(T_n)$ there exists a tree $W_u \in \mathcal{K}$ such that $W_u \leqslant^0 T_n(u)$. By (2.10) there exist an $H(u) < \lambda$ and a tree $S_u \in \mathcal{K}$ such that $S_u \leqslant^0 W_u$ and $S_u \leqslant a_{n,H(u)}$. Therefore by (2.8)(c), there are $Z_u \leqslant^0 S_u$ ($u \in \mathrm{SL}_{r(n)}(T_n)$) such that

$$\bigcup \{Z_u : u \in \mathrm{SL}_{r(n)}(T_n)\} \in \mathscr{K}.$$

Let us denote this tree by T_{n+1}.

Let $T_\infty = \bigcap_{n\in\omega} T_n$. One can readily verify that for each $n \in \omega$, $T_\infty \leqslant^n T_n$, so we have $T_\infty \in \mathscr{K}$. Let $W \leqslant T_\infty$, $W \in \mathscr{K}_0$.

Now, for each $\phi \in {}^\omega\lambda$, we set

$$T^\phi = \bigcap_{n\in\omega} \bigcup \{W(u) : u \in \mathrm{Sl}_{r(n)}(T_\infty) \cap W \;\&\; H(u) = \phi(n)\}.$$

Every T^ϕ is a tree (not necessarily in $\mathscr{K}$), hence $\mathrm{Br}(T^\phi)$ is a closed subset of $\mathrm{Br}(W)$. Observe that $\mathrm{Br}(W) = \bigcup_{\phi\in{}^\omega\lambda} \mathrm{Br}(T^\phi)$. Since the splitting criterion is $(\lambda^{\aleph_0})^+$-complete, we can use Theorem 2.1 and obtain a $T' \leqslant^* W$ such that $T' \leqslant T^\phi$ for some $\phi \in {}^\omega\lambda$. Obviously $T' \in \mathscr{K}$.

It remains to show that for every $n \in \omega$,

$$T' \leqslant \bigvee \{S_u : u \in \mathrm{SL}_{r(n)}(T_\infty) \cap W \;\&\; H(u) = \phi(n)\}.$$

Suppose not. Then there exists a tree $Z \in \mathscr{K}$ such that $Z \leqslant T'$ and $Z \wedge S_u = \mathbf{0}$ for every $u \in \mathrm{SL}_{r(n)}(T_\infty) \cap W$, $H(u) = \phi(n)$. Then

$$\begin{aligned} Z \subseteq T' \subseteq T^\phi &\subseteq \bigcup \{W(u) : u \in \mathrm{SL}_{r(n)}(T_\infty) \cap W,\ H(u) = \phi(n)\} \\ &\subseteq \bigcup \{S_u : u \in \mathrm{SL}_{r(n)}(T_\infty) \cap W,\ H(u) = \phi(n)\}. \end{aligned}$$

Let $f \in \mathrm{Br}(Z)$. Then $f \in \mathrm{Br}(T_\infty) \cap \mathrm{Br}(T^\phi) \cap \mathrm{Br}(W)$. Let m be such that $u = f \restriction m \in \mathrm{SL}_{r(n)}(T_\infty)$. Clearly $H(u) = \phi(n)$. By (2.8)(b) there exists a tree $Z' \in \mathscr{K}$, $Z' \leqslant Z(u)$. Then $Z' \subseteq W(u) \subseteq S_u$, a contradiction. □

By a slight modification of the proof above one can easily prove

Theorem 2.4. *Let $\mathscr{K}$ be a λ-indecomposable fusion closed family of trees and let the splitting criterion be λ^+-complete. Then the complete Boolean algebra* r.o.$(\mathscr{K}, \leqslant)$ *is $(\aleph_0, \lambda, \lambda)$-distributive, provided* $\mathrm{cf}(\lambda) > \omega_0$, *i.e. every function from ω_0 into λ in the generic extension is bounded by a function from the ground model.*

Shelah [23] has obtained a stronger result. Since we shall need it later, we recall it. First using methods very similar to those used in the proof of Theorem 2.1, Shelah [23, p. 364] proved:

Theorem 2.5. *Let λ be a regular uncountable cardinal. Suppose that for each $v \in T$ either the set $\boldsymbol{S}(v)$ is λ^+-complete or $|\boldsymbol{R}(v)| < \lambda$. Then for every mapping $H : T \to \lambda$ there are a $\xi < \lambda$ and a tree $S \leqslant^* T$ such that $H(v) < \xi$ whenever $v \in S$.*

Let us consider the following property of a family of trees $\mathscr{K}$:

(2.11) *for each $v \in \mathrm{SL}_{r(n)}(T)$, $n \in \omega$,* if for each $\xi \in \mathrm{Succ}(v, T)$, $W_\xi \subseteq T(v^\frown\xi)$, $W_\xi \in \mathscr{K}$, then there is a $W_v \in \mathscr{K}$ such that $W_v \leqslant^0 T(v)$ and $W_v \subseteq \bigcup \{W_\xi : \xi \in \mathrm{Succ}(v, T)\}$.

Now, using Theorem 2.5 instead of Theorem 2.1 in a similar way as in Theorem 2.3 we can easily show (cf. [23, p. 365, Theorem 3.8]):

Theorem 2.6. *Let λ be a regular uncountable cardinal and let $\mathcal{K}$ be a λ-indecomposable fusion closed family of trees. Suppose that $\mathcal{K}$ satisfies* (2.11) *and that for each $v \in \boldsymbol{T}$ either the set $\boldsymbol{S}(v)$ is λ^+-complete or $|\boldsymbol{R}(v)| < \lambda$. Then the complete Boolean algebra* r.o.$(\mathcal{K}, \leqslant)$ *is $(\aleph_0, \lambda, \lambda)$-distributive.*

Proof. Let $a_{n,\eta} \in \text{r.o.}(\mathcal{K}, \leqslant)$, $n \in \omega$, $\eta \in \lambda$, and assume that

$$\bigwedge_{n\in\omega} \bigvee_{\eta\in\lambda} a_{n,\eta} = \mathbf{1}.$$

We shall show

$$\bigvee_{\xi\in\lambda} \bigwedge_{n\in\omega} \bigvee_{\eta\in\xi} a_{n,\eta} = \mathbf{1}.$$

We must prove

$$(\forall T \in \mathcal{K})(\exists \xi_0 < \lambda)(\exists S \in \mathcal{K})\, S \leqslant T \,\&\, S \leqslant \bigwedge_{n\in\omega} \bigvee_{\eta\in\xi_0} a_{n,\eta}.$$

Given $T \in \mathcal{K}$, much as in the proof of Theorem 2.3 we first find a tree $T_\infty \in \mathcal{K}$ and a function $H : \bigcup_{n\in\omega} \text{SL}_{r(n)}(T_\infty) \to \lambda$ such that

$$(\forall n \in \omega)(\forall v \in \text{SL}_{r(n)}(T_\infty))(\exists V \in \mathcal{K})\, T_\infty(v) \leqslant V \leqslant \bigvee_{\eta<H(v)} a_{n,\eta}.$$

We proceed by induction. Let $T_0 = T$. Assume that $T_n \in \mathcal{K}$, $H \restriction \bigcup_{k<n} \text{SL}_{r(k)}(T_n)$ have already been defined. Let $u \in \text{SL}_{r(n)}(T_n)$. by (2.8)(b) there is a tree $W_u \in \mathcal{K}$ such that $W_u \leqslant^0 T_n(u)$. If $\boldsymbol{S}(u)$ is λ^+-complete then by (2.10) we can find an $H(u) < \lambda$ and a tree $S_u \in \mathcal{K}$ such that $S_u \leqslant^0 W_u$ and $S_u \leqslant a_{n,H(u)}$. Otherwise, pick for each $\delta \in \text{Succ}(u, T_n)$ a tree $W_\delta \leqslant T_n(u^\frown\delta)$, $W_\delta \in \mathcal{K}$, and $\xi_\delta < \lambda$ such that $W_\delta \leqslant a_{n,\xi_\delta}$. Since λ is regular and $|\text{Succ}(u, T_n)| < \lambda$, we can define

$$H(u) = \sup\{\xi_\delta + 1 : \delta \in \text{Succ}(u, T_n)\} < \lambda.$$

By (2.11) there is an $S_u \leqslant^0 T_n(u)$ with $S_u \in \mathcal{K}$ and $S_u \subseteq \bigcup \{W_\delta : \delta \in \text{Succ}(u, T_n)\}$. Notice that $S_u \leqslant \bigvee_{\delta<H(u)} a_{n,\delta}$.

Now, by (2.8)(c) we can find $Z_u \leqslant^0 S_u$ $(u \in \text{SL}_{r(n)}(T_n))$ such that $\bigcup \{Z_u : u \in \text{SL}_{r(n)}(T_n)\} \in \mathcal{K}$; let us define $T_{n+1} = \bigcup \{Z_u : u \in \text{SL}_{r(n)}(T_n)\}$.

Let $T_\infty = \bigcap_{n\in\omega} T_n$; then $T_\infty \in \mathcal{K}$. Choose $W \leqslant T_\infty$, $W \in \mathcal{K}_0$, and extend the function H to the whole of W in such a way that $H(v) < \lambda$ for every $v \in W$. Using Theorem 2.5 there are a $\xi_0 < \lambda$ and a tree $S \leqslant^* W$ such that $H(v) < \xi_0$ whenever $v \in S$. Clearly $S \in \mathcal{K}$ and

$$S \leqslant \bigwedge_{n\in\omega} \bigvee_{\zeta\in\xi_0} a_{n,\zeta}. \qquad \square$$

Let us remark that conditions (2.10) and (2.11) are closely related to the S-condition of Shelah [23, p. 360].

3. Special trees

In this section we shall investigate properties of some special types of trees considering them as candidates for a forcing notion. We shall be concerned with perfect tree forcing, Namba forcing and their generalizations.

3.1. S-perfect trees

We start with a generalization of the trees which were introduced for forcing purposes by the first author [3, 5] and considered later e.g. in [13, p. 289], [9, 23].

Let K be an at most countable set of uncountable regular cardinals. For each $\kappa \in K$, let J_κ be an ideal over κ. Moreover, let Γ be a mapping of ω onto K such that $\Gamma^{-1}(\{\kappa\})$ is infinite for each $\kappa \in K$.

Consider the basic tree $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$, were $\boldsymbol{T} \subseteq {}^{<\omega}\sup K$ and the ramification $\boldsymbol{R}$ and the splitting criterion $\boldsymbol{S}$ are defined as follows

(3.1) if $\text{length}(v) = n$ then $\boldsymbol{R}(v) = \Gamma(n)$ and $\boldsymbol{S}(v) = \mathcal{P}(\Gamma(n)) - J_{\Gamma(n)}$.

A tree $T \subseteq \boldsymbol{T}$ is said to be *$\boldsymbol{S}$-perfect* if T satisfies conditions (2.6) and (3.2) where

(3.2) for each $v \in T$, for each $\kappa \in K$, there exist a splitting point $s \geqslant v$, $s \in T$ such that $\Gamma(\text{length(s)}) = \kappa$.

The set of all $\boldsymbol{S}$-perfect trees will be denoted by $\text{Pf}(\boldsymbol{S})$.

