Available online at www.sciencedirect.com
ScienceDirect
Annals of Pure and Applied Logic 144 (2006) 73–82
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Uncountable superperfect forcing and minimality

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Available online 14 November 2006

Abstract

Uncountable superperfect forcing is tree forcing on regular uncountable cardinals κ with $\kappa^{<\kappa} = \kappa$, using trees in which the heights of nodes that split along any branch in the tree form a club set, and such that any node in the tree with more than one immediate extension has measure-one-many extensions, where the measure is relative to some κ-complete, nonprincipal normal filter (or p-filter) F. This forcing adds a generic of minimal degree if and only if F is κ-saturated.

Keywords: Uncountable cardinals; Tree forcing; Minimality

In [1], Elizabeth Theta Brown defined a generalization of Miller forcing [5] to uncountable cardinals κ. Among other things, she showed that in certain cases this forcing adds a generic sequence of minimal degree over the ground model. We will extend that result and prove a partial converse.

Miller forcing conditions are ω-trees, subtrees of ${}^{<\omega}\omega$, with the property that every node has either a single immediate successor or infinitely many immediate successors. In the second case, we say the node splits in the tree, or is a splitting node; a further requirement for a tree to be a condition is that every node in the tree has an extension that splits in the tree.

In Brown's generalization of this forcing, conditions are κ-trees, and a splitting node must have not just infinitely many successors, but measure-one many as determined by some filter F on κ. More precisely, assume that κ is a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ and F is a κ-complete nonprincipal filter on κ. We require that if p is a condition in $\mathbb{P}$ and s splits in p, then

$$\{\alpha \in \kappa \mid s^\frown\alpha \in p\} \in F,$$

where $s^\frown\alpha$ denotes the concatenation of s with $\langle\alpha\rangle$. There are further requirements, which we will specify later, on the density of splitting nodes in p. Brown shows that if F is a normal ultrafilter on κ, then $\mathbb{P}$ adds a minimal degree over the ground model [1].

In Theorem 7, we show that if F is not κ-saturated (that is, if it is possible to partition κ into κ-many disjoint sets of F-positive measure), then $\mathbb{P}$ does not add a minimal degree over the ground model. If κ carries a κ-saturated, κ-complete nonprincipal filter, then κ must be measurable in an inner model [4], so in most cases $\mathbb{P}$ will not add a minimal degree.

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doi:10.1016/j.apal.2006.05.012

The proof of this theorem proceeds by showing that $\mathbb{P}$ adds a Cohen generic subset of κ. In Theorem 10 we show that this is the only way $\mathbb{P}$ can fail to add a minimal degree; specifically, we show that any set in the generic extension that is not of the same degree as the generic can actually be added by Cohen forcing over κ.

In the opposite direction, in Theorem 14, we extend Brown's result for normal ultrafilters to show that if F is normal (or even a p-filter), and F is κ-saturated, then $\mathbb{P}$ does add a minimal degree over the ground model. Thus, in the case that F is normal (or a p-filter), κ-saturation is a necessary and sufficient condition for the $\mathbb{P}$-generic to be of minimal degree.

We have described $\mathbb{P}$ as a generalization of Miller forcing. However, Miller forcing can easily be shown to add a generic of minimal degree, and in most cases $\mathbb{P}$ does not do so. A major difference between the two forcing notions, which plays out here, is the nature of the splitting sets. In Miller forcing, splitting sets (the immediate successors of a splitting node) are required only to be infinite, that is, to have positive measure according to the cofinite filter; in $\mathbb{P}$, splitting sets must have measure one according to the filter F. The key distinction is between positive measure and measure one. If F is an ultrafilter, of course, measure one and positive measure coincide; and it is only when F is very close to being an ultrafilter (when F is κ-saturated) that $\mathbb{P}$ can add a generic of minimal degree.

To make a closer analogy, we should consider variants of Miller forcing in which splitting sets are required to be measure one according to some filter on ω. Groszek has investigated the question of when such forcings add generics of minimal degree; some results in this paper are generalizations to κ of results in [3].