Let us remark that every **S**-perfect tree is perfect in the sense of Section 2. If the set Γ has more than one element the converse need not be true.

Since the properties of $\text{Pf}(\boldsymbol{S})$ we are interested in do not depend on the mapping Γ we can always assume that

(3.3) (a) if K is finite, $K = \{\kappa_0, \ldots, \kappa_{n-1}\}$, then $\Gamma(m) = \kappa_i$ where $i \equiv m \pmod n$;
(b) if K is infinite, $K = \{\kappa_0, \kappa_1, \ldots, \kappa_n, \ldots\}$, then Γ is the sequence $\kappa_0, \kappa_1, \kappa_0, \kappa_1, \kappa_2, \kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_0, \kappa_1, \ldots, \kappa_n, \ldots$.

Moreover, if the ideals J_κ are of the form $[\kappa]^{<\kappa}$ for each $\kappa \in K$, we shall simply say K-perfect and $\text{Pf}(K)$ instead of $\boldsymbol{S}$-perfect and $\text{Pf}(\boldsymbol{S})$, respectively. And finally, $\text{Pf}(\kappa_0, \ldots, \kappa_{n-1}) = \text{Pf}(\{\kappa_0, \ldots, \kappa_{n-1}\})$.

One can easily see that in general the set of trees $\text{Pf}(\boldsymbol{S})$ is neither hereditary nor fusion closed. However, $\text{Pf}(\boldsymbol{S})$ contains a nice dense subset. Let T be an $\boldsymbol{S}$-perfect tree. We say that T has an *ordered splitting* if

(3.4) for each splitting point $v \in T$, if $v \in \text{SL}_m(T)$ then $\Gamma(m) = \Gamma(\text{length}((v))$.

It is easily seen that the set $\text{Pf}^{\text{os}}(\boldsymbol{S})$ of all $\boldsymbol{S}$-perfect trees with an ordered splitting is dense in $\text{Pf}(\boldsymbol{S})$.

We start with the following simple lemma.

Lemma 3.1. *The set* $\mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ *is a fusion closed family of trees. Moreover, if for some* κ *the ideal* J_κ *is* λ^+*-complete, then* $\mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ *is* λ*-indecomposable.*

Proof. One can readily verify that $\mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ is fusion closed.

Now, let $T \in \mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ and let $S \leqslant^* T$. Let $v \in S$ be arbitrary. Since S is a tree there exists a branch $f \in \mathrm{Br}(S)$ which goes through the node v. Since $f \in \mathrm{Br}(T)$, for any $\kappa \in K$ there exists $k > \mathrm{length}(v)$ such that $f \restriction k$ is a splitting point of T and $\Gamma(k) = \kappa$. Since $S \leqslant^* T$, the node $f \restriction k$ is a splitting point of S as well. The same argument shows that S has an ordered splitting. Thus $\mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ is hereditary and (2.8)(a) holds.

Without loss of generality we may assume that J_{κ_0} is λ^+-complete. First, let us notice the following simple fact:

(3.5) if $T \in \mathrm{Pf}(\boldsymbol{S})$, $v \in \mathrm{SL}_0(T)$ is such that $\Gamma(\mathrm{length}(v)) = \Gamma(0)$ then there exists a tree $S \in \mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ with $S \leqslant^0 T$.

Let us define the function $r \in {}^\omega\omega$ as follows: if (3.3)(a) holds then we set $r(k) = k \cdot n$ $(k \in \omega)$; if (3.3)(b) holds then we set $r(k) = k(k+3)/2$ $(k \in \omega)$. Obviously, $\Gamma(r(k)) = \kappa_0$ for each $\kappa \in \omega$. Using (3.5) one immediately obtains (2.8)(b) and (c).

We show that (2.10) holds. Let $a_\xi \in \mathrm{r.o.}(\mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})) = \mathrm{r.o.}(\mathrm{Pf}(\boldsymbol{S}))$, $\xi < \lambda$, $\bigvee_{\xi<\lambda} a_\xi = \mathbf{1}$. Given $T \in \mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ consider $s_T \in \mathrm{SL}_0(T) = \mathrm{SL}_{r(0)}(T)$. For each $\xi \in \mathrm{Succ}(s_T, T)$, $T(s_T{}^\frown\xi)$ is an $\boldsymbol{S}$-perfect tree, hence there exist an $f(\xi) < \lambda$ and an $\boldsymbol{S}$-perfect tree T_ξ such that $T_\xi \leqslant a_{f(\xi)}$ and $T_\xi \leqslant T(s_T{}^\frown\xi)$. By the completeness of the ideal J_{κ_0} there exist a set $A \in J^+_{\kappa_0}$, $A \subseteq \mathrm{Succ}(s_T, T)$, and an $\zeta < \lambda$ such that $f(\xi) = \zeta$ for each $\xi \in A$. Let us set $W = \bigcup_{\xi \in A} T_\xi$. Then $W \leqslant^0 T$ and $W \in \mathrm{Pf}(\boldsymbol{S})$, $W \leqslant a_\zeta$. Using (3.5) we get $S \in \mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ with $S \leqslant^0 W$. □

As a consequence of this lemma and Theorems 2.3, 2.4 and 2.6 we obtain

Theorem 3.2. (a) *If for each* $\kappa \in K$ *the ideal* J_κ *is* $(\lambda^{\aleph_0})^+$*-complete, then the complete Boolean algebra* $\mathrm{r.o.}(\mathrm{Pf}(\boldsymbol{S}))$ *is* $(\aleph_0, \lambda)$*-distributive.*

(b) *If for each* $\kappa \in K$ *the ideal* J_κ *is* λ^+*-complete, then* $\mathrm{r.o.}(\mathrm{Pf}(\boldsymbol{S}))$ *is* $(\aleph_0, \lambda, \lambda)$*-distributive, provided* $\mathrm{cf}(\lambda) > \omega_0$.

(c) *If* λ *is a regular uncountable cardinal and for each* $\kappa \in K$ *either the ideal* J_κ *is* λ^+*-complete or* $\kappa < \lambda$, *then* $\mathrm{r.o.}(\mathrm{Pf}(\boldsymbol{S}))$ *is* $(\aleph_0, \lambda, \lambda)$*-distributive.*

Proof. Since $\mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ is dense in $\mathrm{Pf}(\boldsymbol{S})$, (a) and (b) are obvious. One can easily verify that $\mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$ satisfies (2.11) too. Hence by Theorem 2.6 we get (c). □

Let us observe that a regular tree $T \subseteq \boldsymbol{T}$ is $\boldsymbol{S}$-perfect if and only if the set $\mathrm{SS}(T)$ meets each $\Gamma^{-1}(\{\kappa\})$ $(\kappa \in K)$ in an infinite set. As an easy consequence of Theorem 2.2 we get

Theorem 3.3. *If each ideal J_κ, $\kappa \in K$, is $(2^{\aleph_0})^+$-complete then the set of all regular* $\boldsymbol{S}$*-perfect trees is dense in* $\mathrm{Pf}(\boldsymbol{S})$.

The following theorem generalizes the minimality results for $\mathrm{Pf}(\kappa)$ obtained by Prikry ($\kappa = \omega_1$) and the first author ($\kappa > \omega_1$). The proof will follow the obvious idea for proving such results as given e.g. in [13, pp. 284–288].

Theorem 3.4. *Suppose $\kappa_0 = \max K$ and let $J_{\kappa_0} \supseteq [\kappa_0]^{<\kappa_0}$. If for each $\kappa \in K$ the ideal J_κ is σ-complete, then the forcing notion* $\mathrm{Pf}(\boldsymbol{S})$ *is* $\mathrm{cf}(\kappa_0) = \omega_0$*-minimal.*

Proof. Let $\mathrm{Pf}(\boldsymbol{S})$ be constructed in a model M and let G be an M-generic filter on $\mathrm{Pf}(\boldsymbol{S})$. Since the generic function $g = \bigcup \{s_T : T \in G\}$ is a cofinal function from ω to κ_0 and $M[G] = M[g]$, it remains to show that whenever N is a model of ZFC such that $M \subseteq N \subseteq M[G]$ and $N \models \mathrm{cf}(\kappa_0) = \omega_0$, then $N = M[G]$.

Let $\underset{\sim}{f}$ and $T_0 \in G$ be such that

$$T_0 \Vdash \text{“}\underset{\sim}{f} \text{ is a function from } \check{\omega} \text{ to } \check{\kappa}_0\text{”}.$$

We shall show that either f is bounded by a function from the ground model M, or else $M[f] = M[G]$.

For each $n \in \omega$ and each $\zeta < \kappa_0$ let $a(n, \zeta) = \|\underset{\sim}{f}(\check{n}) = \check{\zeta}\|$ and consider

$$\mathscr{D}_0 = \{S \leqslant T_0 : (\forall n \in \omega)\, |\{\zeta \in \kappa_0 : S \wedge a(n, \zeta) \neq \mathbf{0}\}| < \kappa_0\}$$

and

$$\mathscr{D}_1 = \{S \leqslant T_0 : (\forall T \leqslant S)(\exists n \in \omega)\, |\{\zeta \in \kappa_0 : T \wedge a(n, \zeta) \neq \mathbf{0}\}| = \kappa_0\}.$$

It is easily seen that $\mathscr{D}_0 \cup \mathscr{D}_1$ is dense below T_0, hence $(\mathscr{D}_0 \cup \mathscr{D}_1) \cap G \neq \emptyset$.

If $S_0 \in \mathscr{D}_0 \cap G$ let us define the function h by letting, for each $n \in \omega$,

$$h(n) = \sup\{\zeta \in \kappa_0 : S_0 \wedge a(n, \zeta) \neq \mathbf{0}\}.$$

Since κ_0 is regular, $h \in {}^{\omega}\kappa_0$. Obviously $h \in M$ and for each $n \in \omega$

$$S_0 \Vdash \text{“}\underset{\sim}{f}(\check{n}) \leqslant h(n)^{\vee}\text{”}.$$

Now, suppose that $S_0 \in \mathscr{D}_1 \cap G$. We shall show that the set of all conditions $S \leqslant S_0$ such that

$$S \Vdash \text{“}\underset{\sim}{g} \text{ can be recovered from } \underset{\sim}{f}\text{”}$$

is dense below S_0.

Let $T \leqslant S_0$ be arbitrary. First we show

(3.6) if $v \in T$ is a splitting point of T and $n \in \omega$ then there are $n_v \in \omega$, $S_v \leqslant^0 T(v)$ and for each $\xi \in \mathrm{Succ}(v, S_v)$ $\zeta_v^\xi < \kappa_0$ such that
(i) $(\forall \xi \in \mathrm{Succ}(v, S_v))\, S_v(v^\frown \xi) \Vdash \text{“}\underset{\sim}{f}(\check{n}_v) = \check{\zeta}_v^\xi\text{”}$;
(ii) $(\forall \xi, \xi' \in \mathrm{Succ}(v, S_v))\, \xi \neq \xi' \to \zeta_v^\xi \neq \zeta_v^{\xi'}$;
(iii) $(\forall u \in \mathrm{SL}_1(S_v))\, \boldsymbol{R}(u) = \Gamma(n)$.