There is a second difference between Miller forcing and $\mathbb{P}$. Along any cofinal branch through a condition in $\mathbb{P}$, the (lengths of) splitting nodes are club; in particular, they are measure one according to the club filter. Along any cofinal branch through a Miller condition, the splitting nodes are infinite, that is, positive measure according to the cofinite filter. This would lead us to expect $\mathbb{P}$ to be more similar in some ways to Laver forcing on ω, as along a cofinal branch through a Laver condition the splitting nodes are in fact cofinite.

As regards the question of minimality, $\mathbb{P}$ is closer to (the variants of) Miller forcing on ω than to Laver forcing; the similarity to Laver forcing becomes apparent when we consider questions of bounding. Laver forcing adds a generic real that dominates every ground model real on a cofinite set, while a Miller generic merely dominates ground model reals on an infinite set. The $\mathbb{P}$-generic dominates every ground model κ-sequence on a club set. The connection here is closer than the analogy between cofinite and club as both being measure one sets; Cummings and Shelah have shown that if κ is large enough, the bounding and dominating numbers on κ are the same as the club bounding and dominating numbers [2].

1. Preliminaries

Throughout this paper we assume that κ is an uncountable regular cardinal and $\kappa^{<\kappa} = \kappa$.

We let F denote a filter on κ that is nonprincipal (for $\alpha \in \kappa$, we have $\kappa - \{\alpha\} \in F$) and κ-complete (closed under intersections of size less than κ: if $\{X_\gamma \mid \gamma < \alpha\}$ is a subset of F of size $\alpha < \kappa$, then $\bigcap\{X_\gamma \mid \gamma < \alpha\} \in F$). The property of κ-completeness is necessary to ensure that the forcing $\mathbb{P}$ is κ-closed and therefore preserves κ as a regular cardinal. We sometimes refer to sets in F as measure one sets, sets in the dual ideal as measure zero sets, and sets not in the dual ideal as positive measure sets.

The filter F is normal if it is closed under diagonal intersections of κ-sequences: if $\langle X_\gamma \mid \gamma < \kappa\rangle$ is a sequence from F, then the diagonal intersection

$$\{\beta \mid (\forall\gamma < \beta)\ [\beta \in X_\gamma]\},$$

is in F. A weaker property than normality is being a p-filter: The filter F is a p-filter if whenever $\langle X_\gamma \mid \gamma < \kappa\rangle$ is a sequence from F, there is a set $X \in F$ with the property that X is almost contained in every X_γ:

$$(\forall\gamma)\ [|X - X_\gamma| < \kappa].$$

Of course, F is an ultrafilter if κ cannot be partitioned into two disjoint sets of F-positive measure. The filter F is κ-saturated if κ cannot be partitioned into κ-many disjoint sets of F-positive measure. (This is equivalent to the usual definition of κ-saturation under our assumptions on F and κ.)

We use the filter F to define a forcing partial order $\mathbb{P}$. This forcing was defined by Brown in [1]. In the rest of this section we restate some key definitions and properties of the forcing, mostly without proof.

Definition 1. A condition in $\mathbb{P}$ is a tree $p \subseteq {}^{<\kappa}\kappa$ satisfying the following properties:

1. The tree p is downward closed (which is basically what we mean by tree): if $s \in p$ and r is an initial segment of s, then $r \in p$.
2. Every element (node) of p, viewed as a sequence from κ, is strictly increasing.
3. The tree p is closed under limits of sequences of length less than κ: If $\langle s_\gamma \mid \gamma < \alpha \rangle$, for $\alpha < \kappa$, is an increasing sequence of nodes of p, then the limit $\bigcup\{s_\gamma \mid \gamma < \alpha\}$ is also in p.
4. For $s \in p$, we let $E_s^p = \{\alpha \mid s^\frown\alpha \in p\}$, where $s^\frown\alpha$ denotes the concatenation of s with $\langle\alpha\rangle$. Then for all $s \in p$, E_s^p is either a singleton or an element of F. In the second case, we say that s splits in p, or is a splitting node.
5. Every $s \in p$ has an extension that splits in p.
6. If $\langle s_\gamma \mid \gamma < \alpha \rangle$, for $\alpha < \kappa$, is an increasing sequence of splitting nodes of p, then the limit $\bigcup\{s_\gamma \mid \gamma < \alpha\}$ also splits in p.