To prove (3.6), fix a splitting point $v \in T$ and $n \in \omega$. Since $T \leqslant S_0$ and $S_0 \in \mathcal{D}_1$, for each $\xi \in \mathrm{Succ}(v, T)$ there is $n_\xi \in \omega$ such that the set $\{\zeta \in \kappa : T(v^\frown\xi) \wedge a(n_\xi, \zeta) \neq \mathbf{0}\}$ is of size κ_0. Since $\boldsymbol{S}$ is σ-complete there are $X_v \in \boldsymbol{S}(v)$ and $n_v \in \omega$ such that $n_\xi = n_v$ whenever $\xi \in X_v$. Since $|\mathrm{Succ}(v, T)| \leqslant \kappa_0$, one can easily find for each $\xi \in X_v$ a tree $T_\xi \leqslant T(v^\frown\xi)$ and $\zeta_v^\xi < \kappa_0$ such that

(a) $T_\xi \leqslant a(n_v, \zeta_v^\xi)$;

(b) $\xi \neq \xi' \rightarrow \zeta_v^\xi \neq \zeta_v^{\xi'}$;

(c) $\boldsymbol{R}(s_{T_\xi}) = \Gamma(n)$.

Now it suffices to set $S_v = \bigcup\{T_\xi : \xi \in X_v\}$. Then $S_v \leqslant^0 T(v)$ and (i)–(iii) of (3.6) hold.

Now, by induction on $n \in \omega$, we shall construct a sequence of $\boldsymbol{S}$-perfect trees $T_0 \,{}^0\!\geqslant T_1 \,{}^1\!\geqslant \cdots {}^n\!\geqslant T_{n+1} \,{}^{n+1}\!\geqslant \cdots$ and two functions H_0, H_1 as follows.

Let $v = s_T$, $n = 0$. By (3.6) there are $n_v \in \omega$, $S_v \leqslant^0 T$ and ζ_v^ξ ($\xi \in \mathrm{Succ}(v, S_v)$) such that (i)–(iii) are satisfied. Let $T_0 = S_v$ and define

$$H_0(v) = n_v,$$
$$H_1(v^\frown\xi) = \zeta_v^\xi \quad \text{for each } \xi \in \mathrm{Succ}(v, T_0).$$

Suppose that $T_0 \,{}^0\!\geqslant T_1 \,{}^1\!\geqslant \cdots {}^{n-1}\!\geqslant T_n$ and $H_0 \restriction \bigcup_{k\leqslant n} \mathrm{SL}_k(T_k)$, $H_1 \restriction \bigcup_{k\leqslant n} \{v^\frown\xi : v \in \mathrm{SL}_k(T_k),\ \xi \in \mathrm{Succ}(v, T_k)\}$ have been constructed and that for each $k \leqslant n$ and each $u \in \mathrm{SL}_{k+1}(T_k)$ we have $\boldsymbol{R}(u) = \Gamma(k)$. By (3.6) for each $v \in \mathrm{SL}_{n+1}(T_n)$ there are $n_v \in \omega$, $S_v \leqslant^0 T_n(v)$ and $\zeta_v^\xi < \kappa_0$ for all $\xi \in \mathrm{Succ}(v, S_v)$ such that (i) and (ii) hold and $\boldsymbol{R}(u) = \Gamma(n+1)$ whenever $u \in \mathrm{SL}_1(S_v)$. Let

$$T_{n+1} = \bigcup\{S_v : v \in \mathrm{SL}_{n+1}(T_n)\}$$

and define

$$H_0(v) = n_v \quad \text{for each } v \in \mathrm{SL}_{n+1}(T_{n+1}),$$
$$H_1(v^\frown\xi) = \zeta_v^\xi \quad \text{for each } v \in \mathrm{SL}_{n+1}(T_{n+1}),\ \xi \in \mathrm{Succ}(v, T_{n+1}).$$

Let $T_\infty = \bigcap_{n\in\omega} T_n$. It is obvious that $T_\infty \in \mathrm{Pf}(\boldsymbol{S})$. The set $\{T_\infty : T \leqslant S_0\}$ is dense below $S_0 \in G$, hence $T_\infty \in G$ for some $T \leqslant S_0$. One can easily see that if we define (in $M[f]$) the function

$$H : \bigcup_{n\in\omega} \mathrm{SL}_n(T_\infty) \rightarrow T_\infty$$

by letting, for each $n \in \omega$ and $v \in \mathrm{SL}_n(T_\infty)$,

$$H(v) = v^\frown\xi \quad \text{if and only if} \quad f(H_0(v)) = H_1(v^\frown\xi),$$

then we get

$$\{g\} = \bigcap\left\{T_\infty(u) : (\forall v \leqslant u)\ v \in \bigcup_{n\in\omega} \mathrm{SL}_n(T_\infty) \rightarrow H(v) \in T_\infty(u)\right\},$$

and thus $g \in M[f]$. □

In our applications of the forcing notion $\mathrm{Pf}(\boldsymbol{S})$ in Section 6 we shall need

Theorem 3.5. *Let λ be an uncountable regular cardinal and suppose that for some $\kappa \in K$*

(3.7) *there are sets $A_\xi \in J_\kappa$ ($\xi \in \lambda$) such that $\bigcup_{\xi<\lambda} A_\xi = \kappa$ and $\bigcup_{\xi<\zeta} A_\xi \in J_\kappa$ whenever $\zeta < \lambda$.*

Then for any M-generic filter G on Pf($\boldsymbol{S}$), *in $M[G]$ λ is cofinal with ω_0.*

Proof. Let $n_0, n_1, \ldots, n_k, \ldots$ ($k \in \omega$) be an increasing sequence of natural numbers such that $\Gamma(n_k) = \kappa$ for each $k \in \omega$. Suppose that $G \subseteq \mathrm{Pf}(\boldsymbol{S})$ is M-generic and let $g = \bigcup \{s_T : T \in G\}$. Without loss of generality the sets A_ξ ($\xi \in \lambda$) are pairwise disjoint. It is not difficult to see that the function f defined by

$$f(k) = \xi \quad \text{iff} \quad g(n_k) \in A_\xi \quad (k \in \omega)$$

is cofinal in λ. □

3.2. Namba forcing

Another type of trees was studied by Namba [19]. We shall investigate a generalization of this forcing notion as introduced by several authors, e.g. [23].

Let $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ be a basic tree. Let us recall that a tree $T \subseteq \boldsymbol{T}$ is called an $\boldsymbol{S}$-*Namba* tree if it satisfies condition (2.9), i.e. T has a trunk s_T and all nodes above s_T are splitting. If the basic tree is given as in Subsection 3.1 with $J_\kappa = [\kappa]^{<\kappa}$ ($\kappa \in K$), we shall denote by $\mathrm{Nm}(K)$ the set of corresponding $\boldsymbol{S}$-Namba trees.

Lemma 3.6. *The set* Nm($\boldsymbol{S}$) *of all* $\boldsymbol{S}$*-Namba trees is a hereditary fusion closed family of trees. Moreover, if the splitting criterion* $\boldsymbol{S}$ *is* λ^+*-complete, then* Nm($\boldsymbol{S}$) *is* λ*-indecomposable.*

Proof. Obviously, Nm($\boldsymbol{S}$) is hereditary and fusion closed. It is easily seen that conditions (2.8)(b) and (c) are fulfilled as well; it suffices to let $r(n) = n$ for each $n \in \omega$. It remains to verify condition (2.10).

Let $a_\zeta \in \mathrm{r.o.}(\mathrm{Nm}(\boldsymbol{S}))$ ($\zeta < \lambda$) such that $\bigvee_{\zeta<\lambda} a_\zeta = \mathbf{1}$. Suppose $T \in \mathrm{Nm}(\boldsymbol{S})$ and

$$\neg(\exists T_0 \leqslant^0 T)(\exists \zeta < \lambda)(T_0 \in \mathrm{Nm}(\boldsymbol{S}) \ \&\ T_0 < a_\zeta).$$

We shall construct an $S \in \mathrm{Nm}(\boldsymbol{S})$ with $S \leqslant^0 T$ such that for each $t \in S$

$$\neg(\exists T' \leqslant^0 T(t))(\exists \zeta < \lambda)\ T' \leqslant a_\zeta.$$

This will yield a contradiction since then $S \leqslant^0 T$ and S cannot be extended to an element below any a_ζ.

Let

$$S = \{t \in T : \neg(\exists T' \leqslant^0 T(t))(\exists \zeta < \lambda)\ T' \leqslant a_\zeta\}.$$

We shall show that $S \in \mathrm{Nm}(\boldsymbol{S})$. By assumption, $s_T \in S$. Let $t \geqslant s_T$, $t \in S$ and suppose $\mathrm{Succ}(t, S) \notin \boldsymbol{S}(t)$. This means that

$$\{\xi \in \mathbf{R}(t): t^\frown\xi \in T \;\&\; (\exists T' \leqslant^0 T(t^\frown\xi))(\exists \zeta < \lambda)\; T' \leqslant a_\zeta\} \in \boldsymbol{S}(t).$$

For each $\zeta < \lambda$ put

$$X_\zeta = \{\xi \in \boldsymbol{R}(t): t^\frown\xi \in T \;\&\; (\exists T' \leqslant^0 T(t^\frown\xi))\; T' \leqslant a_\zeta\}.$$

Since the splitting criterion $\boldsymbol{S}$ is λ^+-complete, there is a $\zeta_0 < \lambda$ such that $X_{\zeta_0} \in \boldsymbol{S}(t)$. Now, for each $\xi \in X_{\zeta_0}$ choose a $T^\xi \leqslant^0 T(t^\frown\xi)$ such that $T^\xi \leqslant a_{\zeta_0}$. Then $T' = \bigcup_{\xi \in X_{\zeta_0}} T^\xi \in \mathrm{Nm}(\boldsymbol{S})$, $T' \leqslant^0 T(t)$ and $T' \leqslant a_{\zeta_0}$. This implies that $t \notin S$, a contradiction. □

As a consequence of this lemma and Theorems 2.3 and 2.4 we obtain:

Theorem 3.7. (a) *If the splitting criterion is* $(\lambda^{\aleph_0})^+$*-complete, then the complete Boolean algebra* r.o.$(\mathrm{Nm}(\boldsymbol{S}))$ *is* $(\aleph_0, \lambda)$*-distributive.*

(b) *If the splitting criterion is* λ^+*-complete, then* r.o.$(\mathrm{Nm}(\boldsymbol{S}))$ *is* $(\aleph_0, \lambda, \lambda)$*-distributive, provided* $\mathrm{cf}(\lambda) > \omega_0$.

In the same way as in the case of $\boldsymbol{S}$-perfect trees (Theorem 3.5) one can easily prove the following:

Theorem 3.8. *Suppose that the basic tree is as in subsection* 3.1 *and assume that for some* $\kappa \in K$ *and an uncountable regular cardinal* λ *condition* (3.7) *is satisfied. Then for every* M*-generic filter* G *on* $\mathrm{Nm}(\boldsymbol{S})$, *in* $M[G]$ λ *is cofinal with* ω_0.

4. Perfect trees versus Namba forcing

As we showed in Section 3 both forcing notions $\mathrm{Pf}(\boldsymbol{S})$ and $\mathrm{Nm}(\boldsymbol{S})$ have the same distributivity properties (Theorems 3.2, 3.5 and 3.7, 3.8). However, there are properties which can distinguish both forcing notions. Magidor and Shelah [23, pp. 366–368] proved that under CH the forcing notions $\mathrm{Nm}(\omega_2)$ and $\mathrm{Pf}(\omega_2)$ are different: the former one adds a fast growing function from ω_0 to ω_2, the latter one does not. This result can be generalized.

Theorem 4.1. *If* J_κ *is* $(2^{\aleph_0})^+$*-complete for each* $\kappa \in K$ *and if* $\sup K \in K$, *then whenever* G *is an* M*-generic filter on* $\mathrm{Pf}(\boldsymbol{S})$, *the generic extension* $M[G]$ *does not contain a fast growing function from* ω_0 *into* $\sup K$.

Proof. Let $\kappa_0 = \max K$. If G is an M-generic filter on $\mathrm{Pf}(\boldsymbol{S})$ then $g = \bigcup \{s_T : T \in G\} \in {}^\omega\kappa_0$ is unbounded in κ_0. So by Lemma 1.11 it is sufficient to show that the generic function g has no fast growing part. Or more formally, we have to prove

$$(\forall T \in \mathrm{Pf}(\boldsymbol{S}))(\forall A \in [\omega]^{\omega})(\exists S \leqslant T)(\exists H \in {}^{\kappa_0}\kappa_0 \cap M)(\forall m \in \omega)$$
$$(\exists k \in \omega)\, k \geqslant m \;\&\; S \Vdash \text{“}H(\underset{\sim}{g}(c_A(k))^{\vee} > \underset{\sim}{g}(c_A(k+1)^{\vee})\text{”},$$

where for every $X \in [\omega]^{\omega}$ c_X is the counting function of X, i.e. $c_X(0) = \min X$, $c_X(n+1) = \min\{k \in X : k > c_X(n)\}$ $(n \in \omega)$.

Let $T \in \mathrm{Pf}(\boldsymbol{S})$ and $A \in [\omega]^{\omega}$ be arbitrary. As the set of all regular K-perfect trees is dense in $\mathrm{Pf}(\boldsymbol{S})$ by Theorem 3.3, without loss of generality we may assume that T is regular and has an ordered splitting.

For each branch $f \in \mathrm{Br}(T)$ define

$$A_f = \{m \in A : f(m) > \max\{f(k) : k \in A \;\&\; k < m\}$$

and for each $B \subseteq A$ let

$$E_B = \{f \in \mathrm{Br}(T) : A_f = B\}.$$

By Theorem 2.1 there are $T_0 \leqslant^* T$ and $B_0 \subseteq A$ such that $f \in E_{B_0}$ whenever $f \in \mathrm{Br}(T_0)$. Now there are two possible cases:

Case I: B_0 *is finite* (obviously B_0 is always nonempty). Let $B = A - B_0$, $n_0 = \max B_0$ and define

$$H(\alpha) = \gamma_0 \quad (\alpha \in \kappa_0),$$

where $\gamma_0 = v_0(n_0)$ for some $v_0 \in T_0$ such that $\mathrm{length}(v_0) > n_0$. Let $S \in \mathrm{Pf}(\boldsymbol{S})$ be such that $S \leqslant T_0(v_0)$. Clearly for each $k \in B$ and each $v \in S \cap {}^{k+1}\kappa_0$ we have $v(k) < \gamma_0$ and $H(v(k)) = \gamma_0$. In particular, for each k, $m \in A$, if $n_0 < k < m$ then

$$S \Vdash \text{“}\underset{\sim}{g}(\check{m}) < \check{H}(\underset{\sim}{g}(\check{k})) = \check{\gamma}_0\text{”}.$$

Case II: B_0 *is infinite*. Find $B_1 \in [B_0]^{\omega}$ and a regular $\boldsymbol{S}$-perfect tree $T_1 \leqslant T_0$ such that if $D = \mathrm{SS}(T_1)$ then

(i) $(\exists^{\infty} n)(\forall v \in T_1)\, \mathrm{length}(v) = c_D(2n+1) \rightarrow \boldsymbol{R}(v) = \kappa_0$,

(ii) $(\forall n \in \omega)(\forall v \in T_1)\, \mathrm{length}(v) = c_D(2n) \rightarrow \boldsymbol{R}(v) = \kappa_0$,

(iii) $(\forall n \in \omega)\, c_{B_1}(2n) < c_{B_1}(2n+1) < c_D(2n) < c_D(2n+1) < c_{B_1}(2n+2)$.

For each $v \in T_1 \cap {}^{c_D(2n)}\kappa_0$ pick $\xi_v \in \mathrm{Succ}(v, T_1)$, $\xi_v > \max \mathrm{ran}(v)$ and set

$$S_n = \bigcup \{T_1(v^\frown \xi_v) : v \in \mathrm{SL}_{2n}(T_1)\}.$$

One can easily check that $S = \bigcap_{n \in \omega} S_n$ is an $\boldsymbol{S}$-perfect tree and $S \leqslant T_1$. Let us define for each $\xi \in \kappa_0$

$$H(\xi) = \min\Big\{\gamma \in \kappa_0 : \Big(\forall u \in S \cap \bigcup_{k \in \omega} {}^{k}\xi\Big)(\exists n \in \omega)(\exists v \geqslant u)$$
$$v \in \mathrm{SL}_{2n}(T_1) \;\&\; \gamma > \xi_v\Big\}.$$

Now S and H are as required: obviously $H \in M$ and for each $n \in \omega$ and each $u \in S$ such that length$(u) > c_{B_1}(2n+1)$ we have

$$H(u(c_{B_1}(2n))) > \xi_v > u(c_{B_1}(2n+1)),$$

where $v \geq u \upharpoonright c_{B_1}(2n)$ is such that $v \in \mathrm{SL}_{2n}(T_1) \cap S$.

Since $B_1 \subseteq B_0 \subseteq A$, if $c_{B_1}(2n) = c_A(k)$ then $c_{B_1}(2n+1) \geq c_A(k+1)$ and $u(c_{B_1}(2n+1)) \geq u(c_A(k+1))$. This means

$$S \Vdash \text{“}\underset{\sim}{g} \text{ is not fast growing on } \check{A}\text{”}. \quad \square$$

On the other hand, by a simple computation we get

Theorem 4.2. *Let the basic tree* $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ *be as in subsection* 3.1. *If for some* $\kappa \in K$ *and an uncountable regular cardinal* λ (3.7) *holds, then for every* M*-generic filter* G *on* $\mathrm{Nm}(\boldsymbol{S})$ *the generic extension* $M[G]$ *contains a fast growing function from* ω_0 *into* λ.

By Theorem 3.4 the forcing notion $\mathrm{Pf}(\boldsymbol{S})$ *is* $\mathrm{cf}(\kappa) = \omega_0$-minimal provided $\kappa = \max K$ and the splitting criterion $\boldsymbol{S}$ is σ-complete. If $\kappa = \lambda^+$ for some uncountable cardinal λ and if the splitting criterion $\boldsymbol{S}$ is $(\lambda^{\aleph_0})^+$-complete (e.g. under GCH, if λ is regular and $\boldsymbol{S}$ is κ-complete) then $\mathrm{Nm}(\boldsymbol{S})$ is also $\mathrm{cf}(\kappa) = \omega_0$-minimal (by Theorems 1.1 and 3.7). However, as we shall now show, if κ is a limit cardinal, then, even if the splitting criterion is κ-complete, the forcing notion $\mathrm{Nm}(\boldsymbol{S})$ need not be $\mathrm{cf}(\kappa) = \omega_0$-minimal.

First, let us recall the definition of Prikry's forcing Pr (see [20]). Let J be a normal prime ideal on a measurable cardinal κ. The forcing conditions are pairs (s, A) where $s \in [\kappa]^{<\omega}$, $A \in [\kappa]^{\kappa} - J$ and $\max s < \min A$, ordered by $(s, A) \leq (t, B)$ iff $s \supseteq t$, $A \subseteq B$ and $s - t \subseteq B$. Prikry [20] showed that the forcing Pr changes the cofinality of the cardinal κ to ω_0 without adding new subsets of smaller cardinals. Dehornoy [7] proved that r.o.(Pr) is isomorphic to r.o.($\mathrm{Nm}(\boldsymbol{S})$), where $\boldsymbol{T} = {}^{<\omega}\kappa$, $\boldsymbol{R}(v) = \kappa$ and $\boldsymbol{S}(v) = \mathcal{P}(\kappa) - J$ for each $v \in \boldsymbol{T}$, provided J is a normal prime ideal on κ. We shall show that in this case $\mathrm{Nm}(\boldsymbol{S})$ (and hence Pr) is not $\mathrm{cf}(\kappa) = \omega_0$-minimal.

So let us assume that κ is a measurable cardinal, J is a normal prime ideal on κ and

(4.1) $\quad \boldsymbol{R}(v) = \kappa, \qquad \boldsymbol{S}(v) = J^+ \quad$ for each $v \in \boldsymbol{T} = {}^{<\omega}\kappa$.

Let Pn be the set of all $\boldsymbol{S}$-Namba trees T which satisfy the following condition:

(4.2) (i) there is a set $A_T \in J^+$ such that for each $v \geq s_T$, $v \in T$ iff $v = s_T{}^\frown u$ for some increasing sequence $u \in {}^{<\omega}A_T$;
(ii) for each $v \in T$ and each $\xi \in \mathrm{Succ}(v, T)$, if $v \geq s_T$ then $\xi > \max \mathrm{ran}(v)$.

Since J is a normal prime ideal it is not hard to show that Pn is a dense subset of $\mathrm{Nm}(\boldsymbol{S})$. To simplify the notation we shall identify a finite set $s \in [\kappa]^{<\omega}$ with the unique strictly increasing function from length(s) onto s. Thus, under this identification, if $A \in J^+$ and $v \in {}^{<\omega}\kappa$ then $v^\frown[A]^{<\omega}$ is a tree $T \in \mathrm{Pn}$ such that $s_T = v$ and $A_T = A$. Obviously, for each $T, S \in \mathrm{Pn}$ we have $T \leq S$ if and only if $T \subseteq S$ and $A_T \subseteq S$.

For a proof of the following lemma see [13, p. 266].

Lemma 4.3. *Let B be a complete subalgebra of a complete Boolean algebra D. Then B is locally equal to D if and only if $M[G] = M[G \cap B]$ whenever G is an M-generic filter on D.*

Now we shall prove the promised result.

Theorem 4.4. *Let J be a normal prime ideal on a measurable cardinal κ and let $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ be as in (4.1). If G is an M-generic filter on* $\mathrm{Nm}(\boldsymbol{S})$, *then $M[G] \supseteq M$ is not a* $\mathrm{cf}(\kappa) = \omega_0$*-minimal extension.*

Remark. Note that by Theorem 3.8 $M[G] \supseteq M$ is a $\mathrm{cf}(\kappa) = \omega_0$-extension.

Proof. Since Pn is dense in $\mathrm{Nm}(\boldsymbol{S})$, we must find a complete Boolean subalgebra B^* of $B = \mathrm{r.o.(Pn)}$ which is not locally equal to r.o.(Pn) yet the forcing with B^* changes the cofinality of κ to ω_0.