The partial ordering $\mathbb{P}$ is ordered by $p \leq q \iff p \subseteq q$.

That is, the conditions in $\mathbb{P}$ are trees, consisting of sequences from κ of length less than κ, satisfying certain closure and branching conditions. Stronger conditions are subtrees. A cofinal branch through a condition p, that is, a maximal linearly-ordered subset of p, can be identified with a function from κ to κ. If $G \subseteq \mathbb{P}$ is a generic set, we identify G with a "generic function" $g : \kappa \to \kappa$, which is the unique cofinal branch common to all the trees in G.

Proposition 2. *The partial ordering $\mathbb{P}$ is κ-closed: If $\langle p_\gamma \mid \gamma < \alpha \rangle$ is a decreasing sequence from $\mathbb{P}$ of length $\alpha < \kappa$, then its limit $\bigcap \langle p_\gamma \mid \gamma < \alpha \rangle$ is a condition in $\mathbb{P}$.*

A consequence of this proposition is that $\mathbb{P}$ preserves cardinals up to and including κ. If $2^\kappa = \kappa^+$, preservation of κ^{++} and larger cardinals follows by standard counting arguments. Preservation of κ^+ relies on a fusion argument: Although $\mathbb{P}$ is not closed under limits of κ sequences, it is closed under limits of certain κ sequences called fusion sequences, and this is enough to show that κ^+ is preserved (which we will not show here). The fusion argument is also a critical tool in proving minimality of the generic. In the rest of this section we give the definitions and facts we will need for this purpose.

Definition 3. If p is a condition in $\mathbb{P}$, trunk(p) is the minimal node that splits in p.

If $s \in p$, p_s is the maximal subtree of p whose trunk extends s;

$$p_s = \{t \in p \mid t \subseteq s \ \vee \ s \subset t\}.$$

The set of splitting nodes of p is

$$\text{split}(p) = \{s \mid s \text{ splits in } p\}.$$

The αth splitting level of p, $\text{split}_\alpha(p)$, is defined by

$$\text{split}_0(p) = \{\text{trunk}(p)\},$$

$$\text{split}_{\alpha+1}(p) = \{s \in \text{split}(p) \mid (\exists t \in \text{split}_\alpha(p))[t \subset s \ \& \ (\forall r)\ [t \subset r \subset s \implies r \notin \text{split}(p)]]\},$$

and for λ a limit ordinal,

$$\text{split}_\lambda(p) = \left\{s \mid (\exists\{s_\alpha \mid \alpha < \lambda\})[((\forall\alpha < \lambda)\ [s_\alpha \in \text{split}_\alpha(p)]) \ \& \ s = \bigcup\{s_\alpha \mid \alpha < \lambda\}]\right\}.$$

Finally, we define

$$p \leq_\alpha q \iff p \leq q \ \& \ \text{split}_\alpha(p) = \text{split}_\alpha(q).$$

Note that $p \leq_\alpha q$ means that p is a subtree of q, p and q are the same up to the αth splitting level, and every node in $\text{split}_\alpha(q)$ also splits in p. In particular, $p \leq_0 q$ means that p is a subtree of q with the same trunk.

Proposition 4 (*Fusion Lemma*). *Let* LOR *be the class of limit ordinals. If $\langle p_\gamma \mid \gamma < \kappa \rangle$ is a fusion sequence from $\mathbb{P}$, that is, if it satisfies:*

$$(\forall\gamma)\ [p_{\gamma+1} \leq_\gamma p_\gamma]$$

$$(\forall\alpha \in \text{LOR})\ \left[p_\alpha = \bigcap \langle p_\gamma \mid \gamma < \alpha \rangle\right],$$

then its limit or fusion, $p = \bigcap \langle p_\gamma \mid \gamma < \kappa \rangle$, *is a condition in* $\mathbb{P}$. *Furthermore,*

$$(\forall \gamma)\ \left[p \leq_\gamma p_\gamma \right].$$

Proposition 5. *If* $p \in \mathbb{P}$ *and* $\{q(s) \mid s \in \text{split}_\alpha(p)\}$ *is a collection of conditions such that*

$$(\forall s \in \text{split}_\alpha(p))\ [q(s) \leq_0 p_s],$$

then

$$q = \bigcup \{q(s) \mid s \in \text{split}_\alpha(p)\} \leq_\alpha p.$$

Furthermore, for $s \in \text{split}_\alpha(p)$, $q_s = q(s)$.

Proposition 6 incorporates into a single proposition the applications of the fusion method we will need.

Proposition 6. *If* φ *is a property of conditions satisfying*

$$(\forall p)\ (\exists q \leq_0 p)\ [\varphi(q)]\ \ \&$$
$$(\forall q)\ (\forall r \leq_0 q)\ [\varphi(q) \implies \varphi(r)],$$

then $\{q \mid (\forall s \in \text{split}(q))\ [\varphi(q_s)]\}$ *is a dense subset of* $\mathbb{P}$.

Proof. Given $p \in \mathbb{P}$, produce the desired $q \leq p$ by constructing a fusion sequence $\langle p_\gamma \mid \gamma < \kappa \rangle$. Let

$$p_0 = p,$$

and for $\alpha \in LOR$,

$$p_\alpha = \bigcap \langle p_\gamma \mid \gamma < \alpha \rangle.$$

Given p_γ, for each $s \in \text{split}_\gamma(p_\gamma)$, choose $q(s) \leq_0 (p_\gamma)_s$ with the property $\varphi(q(s))$. Then by Proposition 5, we can set

$$p_{\gamma+1} = \bigcup \{q(s) \mid s \in \text{split}_\gamma(p_\gamma)\} \leq_\gamma p_\gamma.$$

By construction, we have

$$(\forall s \in \text{split}_\gamma(p_{\gamma+1}))\ [\varphi((p_{\gamma+1})_s)].$$

Now apply Proposition 4 to set

$$q = \bigcap \langle p_\gamma \mid \gamma < \kappa \rangle.$$

For every $\gamma < \kappa$, since $q \leq_\gamma p_{\gamma+1}$, it follows that

$$(\forall s \in \text{split}_\gamma(q))\ [q_s \leq_0 (p_{\gamma+1})_s],$$

so by the properties of φ,

$$(\forall s \in \text{split}_\gamma(q))\ [\varphi(q_s)],$$

and the condition q has the desired properties.

Note: Not only is it the case that $q \leq_\gamma p_{\gamma+1}$, but actually $q \leq_{\gamma+1} p_{\gamma+1}$; it follows from this that for every s in $\text{split}_\gamma(q)$, $E^q_s = E^{p_{\gamma+1}}_s$. Therefore, if instead of

$$(\forall q)\ (\forall r \leq_0 q)\ [\varphi(q) \implies \varphi(r)],$$

φ satisfies the weaker property

$$(\forall q)\ (\forall r \leq_0 q)\ [(\varphi(q)\ \&\ E^q_{\text{trunk}(q)} = E^r_{\text{trunk}(q)}) \implies \varphi(r)],$$

this proposition still holds. □

If x and y are subsets of the ground model M in the generic extension $M[g]$, we define $x \leq_M y \iff x \in M[y]$; this induces the ordering of M-degrees, or degrees over M, on all subsets of M in $M[g]$. The generic g is of minimal M-degree if, for every $x \subset M$ in $M[g]$, either $x \in M$ or $g \in M[x]$.

Every $x \subset M$ in $M[g]$ realizes a term τ such that $1_{\mathbb{P}} \Vdash$ "$\tau \subset M$". This means that in considering subsets of M in the generic extension, we need only consider such terms, "terms for subsets of M". We will sometimes blur the distinction between elements of $M[g]$ and terms.