For each $T \in \mathrm{Pn}$ let us define a tree $T^* \subseteq T$ as follows

$v \in T^*$ iff there exists a $u \in T$ such that length $(u) >$ length(v) and for each $n \in \omega$, if length$(v) > n$ then $v(n)$ is such that
(i) $u(n-1) < v(n) < u(n+1)$ whenever n is odd and $u(n-1) < u(n+1) - 1$;
(ii) $v(n) = u(n)$ otherwise.

Let $\mathrm{Pn}^* = \{T^* : T \in \mathrm{Pn}\}$. It is not difficult to see that $(\mathrm{Pn}^*, \leq)$ is a separative partially ordered set. Let us define a function $\psi : \mathrm{Pn} \to \mathrm{Pn}^*$ by letting

$$\psi(T) = T^* \quad (T \in \mathrm{Pn}).$$

We claim that ψ is a normal mapping. Obviously ψ is order preserving and onto, so it remains to show that whenever $T \in \mathrm{Pn}$ and $S^* \leq T^*$, there exists $R \leq T$ such that $R^* \leq S^*$. Define

$$u \in R \quad \text{iff} \quad (u \in T)\ \&\ (\exists v \in S)(\forall n \in \omega)\ 2n \in \mathrm{dom}(v) \cap \mathrm{dom}(u) \to v(2n) = u(2n).$$

It can be easily checked that $R^* = S^*$. Thus ψ is a normal mapping. And so by Lemma 1.9 $B^* = \bar{\psi}(\mathrm{r.o.(Pn^*)})$ is a complete subalgebra of $B = \mathrm{r.o.(Pn)}$.

It is not hard to verify that forcing with Pn^* changes the cofinality of κ to ω_0. Thus if we show that B^* is not locally equal to B, our proof will be complete. To do this it suffices to find an element $b \in B$ such that

$$B_a = \{c \wedge a : c \in B\} \neq B_a^* = \{c \wedge a : c \in B^*\}$$

whenever $a \in B$ and $a \leq b$. We will show that $b = \mathbf{1}$ works.

Let $a \in B$ be an arbitrary non-zero element. Since Pn is dense in B, there is $T_0 \leq a$, $T_0 \in \text{Pn}$ such that $n = \text{length}(s_{T_0})$ is odd. Let $A_{T_0} \in J^+$ be such that $T_0 = s_{T_0}{}^\frown[A_{T_0}]^{<\omega}$ and let $A_{T_0} = \{\xi_0, \xi_1, \xi_2, \ldots, \xi_\eta, \ldots\}$ be an enumeration of the set A_{T_0} such that $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_\eta < \cdots$ $(\eta \in \kappa)$. Let us set

$$S_0 = s_{T_0}{}^\frown\xi_0{}^\frown\xi_2{}^\frown[A_{T_0} - \{\xi_0, \xi_1, \xi_2\}]^{<\omega} \quad \text{and}$$
$$S_1 = s_{T_0}{}^\frown\xi_1{}^\frown\xi_2{}^\frown[A_{T_0} - \{\xi_0, \xi_1, \xi_2\}]^{<\omega}.$$

Obviously $S_0, S_1 \leq T_0 \leq a$, hence $S_0, S_1 \in B_a$ and $S_0 \wedge S_1 = \mathbf{0}$. It is sufficient to show that $S_0 \notin B_a^*$.

Suppose $S_0 \in B_a^*$. Then $S_0 = T_0 \wedge \bar{\psi}(u)$ for some $u \in \text{r.o.}(\text{Pn}^*)$. Let $A \subseteq \text{Pn}^*$ be such that $u = \bigvee A$. Hence

$$S_0 = T_0 \wedge \bigvee \{S \in \text{Pn} : (\exists T \in A)\, S^* \leq T\}.$$

Since $S_0 \neq \mathbf{0}$, there exist $T \in A$ and $S \in \text{Pn}$ such that $S^* \leq T$ and $S \leq S_0$. Let us remark that for any $v \in S$, $\text{length}(v) > n$, we have $v(n) = \xi_0$. We define a tree W as follows

$$v \in W \leftrightarrow (\exists u)(u \in S \;\&\; (\forall i < \text{length}(v))(i \neq n \rightarrow v(i) = u(i))$$
$$\&\; (n < \text{length}(v) \rightarrow v(n) = \xi_1)).$$

It is easy to see that $W^* = S^*$. So $W^* \leq T$ and therefore $W \leq S_0$. On the other hand $W \leq S_1$, contradicting $S_0 \wedge S_1 = \mathbf{0}$. $\square$

Remark. The above proof shows essentially that the generic function $g : \omega \rightarrow \kappa$ cannot be recovered from its even part $\{g(2n) : n \in \omega\}$.

Let us close this section with a few remarks on the product $P \times Q$ where P and Q are $\text{Nm}(\kappa)$ or $\text{Pf}(\kappa)$ (i.e. the basic tree is $\boldsymbol{T} = {}^{<\omega}\kappa$, $\boldsymbol{R}(v) = \kappa$, $\boldsymbol{S}(v) = [\kappa]^\kappa$ for each $v \in \boldsymbol{T}$). The forcing $P \times Q$ changes the cofinality of κ to ω_0, the corresponding generic extension is never $\text{cf}(\kappa) = \omega_0$-minimal and hence, even if forcing with neither P nor Q adds reals, forcing with the product may add reals. Actually, for $\kappa = \aleph_2$ by Theorem 1.1 the forcing $P \times Q$ must add a real. Moreover, Hart observed that forcing by $\text{Nm}(\kappa) \times \text{Nm}(\kappa)$ adds a Cohen real. Indeed, the mapping ψ defined on all pairs $(T, S) \in \text{Nm}(\kappa) \times \text{Nm}(\kappa)$ with $\text{length}(s_T) = \text{length}(s_S)$ by

$$\psi(T, S) = p \quad \text{iff} \quad \text{(i) } p \in {}^n2, \text{ where } n = \text{length}(s_T), \text{ and}$$
$$\text{(ii) } p(k) = 0 \text{ iff } s_T(k) < s_S(k) \text{ for each } k < n,$$

is normal.

5. Collapsing κ^+ to $\mathfrak{h}$

Let κ be an uncountable regular cardinal and let J be an ideal on κ such that $J \supseteq [\kappa]^{<\kappa}$. Let the basic tree $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ be such that $\boldsymbol{T} = {}^{<\omega}\kappa$, $\boldsymbol{R}(v) = \kappa$ and $\boldsymbol{S}(v) = \mathcal{P}(\kappa) - J$ for each $v \in \boldsymbol{T}$. It is easily seen that both $\text{Pf}(\boldsymbol{S})$ and $\text{Nm}(\boldsymbol{S})$ satisfy the $(2^\kappa)^+$-cc and do not satisfy the κ-cc. Even if we assume GCH, in general we cannot give a better estimate than κ^{++}-cc. Actually, if κ is a successor cardinal, then this estimate cannot be improved since e.g. under GCH both forcing notions collapse κ^+ to κ by Theorems 1.4, 3.2, 3.5 and 3.7, 3.8 (and as we already mentioned, this result is also proved by Shelah [23, p. 440]).

By Jensen's Covering Lemma, if $0^\#$ does not exist then whenever the cofinality of a regular uncountable cardinal κ is changed to ω_0, κ must be collapsed. On the other hand, if κ is a measurable cardinal and J a normal prime ideal on κ, then the forcing notion Nm($\boldsymbol{S}$) for $\boldsymbol{S}$ defined by $\boldsymbol{S}(v) = \mathcal{P}(\kappa) - J$ ($v \in \boldsymbol{T}$) is equivalent to Prikry's forcing Pr and it does not collapse cardinals. So it is natural to ask the following question: if $\boldsymbol{S}$ is as above, does forcing with Pf($\boldsymbol{S}$) preserve cardinals?

In general the answer is negative. Let us denote by $\mathfrak{h}$ the least cardinal λ such that the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ fails to be $(\lambda, 2^{\aleph_0})$-distributive. In [2] the authors showed that r.o.($\mathcal{P}(\omega)/\text{fin}$) collapses $2^{\aleph_0}$ to $\mathfrak{h}$. By Theorem 2.2 (compare [5], p. 48, Balcar's theorem]) one can easily see that the complete Boolean algebra r.o.($\mathcal{P}(\omega)/\text{fin}$) can be completely embedded into r.o.(Pf($\boldsymbol{S}$)) provided $\boldsymbol{S}(v) = J$ for each $v \in \boldsymbol{T}$ and J is $(2^{\aleph_0})^+$-complete: the complete embedding is induced by mapping each set $X \in [\omega]^\omega$ to the Boolean union of the set of all regular $\boldsymbol{S}$-perfect trees T such that SS(T) is equal to X modulo a finite set. So if $\mathfrak{h} < 2^{\aleph_0}$, $2^{\aleph_0}$ is collapsed to $\mathfrak{h}$. However, this argument does not work if we assume e.g. CH.

Our goal in this section is to prove the following theorem which shows that under some assumptions, when forcing with Pf($\boldsymbol{S}$), not only $2^{\aleph_0}$ but κ^+ is collapsed to $\mathfrak{h}$ as well.

Theorem 5.1. *Let the basic tree* $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ *be as in subsection* 3.1, *let* $\kappa = \sup K \in K$ *and let* $J_\kappa \supseteq [\kappa]^{<\kappa}$. *If* $2^\kappa = \kappa^+$ *and if the splitting criterion* $\boldsymbol{S}$ *is* $(2^{\aleph_0})^+$*-complete, then* Pf($\boldsymbol{S}$) *collapses* κ^+ *to* $\mathfrak{h}$.

Remark. Let us note that under the assumption $2^\kappa = \kappa^+$ the forcing notion Pf($\boldsymbol{S}$) satisfies the κ^{++}-cc and thus all cardinals above κ^+ are preserved.

Let the basic tree $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ be as in the assumptions of Theorem 5.1. If T is a regular $\boldsymbol{S}$-perfect tree and if $\lambda \in K$ then we denote

$$\text{SS}_\lambda(T) = \text{SS}(T) \cap \Gamma^{-1}(\{\lambda\}).$$

So for $v \in T$ with length(v) $\in \text{SS}_\lambda(T)$ we have $\boldsymbol{R}(v) = \lambda$.

Let us recall that for every infinite set $A \subseteq \omega$, c_A is the counting function of A.

We shall prove Theorem 5.1 in a sequence of lemmas. Until the end of the proof of Theorem 5.1 we shall assume that the basic tree $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ as described above is fixed (i.e. $\kappa = \max K$, $J_\kappa \supseteq [\kappa]^{<\kappa}$ and $\boldsymbol{S}$ is $(2^{\aleph_0})^+$-complete) and that $2^\kappa = \kappa^+$.

Lemma 5.2. *For every $\boldsymbol{S}$-perfect tree T there is a regular $\boldsymbol{S}$-perfect tree S such that $S \leqslant T$ and*

(5.1) *if* $A = \mathrm{SS}(S)$ *then* $\mathrm{SS}_\kappa(S) = A - \{c_A(3m+2) : m \in \omega\}$.