2. Non-minimality

Theorem 7. *Suppose that κ can be partitioned into κ-many disjoint F-positive measure sets. Then forcing with $\mathbb{P}$ adds a Cohen generic subset of κ.*

In particular, this implies that the $\mathbb{P}$-generic g is not of minimal degree over the ground model, as the even and odd parts of a Cohen generic are of incomparable degree over the ground model.

Proof. Let $\mathbb{Q}$ denote the forcing to add a Cohen generic subset of κ; conditions in $\mathbb{Q}$ are sequences s in ${}^{<\kappa}\kappa$, ordered by end-extension. Note that $\mathbb{Q}$ is κ-closed and, by assumption on κ, has size κ.

By assumption, we can partition κ into κ-many disjoint sets of positive measure, which we can index by elements of ${}^{<\kappa}\kappa$:

$$\kappa = \bigcup\{X_s \mid s \in {}^{<\kappa}\kappa\}.$$

From the $\mathbb{P}$-generic g, we define a new sequence $f[g]$ as follows. Given $\gamma < \kappa$, we let

$$f(\gamma) = s \iff \gamma \in X_s,$$

and for any function $h : \alpha \to \kappa$, $\alpha \leq \kappa$, we let $f[h]$ be the concatenation of

$$\langle f(h(\beta)) \mid \beta < \alpha\rangle.$$

Because of the regularity of κ, $f[g]$ is a κ-length sequence, so $f[g]$ has the correct form to be a $\mathbb{Q}$-generic.

To show that $\mathbb{P}$ forces $f[g]$ to be a $\mathbb{Q}$-generic, it suffices to show that for every $p \in \mathbb{P}$ and every dense set $D \subseteq \mathbb{Q}$, there is an extension $r \leq p$ such that

$$r \Vdash \text{“}f[g] \text{ meets } D\text{”}.$$

Note that if q is a condition in $\mathbb{P}$ and $t \subseteq \operatorname{trunk}(q)$, then

$$q \Vdash \text{“}f[t] \subseteq f[g]\text{”}.$$

Let p and D be given. Let s be the trunk of p; then $f[s] \in \mathbb{Q}$. Because D is dense in $\mathbb{Q}$, there is a condition $r \supset f[s]$ such that $r \in D$. We can write $r = f[s]^\frown u$ for some $u \in {}^{<\kappa}\kappa$.

Now because X_u is of positive measure and E_s^p is of measure one, there is some $\alpha \in X_u \cap E_s^p$. Let $q = p_{s^\frown\alpha}$. Now $f[s^\frown\alpha] = f[s]^\frown f(\alpha) = r$, so

$$q \Vdash \text{“}r \subset f[g]\text{”},$$

as desired. $\square$

Theorem 10 shows that adding a Cohen subset of κ is essentially the only way in which $\mathbb{P}$ can fail to add a minimal degree. Specifically, we show that any set of intermediate degree between the ground model and the $\mathbb{P}$-generic can be added by Cohen forcing.

Lemma 9 isolates a strategy that is used in showing that a set is not of intermediate degree. It will be useful in the next section as well as in the proof of Theorem 10.

Definition 8. If τ is a term for a subset of M, and p and q are conditions, we say that $p \perp_\tau q$ if

$$(\exists x \in M)\ [(p \Vdash x \in \tau\ \&\ q \Vdash x \notin \tau) \ \vee\ (p \Vdash x \notin \tau\ \&\ q \Vdash x \in \tau)].$$

That is, $p \perp_\tau q$ if p and q force incompatible facts about τ. If this is the case, then by knowing $\tau[G]$, the realization of τ in $M[G]$, we can distinguish which of the alternatives $p \in G$ or $q \in G$ can possibly be true.

Lemma 9. *If τ is a term for a subset of the ground model M, and $p \in \mathbb{P}$ has the property:*

$$(\forall s \in \text{split}(p))\ (\forall \alpha \neq \beta \in E_s^p)\ [p_{s^\frown\alpha} \perp_\tau p_{s^\frown\beta}],$$

then p forces that $g \in M[\tau]$.