Proof. By Theorem 3.3 there exists a regular $\boldsymbol{S}$-perfect tree W with $W \leqslant T$. By the definition of $\boldsymbol{S}$-perfect tree, for every $\lambda \in K$ the set $\mathrm{SS}_\lambda(W)$ is infinite. Therefore by induction one can easily find a set $A \subseteq \mathrm{SS}(W)$ such that $A \cap \mathrm{SS}_\lambda(W)$ is infinite for each $\lambda \in K$ and $\mathrm{SS}_\kappa(W) \cap A = A - \{c_A(3m+2) : m \in \omega\}$.

Now, take any regular $\boldsymbol{S}$-perfect tree $S \leqslant W$ such that $\mathrm{SS}(S) = A$; S is as required. □

Let $\mathrm{Pf}^{\mathrm{r}}_\kappa(\boldsymbol{S})$ denote the set of all regular $\boldsymbol{S}$-perfect trees satisfying (5.1). By the lemma which we have just proved $\mathrm{Pf}^{\mathrm{r}}_\kappa(\boldsymbol{S})$ is dense in $\mathrm{Pf}(\boldsymbol{S})$.

We shall need the following notion. Let $T \subseteq \boldsymbol{T}$ be a tree. For each $t \in T$ let $\mathrm{sp}_T(t) \in T$ be such that $\mathrm{sp}_T(t)$ is a splitting point of T, $\mathrm{sp}_T(t) \geqslant t$ and no $u \in T$, $t \leqslant u < \mathrm{sp}_T(t)$ is splitting.

Lemma 5.3. *Let $T \in \mathrm{Pf}^{\mathrm{r}}_\kappa(\boldsymbol{S})$, $B = \mathrm{SS}_\kappa(T)$ and let $\{S_\zeta : \zeta \in \kappa\}$ be a set of regular $\boldsymbol{S}$-perfect trees such that $\mathrm{SS}_\kappa(S_\zeta) = \{c_B(2n) ; n \in \omega\}$ for each $\zeta \in \kappa$. Then there exists a regular $\boldsymbol{S}$-perfect tree $S \leqslant T$ such that $\mathrm{SS}_\kappa(S) = \{c_B(2n) : n \in \omega\}$ and $S \wedge S_\zeta = \mathbf{0}$ for all $\zeta \in \kappa$.*

Proof. Let $A = \mathrm{SS}(T)$.

For each $n \in \omega$ and each $v \in T \cap {}^{c_A(3n)}\kappa$ let us fix a function f_v with domain $\mathrm{Succ}(v, T)$ such that

(i) $f_v(\xi) \in \mathrm{Succ}(\mathrm{sp}_T(v^\frown\xi))$ for each $\xi \in \mathrm{Succ}(v, T)$,

and in addition

(ii) $|\{\xi \in \mathrm{Succ}(v, T) : \mathrm{sp}_T(v^\frown\xi)^\frown f_v(\xi) \in S_\zeta\}| < \kappa$ for each $\zeta < \kappa$.

This is always possible: since $\mathrm{length}(\mathrm{sp}_T(v^\frown\xi)) = c_A(3n+1) = c_B(2n+1)$, the node $\mathrm{sp}_T(v^\frown\xi)$ has at most one immediate successor in every S_ζ and there are only κ many S_ζ's.

Define

$$T' = \bigcup \{T(\mathrm{sp}_T(s_T{}^\frown\xi)^\frown f_{s_T}(\xi)) : \xi \in \mathrm{Succ}(s_T, T)\}$$

and let

$$T^{(0)} = T',$$

$$T^{(n)} = \bigcup \{T'(t) : t \in T^{(n-1)} \cap {}^{c_A(3n)}\kappa\} \quad \text{for } n > 0,$$

and let

$$S = \bigcap_{n \in \omega} T^{(n)} \leqslant T.$$

One can easily check that S is regular $\boldsymbol{S}$-perfect tree and $\mathrm{SS}_\kappa(S)=\{c_B(2n):n\in\omega\}$. Since (by (ii)) each splitting point t of S has less than κ immediate successors in each S_ζ, we have $S\cap S_\zeta=\mathbf{0}$ for each $\zeta<\kappa$. $\square$

Before going further, let us recall a few definitions. Let $(P,\leqslant)$ be a partially ordered set and let $Q\subseteq P$. A set $R\subseteq P$ is said to be a *disjoint refinement* of Q if the family R is pairwise incompatible and for every $q\in Q$ there is an $r\in R$ such that $r\leqslant q$.

Let us consider $\mathcal{P}(\omega)/\mathrm{fin}$. We say that $\Theta=\{H_\xi:\xi\in\mathfrak{h}\}$ is a *base matrix* for $\mathcal{P}(\omega)/\mathrm{fin}$ if H_ξ is a maximal almost disjoint (i.e. MAD) family on ω for each $\xi\in\mathfrak{h}$ and $\bigcup\Theta$ is dense in $\mathcal{P}(\omega)/\mathrm{fin}$. It is shown in [2] that a base matrix for $\mathcal{P}(\omega)/\mathrm{fin}$ always exists.

Lemma 5.4. *Let A be an infinite subset of $\Gamma^{-1}(\{\kappa\})$ and let $\mathcal{P}_A=\{T\in\mathrm{Pf}^\Gamma_\kappa(\boldsymbol{S}):\mathrm{SS}_\kappa(T)=A\}$. Then there exists a disjoint refinement of $\mathcal{P}_A$ of cardinality κ^+.*

Proof. Since $2^\kappa=\kappa^+$, the cardinality of the set $\mathcal{P}_A$ is at most κ^+; so let $\{T_\xi:\xi<\kappa^+\}$ be an enumeration of $\mathcal{P}_A$ (with repetitions, if necessary).

We shall proceed by induction. Let $S_0\leqslant T_0$ be such that S_0 is a regular $\boldsymbol{S}$-perfect tree and $\mathrm{SS}_\kappa(S_0)=\{c_A(2n):n\in\omega\}$.

Let $0<\xi<\kappa^+$ and assume that regular $\boldsymbol{S}$-perfect trees S_ζ, $\zeta<\xi$, have already been chosen such that for each $\zeta<\xi$, $S_\zeta<T_\zeta$, $\mathrm{SS}_\kappa(S_\zeta)=\{c_A(2n):n\in\omega\}$ and $S_\zeta\wedge S_{\zeta'}=\mathbf{0}$ whenever $\zeta\neq\zeta'<\xi$.

Since $\xi<\kappa^+$, find by Lemma 5.3 a regular $\boldsymbol{S}$-perfect tree $S_\xi\leqslant T_\xi$ such that $\mathrm{SS}_\kappa(S_\xi)=\{c_A(2n):n\in\omega\}$ and $S_\xi\wedge S_\zeta=\mathbf{0}$ whenever $\zeta<\xi$.

The family $\{S_\xi:\xi\in\kappa^+\}$ forms a disjoint refinement of $\mathcal{P}_A$. $\square$

Lemma 5.5. *There exists a family $\{\mathcal{A}_\xi:\xi<\mathfrak{h}\}$ of maximal antichains of $\mathrm{Pf}^\Gamma_\kappa(\boldsymbol{S})$ of cardinality κ^+ such that $\bigcup\{\mathcal{A}_\xi:\xi\in\mathfrak{h}\}$ is dense in $\mathrm{Pf}^\Gamma_\kappa(\boldsymbol{S})$.*

Proof. Let $\Theta=\{H_\xi:\xi\in\mathfrak{h}\}$ be a base matrix for $\mathcal{P}(\omega)/\mathrm{fin}$. For each $\xi<\mathfrak{h}$ let $\{H^n_\xi:n\in\omega\}$ be as follows:

$$H^0_\xi=H_\xi,$$
$$H^n_\xi=\{A-\{0,1,\ldots,n-1\}:A\in H_\xi\}\quad\text{for } n>0.$$

It is obvious that each H^n_ξ is a MAD family on ω and if $B\in[\omega]^\omega$ is such that there exists $A\in H_\xi$, $A\subseteq^* B$, then there exists $n\in\omega$ such that $A-\{0,1,\ldots,n-1\}\subseteq B$. Let

$$\bar{\Theta}=\{H^n_\xi:n<\omega,\ \xi\in\mathfrak{h}\}.$$

Now by Lemma 5.4 we can find for each $n\in\omega$, $\xi\in\mathfrak{h}$ and $A\in H^n_\xi$ a disjoint refinement of the set $\mathcal{P}_{A(\kappa)}$ of cardinality κ^+, where $A(\kappa)=\{c_{\Gamma^{-1}(\{\kappa\})}(n):n\in A\}$. Let $\mathcal{R}_A$ be such a refinement; without loss of generality we may assume that

$$\mathcal{R}_A\subseteq\mathrm{Pf}^\Gamma_\kappa(\boldsymbol{S})\quad\text{and}\quad\bigvee\mathcal{R}_A=\bigvee\mathcal{P}_{A(\kappa)}.$$

Let $\mathscr{A}^n_\xi = \bigcup\{\mathscr{R}_A : A \in H^n_\xi\}$. Obviously every $\mathscr{A}^n_\xi$ ($n \in \omega$, $\xi \in \mathfrak{h}$) is a maximal antichain in $\mathrm{Pf}^r_\kappa(\boldsymbol{S})$ and $|\mathscr{A}^n_\xi| = \kappa^+$. It remains to show that $\bigcup\{\mathscr{A}^n_\xi : n \in \omega,\ \xi \in \mathfrak{h}\}$ is dense in $\mathrm{Pf}^r_\kappa(\boldsymbol{S})$.

Let $T \in \mathrm{Pf}^r_\kappa(\boldsymbol{S})$ and let $\mathrm{SS}_\kappa(T) = A$. Θ is a base matrix hence there are $\xi \in \mathfrak{h}$, $n \in \omega$ and $B \in H^n_\xi$ such that $B \subseteq A$. Let $T' \leqslant T$ be such that $\mathrm{SS}_\kappa(T') = B$. Then $B = C(\kappa)$ for some $C \in [\omega]^\omega$ and $\mathscr{R}_C$ is a disjoint refinement of $\mathscr{P}_B$, therefore there exists an $S \in \mathscr{R}_C$ such that $S \leqslant T' \leqslant T$. □

Proof of Theorem 5.1. By Lemma 5.5 we can find a family $\{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}$ of maximal antichains in $\mathrm{Pf}^r_\kappa(\boldsymbol{S})$ such that $\{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}$ is a dense subset of $\mathrm{Pf}^r_\kappa(\boldsymbol{S})$ (and hence of $\mathrm{Pf}(\boldsymbol{S})$). For each tree $T \in \bigcup\{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}$ let $\{S_{\zeta,T} : \zeta \in \kappa^+\}$ be a fixed maximal antichain below T in $\mathrm{Pf}^r_\kappa(\boldsymbol{S})$ (by Lemmas 5.2 and 5.3 it exists).

If G is an M-generic filter on $\mathrm{Pf}(\boldsymbol{S})$ let us define in $M[G]$ a function f as follows:

$$f(\xi) = \zeta \quad \text{iff} \quad T \in \mathscr{A}_\xi \cap G \quad \text{and} \quad S_{\zeta,T} \in G.$$

One can easily check that f is a function from $\mathfrak{h}^M$ onto $(\kappa^+)^M$:

(i) f is a function because every family $\mathscr{A}_\xi$ is a maximal antichain and hence exactly one $T_\xi \in \mathscr{A}_\xi$ belongs to G; if $T \in G$ then the same holds for the $S_{\zeta,T}$'s.

(ii) To prove that f is onto it suffices to show that the set $\{S_{\zeta,T} : T \in \bigcup\{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}\}$ is dense in $\mathrm{Pf}(\boldsymbol{S})$ for each $\zeta \in \kappa^+$. Let S be arbitrary. Since $\bigcup\{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}$ is dense, there exist $\xi \in \mathfrak{h}$ and $T \in A_\xi$ such that $T \leqslant S$; but then $S_{\zeta,T} \leqslant T \leqslant S$. □

Let us remark that the technique for embedding the algebra $\mathscr{P}(\omega)/\mathrm{fin}$ into r.o.($\mathrm{Pf}(\boldsymbol{S})$) described at the beginning of this section does work only for $|K| = 1$; e.g. if $|K| = 2$ then the set A of all even natural numbers is not a splitting set of any $\boldsymbol{S}$-perfect tree. Nevertheless, for every K we can prove the corresponding generalization of Balcar's theorem:

Theorem 5.6. *Let the basic tree be as in subsection* 3.1 *and let* $\boldsymbol{S}$ *be* $(2^{\aleph_0})^+$-*complete. Then* r.o.($\mathscr{P}(\omega)/\mathrm{fin}$) *can be completely embedded into* r.o.($\mathrm{Pf}(\boldsymbol{S})$).

Proof. Fix a $\kappa \in K$ and let $A = \Gamma^{-1}(\{\kappa\}) \in [\omega]^\omega$. For any $X \subseteq \omega$ set

$$\pi(X) = \bigvee\{\tau \in \mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S}) : \mathrm{SS}_\kappa(T) = \{c_A(n) : n \in X\} \bmod \mathrm{fin}\}.$$

One can easily show that π is the required embedding. □

We are not able to prove or disprove a similar result for Namba forcing. On the one hand, in the case of a measurable cardinal κ and a normal prime ideal on κ Namba forcing cannot contain $\mathscr{P}(\omega)/\mathrm{fin}$. On the other hand, we have the following partial result:

Theorem 5.7. *If* $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ *and* $\boldsymbol{S}(v) = [\omega_2]^{\omega_2}$ *for each* $v \in \boldsymbol{T}$ *then* r.o.($\mathrm{Nm}(\boldsymbol{S})$) *contains a complete subalgebra isomorphic to* r.o.($\mathscr{P}(\omega)/\mathrm{fin}$).

Proof. Let M be a transitive model of ZFC such that $M \models 2^{\aleph_0} = \aleph_1$ & $2^{\aleph_1} = \aleph_2$. In M, consider Nm(**S**) and let G be M-generic on Nm(**S**). By Theorem 3.7, we have $\bigcup_{\xi<\omega_1} {}^{\xi}\omega_1 \cap M[G] \subseteq M$ and by Theorem 3.8, $M[G] \models |\aleph_2^M| = \aleph_1$.

Since $\mathcal{P}(\omega) \cap M[G] \subseteq M$, the partially ordered set $P = (\mathcal{P}(\omega)/\text{fin})^{M[G]}$ belongs to M. Let $\mathcal{D} = \{D \in M : D \text{ is dense in } P\}$. Clearly $\mathcal{D} \in M$ and $M \models |\mathcal{D}| = \aleph_2$, hence $M[G] \models |\mathcal{D}| = \aleph_1$.

Now we work in $M[G]$. Let $\mathcal{D} = \{D_\xi : \xi < \omega_1\}$. We proceed by induction. Let $a_0 \in D_0$ be arbitrary. Assume that we have already chosen $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_\zeta \geqslant \cdots$ for $\zeta < \xi < \omega_1$ such that $a_\zeta \in D_\zeta$ for each $\zeta < \xi$. By assumption the sequence $\{a_\zeta : \zeta < \xi\}$ belongs to M. Since P is σ-closed there exists an $a_\xi \in D_\xi$ such that $a_\xi \leqslant a_\zeta$ for each $\zeta < \xi$. Let $H = \{\mathcal{A}_\xi : \xi \in \omega_1\}$. Clearly H is an M-generic set on P and $H \in M[G]$. It follows that $M[H] \subseteq M[G]$ and therefore r.o.$(\mathcal{P}(\omega)/\text{fin})$ can be completely embedded into r.o.(Nm(**S**)) (cf. [13, pp. 265–267]). □

6. Some generic extensions

In this section we shall present some constructions of mainly minimal generic extensions based on the forcing notions introduced in Section 3. Let M be a fixed transitive model of ZFC.

6.1. $|\kappa| = \aleph_0$-minimal extensions

Prikry and Abraham [1] proved that both forcing notions Pf(ω_1) and Nm(ω_1) are $|\aleph_1^M| = \aleph_0$-minimal. As we have already mentioned, under the axiom of constructibility Sacks [22] constructed a $|\kappa| = \aleph_0$-minimal extension for every regular cardinal κ.

Using the results of Section 3 we shall now construct a $|\kappa| = \aleph_0$-minimal extension of a transitive model of ZFC for every uncountable regular cardinal κ such that $\kappa = 2^{\underline{\kappa}}$. Suppose κ is an uncountable regular cardinal and J is a σ-complete ideal on κ possessing the following property

(6.1) (i) $[\kappa]^{<\kappa} \subseteq J$, and
(ii) for each uncountable regular cardinal $\lambda < \kappa$ there is a system $\{A_\xi : \xi < \lambda\} \subseteq J$ such that $\bigcup_{\xi<\lambda} A_\xi = \kappa$ and $\bigcup_{\xi<\zeta} A_\xi \in J$ whenever $\zeta < \lambda$.

Theorem 6.1. *Let κ be an uncountable regular cardinal in M such that $\kappa = 2^{\underline{\kappa}}$. Suppose that the basic tree* (**T**, **R**, **S**) *in M is such that* **S**$(v) = J^+$ *for each* $v \in$ **T**, *where J is a σ-complete ideal on κ satisfying* (6.1). *Let G be an M-generic filter on* Pf(**S**). *Then*

(a) $M[G] \models |\kappa| = \aleph_0$,
(b) $M[G] \supseteq M$ *is a* $|\kappa| = \aleph_0$*-minimal extension.*

Moreover, if $M \models 2^\kappa = \kappa^+$, *then*

(c) κ^+ *is a cardinal in* $M[G]$ (*i.e.* $\kappa^+ = \omega_1^{M[G]}$).

Proof. By Theorem 3.5 every uncountable regular cardinal $\lambda \leq \kappa$ in M is cofinal with ω_0 in $M[G]$, therefore κ must be countable in $M[G]$ and (a) holds.

Since J is σ-complete and $[\kappa]^{<\kappa} \subseteq J$, (b) follows from Theorem 3.4.

If $M \models 2^\kappa = \kappa^+$ and κ^+ fails to be a cardinal in $M[G]$, then κ^+ must be countable in $M[G]$. As the forcing notion $\mathrm{Pf}(\boldsymbol{S})$ is of size κ^+ in M, there exists an M-generic filter H on the collapsing algebra $\mathrm{Col}(\aleph_0, \kappa^+)$ (for the definition see e.g. [13, p. 276]). But this contradicts the $|\kappa| = \aleph_0$-minimality of $M[G]$ and we have proved (c). □

The existence of an ideal J on κ as in Theorem 6.1 is guaranteed by the following lemma.

Lemma 6.2. *Let κ be an uncountable regular cardinal such that $\kappa = 2^{\underline{\kappa}}$. Then there exists a σ-complete ideal satisfying* (6.1).

Proof. We shall construct a σ-complete ideal satisfying (6.1) on a set C of cardinality κ.

For every uncountable regular cardinal μ let

$$R(\mu) = \{\lambda < \mu : \lambda \text{ is an uncountable regular cardinal}\}$$

and

$$C_\mu = \{f \in {}^{R(\mu)}\mu : (\forall \lambda \in R(\mu))\, f(\lambda) \in \lambda\}.$$

The set C is defined as follows:

(i) if $\kappa = \mu^+$ then $C = C_\kappa$,

(ii) if κ is a limit cardinal then

$$C = \{f \in {}^{R(\kappa)}\kappa : (\exists \mu \in R(\kappa))(f \restriction \mu \in C_\mu \;\&\; (\forall \lambda \geq \mu)(\lambda < \kappa \rightarrow f(\lambda) = 0))\}.$$

Now, for each $\lambda \in R(\kappa)$ and each $\xi < \lambda$ set

$$A^\lambda_\xi = \{f \in C : f(\lambda) = \xi\}$$

and define $J \subseteq \mathscr{P}(C)$ by

$$X \in J \quad \text{iff} \quad \text{there are a countable set } Y \subseteq R(\kappa) \text{ and ordinals } \zeta_\lambda < \lambda \text{ for each } \lambda \in Y \text{ such that } X \subseteq \bigcup_{\lambda \in Y} \bigcup_{\xi \in \zeta_\lambda} A^\lambda_\xi.$$

It is easy to verify that J is a σ-complete ideal satisfying (6.1). □

Remark. Note that the assumption $\kappa = 2^{\underline{\kappa}}$ in Theorem 6.1 was needed to ensure the existence of an ideal satisfying (6.1). In the case that $\kappa < \aleph^M_{\omega_1}$ is an uncountable regular cardinal we can use the forcing $\mathrm{Pf}(R(\kappa^+))$, thus avoiding the assumption $\kappa = 2^{\underline{\kappa}}$.

6.2. $\mathrm{cf}(\kappa)=\omega_0$*-minimal extensions*

As can be easily seen the extension constructed in subsection 6.1 is in fact $\mathrm{cf}(\kappa)=\omega_0$-minimal.

By Theorem 3.4 the forcing notion $\mathrm{Pf}(\kappa)$ is $\mathrm{cf}(\kappa)=\omega_0$-minimal for every uncountable regular cardinal κ. Moreover, if $\kappa>\aleph_1$ then $\aleph_1^M$ remains a cardinal in the generic extension, and if $\kappa>2^{\aleph_0}$ and $2^\kappa=\kappa^+$ and κ^+ is collapsed to $\mathfrak{h}$. So, assuming the generalized continuum hypothesis, κ^{++} becomes $\aleph_2$ of the generic extension.

By Theorems 1.1, 3.7 and 3.8 the forcing notion $\mathrm{Nm}(\kappa^+)$ is $\mathrm{cf}(\kappa^+)=\omega_0$-minimal for every uncountable cardinal κ such that $\kappa^{\aleph_0}=\kappa$. On the other hand, Theorem 4.4 gives an example where Namba forcing is not $\mathrm{cf}(\kappa)=\omega_0$-minimal (for a measurable cardinal κ).

Other examples of $\mathrm{cf}(\kappa)=\omega_0$-minimal extensions are discussed in 6.3 and 6.4.

6.3. $|\kappa|=\aleph_1$*-minimal extensions*

Under CH and $2^\kappa=\kappa^+=\lambda$, the forcing notion $\mathrm{Pf}(\kappa)$ gives both a $|\kappa|=\aleph_1$-minimal and a $|\lambda|=\aleph_1$-minimal extension in which $\aleph_1$ is preserved (by Theorems 3.4 and 5.1). We shall now construct a $|\kappa|=\aleph_1$-minimal extension which changes cofinalities of prescribed cardinals and does not collapse $\aleph_1$.

Theorem 6.3. *Assume* GCH. *Let $\kappa>\aleph_1$ be a regular cardinal and let R be a set of regular cardinals larger than $\aleph_1$ and not larger than κ such that R is a disjoint union $\bigcup_{\mu\in A}R_\mu$, where*

(i) *A is an at most countable set of regular cardinals,*

(ii) *for each $\mu\in A$, R_μ is the set of all regular cardinals λ such that $\mu\leqslant\lambda\leqslant\sup R_\mu$ and $\sup R_\mu\in R_\mu$,*

(iii) *$\sup A\in A$ and $\kappa\in \mathrm{R}_{\sup A}$.*

Then there exists a forcing notion P which is

(a) *both $|\kappa^+|=\aleph_1$-minimal and $|\kappa|=\aleph_1$-minimal,*

(b) *for every regular cardinal $\lambda\leqslant\kappa$ in the ground model,* $\mathrm{cf}(\lambda)=\omega_0$ *in the extension if and only if $\lambda\in R$,*

(c) *$\aleph_1$ and cardinals larger than κ^+ are preserved.*

Proof. Let $K=\{\sup R_\mu:\mu\in A\}$. Consider R_μ and let $\nu=\sup R_\mu$. In a similar way as in Theorem 6.1 we construct a μ-complete ideal J_ν on ν such that $J_\nu\supseteq[\nu]^{<\nu}$ and for each regular cardinal λ, $\mu\leqslant\lambda\leqslant\nu$, there exists a system $\{A_\xi:\xi\in\lambda\}\subseteq J_\nu$ such that $\bigcup_{\xi<\lambda}A_\xi=\nu$ and $\bigcup_{\xi<\zeta}A_\xi\in J_\nu$ for each $\zeta<\nu$. Now, using K and $\{J_\nu:\nu\in K\}$ we construct a basic tree $(\boldsymbol{T},\boldsymbol{R},\boldsymbol{S})$ as in subsection 3.1.

By Theorems 3.2, 3.4, 3.5 and 5.1 one can easily see that $\mathrm{Pf}(\boldsymbol{S})$ is the required forcing notion. $\square$

6.4. A strong minimality property

Let us begin with the following simple observation: let $M \subseteq N$ be an extension, let κ be an uncountable cardinal in M such that $N \models |\kappa| = \aleph_0$. Then for any cardinal $\lambda < \kappa$ in M there exists a model W, $M \subseteq W \subseteq N$, such that $\aleph_1^W$ is the successor of λ in M, assuming GCH in M.

We shall show that if we replace $\aleph_0$ by $\aleph_1$ and $\aleph_1^W$ by $\aleph_2^W$ this observation need not be true.

Theorem 6.4. *Suppose $M \models$ GCH and let in M κ be a regular cardinal, $\aleph_1^M < \kappa < \aleph_{\omega_1}^M$. Then there is a forcing notion $P \in M$ such that for every M-generic filter G on P the following holds:*

(a) $M[G] \models \aleph_1 = \aleph_1^M$,

(b) *the successor of κ in M is of size $\aleph_1$ in $M[G]$ and $\kappa^{++} = \aleph_2^{M[G]}$,*

(c) *if W is a model, $M \subseteq W \subseteq M[G]$, such that some cardinal $\mu \in M$, $\aleph_1 < \mu \leqslant \kappa^+$ is of size $\aleph_1$ in W, then $W = M[G]$ (i.e. $M[G]$ is $|\mu| = \aleph_1$-minimal for every μ, $\aleph_1 < \mu \leqslant \kappa^+$).*

Proof. Let $R = \{\mu : \mu$ a regular cardinal in M, $\aleph_1 < \mu \leqslant \kappa\}$. Since $\kappa < \aleph_{\omega_1}$, R can be written in the form used in Theorem 6.3. Let P be the forcing notion used in the proof of Theorem 6.3. Assertions (a) and (b) then follow from Theorem 6.3.

Notice that the splitting criterion S *is* $(2^{\aleph_0})^+$-complete, therefore no new reals are added and the continuum hypothesis holds in $M[G]$.

Now, assume $M \subseteq W \subseteq M[G]$ and for some cardinal μ of M, $\aleph_1^M < \mu \leqslant \kappa^+$, we have $W \models |\mu| = \aleph_1$. Then $\aleph_2^W > \aleph_2^M$. By Theorem 6.3(a) it suffices to show that $\aleph_2^W = \aleph_2^{M[G]}$.

Since $\aleph_2^{M[G]}$ is a regular cardinal in W, it cannot be cofinal with ω_0 in M. So $\aleph_2^{M[G]} = \aleph_{\xi+1}^M$ for some $\xi < \omega_1$. By Theorem 1.7, ξ cannot be a limit (since CH holds both in $M[G]$ and W). Therefore $\aleph_\xi^M$ is a regular cardinal in M. Since $W \models |\aleph_\xi^M| = \aleph_1$, the cofinality of $\aleph_\xi^M$ in W is ω_0 or ω_1.

Assume $\aleph_\xi^M \leqslant \kappa$. Then the cofinality of $\aleph_\xi^M$ in $M[G]$ is $\aleph_0$ (Theorem 6.3(b)). Hence $\mathrm{cf}(\aleph_\xi^M)$ in W is ω_0 as well. By Theorem 1.4 (applied to $M \subseteq W$) we obtain

$$W \models 2^{\aleph_\xi^M} \cap M| \leqslant |\aleph_\xi^M|^{\aleph_0} = \aleph_1,$$

which is a contradiction since

$$W \models |2^{\aleph_\xi^M} \cap M| = \aleph_{\xi+1}^M = \aleph_2.$$

So we have proved that $\aleph_\xi^M > \kappa$ and consequently $\aleph_2^W = \aleph_2^{M[G]}$. □

7. Problems and comments

We present some open questions which are closely related to the results obtained in this paper.

In connection with Theorem 6.3 the following questions arise.

Question 1. Is there a $|\aleph_2| = \aleph_1$-minimal extension which is not a $|\aleph_3| = \aleph_1$-extension?

Question 2. Is there a $|\aleph_3| = \aleph_1$-minimal extension preserving $\aleph_4$ which is not $|\aleph_2| = \aleph_1$-minimal?

In connection with Theorem 6.4 it is natural to ask

Question 3. Can the assumption $\kappa < \aleph_{\omega_1}$ be omitted in Theorem 6.4?

Question 4. For a given model M, is there an extension N such that $\aleph_1^M = \aleph_1^N$ and, say, $\aleph_2^N = \aleph_6^M$ and for any model W, $M \subseteq W \subseteq N$, if $\mathrm{Card}^M \neq \mathrm{Card}^W$ then $W = N$?

A natural candidate for forcing such an extension is the forcing notion of the form used in the proof of Theorem 6.1.

The forcing notion $\mathrm{Pf}(\omega_2, \omega_4)$ is not strongly minimal. More precisely we have the following.

Theorem 7.1. *Assume* GCH. *Then* $\mathrm{r.o.}(\mathrm{Pf}(\omega_2))$ *can be completely embedded in* $\mathrm{r.o.}(\mathrm{Pf}(\omega_2, \omega_4))$.

Remark. Obviously $\mathrm{r.o.}(\mathrm{Pf}(\omega_2))$ cannot be embedded in $\mathrm{r.o.}(\mathrm{Pf}(\omega_2, \omega_3))$.

Proof. Using the assumption $2^{\aleph_2} = \aleph_3$ we shall find a 'nice' dense subset of $\mathrm{Pf}(\omega_2, \omega_4)$. Let us recall that the basic tree $(\boldsymbol{T}, \boldsymbol{R}, \boldsymbol{S})$ is such that for each $v \in \boldsymbol{T}$, $\boldsymbol{R}(v) = \omega_2$ and $\boldsymbol{S}(v) = [\omega_2]^{\omega_2}$ whenever $\mathrm{length}(v)$ is even, and $\boldsymbol{R}(v) = \omega_4$, $\boldsymbol{S}(v) = [\omega_4]^{\omega_4}$ whenever $\mathrm{length}(v)$ is odd.

Let T be a regular $\boldsymbol{S}$-perfect tree with $A = \mathrm{SS}(T)$ such that every even (odd) splitting point is followed by an odd (even) one, i.e. $T \in \mathrm{Pf}^{\mathrm{os}}(\boldsymbol{S})$. Such trees form a dense subset of $\mathrm{Pf}(\omega_2, \omega_4)$. Let v be a splitting point of T such that $n = \mathrm{length}(v)$ is odd. Let m be the first element of A with $m > n$. Then m is even. Let $E = \{k \in \omega\colon k \text{ is even and } n+1 \leqslant k \leqslant m\}$. For $\xi \in \mathrm{Succ}(v, T)$ denote

$$H(v, T, \xi) = \{u \restriction E : \mathrm{length}(u) \geqslant m+2 \;\&\; u \supseteq v^\frown \xi \;\&\; u \in T\}.$$

The tree T is said to be nice if for each splitting point v of odd length n, each ξ, $\eta \in \mathrm{Succ}(v, T)$ we have

$$H(v, T, \xi) = H(v, T, \eta).$$

Since for each $v \in \mathrm{SL}_{2n}(T)$ $(n \in \omega)$ the function H has at most $2^{\aleph_2} = \aleph_3$ possible values, using fusion one can easily show that the set of all nice trees is dense in $\mathrm{Pf}(\omega_2, \omega_4)$.

For every nice tree T we set

$$\psi(T) = \{v \in {}^{<\omega_0}\omega_2 : (\exists u \in T)(\forall k \in \omega)(k < \mathrm{length}(v) \rightarrow v(k) = u(2k))\}.$$

It is easy to see that ψ is a normal mapping from the set of all nice trees onto the set of all regular trees in $\mathrm{Pf}(\omega_2)$; thus ψ induces a complete embedding of r.o.$(\mathrm{Pf}(\omega_2))$ into r.o.$(\mathrm{Pf}(\omega_2, \omega_4))$. $\square$

The following question is rather technical.

Question 5. Let J be an ideal on $\aleph_4$ possessing the property (6.1)(ii) for $\lambda = \aleph_2$ and $\lambda = \aleph_4$ but not for $\lambda = \aleph_3$. Does the forcing notion $\mathrm{Pf}(J)$ change the cofinality of $\aleph_3$ to ω_0?

We finish with a simple but probably difficult problem related to Theorem 1.6.

Question 6. Is there an extension $N \supseteq M$ such that $\aleph_1^N = \aleph_1^M$ and $\aleph_2^N = \aleph_{\omega+1}^M$ (and consequently $N \models 2^{\aleph_0} \geqslant \aleph_2$)?

An affirmative answer to this question yields a solution to the Jensen–Solovay problem [14] on violating CH by adding a real (compare the solution by Shelah and Woodin [24]).

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