Proof. In this case, whenever s splits in p, and r and t are two different immediate extensions of s in p, we have that p_r and p_t force incompatible facts about τ; thus, if we know g is a generic branch through p and $s \subset g$, from τ we can identify the unique immediate extension of s contained in g. In this way we can use τ to trace the generic branch through p, determining which way g turns at every splitting node. More precisely, p forces that

$$g = \bigcup\{s \in p \mid (\forall x)\ [(p_s \Vdash \text{“}x \in \tau\text{”} \Longrightarrow x \in \tau)\ \&\ (p_s \Vdash \text{“}x \notin \tau\text{”} \Longrightarrow x \notin \tau)]\}. \quad \square$$

Theorem 10. *If τ is any element of $M[g]$, then either $g \in M[\tau]$, $\tau \in M$, or τ is added by a κ-closed forcing of size κ.*

We will call a κ-closed forcing of size κ a κ-Cohen forcing. By “τ is added by a κ-Cohen forcing”, we mean that there is a κ-Cohen forcing in M that is equivalent to a two-step iteration $\mathbb{R}_1 \star \mathbb{R}_2$ such that τ is equivalent to the $\mathbb{R}_1$-generic. In particular, by general forcing technology, if $G_\mathbb{C}$ is a κ-Cohen generic, every subset of M in $M[G_\mathbb{C}]$ is added by a κ-Cohen forcing.

Proof. Suppose that τ is a term for a subset of M in $M[g]$ that is not in M and not added by a κ-Cohen forcing. Beginning with a condition p, we find $q \leq p$ such that q forces $g \in M[\tau]$.

We know, by general forcing technology, that τ is equivalent to a generic for some partial ordering $\mathbb{Q}$, so we can safely assume τ denotes a $\mathbb{Q}$-generic. We can also assume that the $\mathbb{Q}$-generic is forced (by $1_\mathbb{Q}$) not to be added by a κ-Cohen forcing. (This is because “$G_\mathbb{Q}$ is added by a κ-Cohen forcing” can be evaluated in $M[G_\mathbb{Q}]$.)

Claim 1: If p is a condition with trunk s, then there is a condition $r \leq_0 p$ such that one of the following two conditions holds:

1. $(\forall \alpha \neq \beta \in E_s^r)\ [r_{s^\frown\alpha} \perp_\tau r_{s^\frown\beta}]$.
2. $(\forall m.a.c.\ A \subseteq \mathbb{Q})\ (\exists \alpha \in E_s^r)\ (\exists x \in A)\ [r_{s^\frown\alpha} \Vdash \text{“}x \in \tau\text{”}]$, where $m.a.c.$ denotes “maximal antichain”.

Proof of Claim 1: Enumerate $E_s^p = \{\eta(\alpha) \mid \alpha < \kappa\}$. By induction on α produce conditions $r^\alpha(\beta) \leq p_{s^\frown\eta(\beta)}$ for $\beta \geq \alpha$:

Set $r^0(\beta) = p_{s^\frown\eta(\beta)}$, and if $\alpha \in LOR$ has been reached, for $\beta \geq \alpha$ set $r^\alpha(\beta) = \bigcap\{r^\gamma(\beta) \mid \gamma < \alpha\}$.

If, for $\alpha < \kappa$, the condition $r^\alpha = \bigcup\{r^\alpha(\beta) \mid \beta \geq \alpha\}$ satisfies condition 2, then set $r = r^\alpha$; this is the desired condition.

Otherwise, as condition 2 fails, we can choose a maximal antichain $A \subseteq \mathbb{Q}$ such that

$$(\forall \beta \geq \alpha)(\forall x \in A)[r^\alpha(\beta) \not\Vdash \text{“}x \in \tau\text{”}].$$

Because τ is forced to be $\mathbb{Q}$-generic, we can choose $x(\alpha) \in A$ and $r_{s^\frown\eta(\alpha)} \leq r^\alpha(\alpha)$ so that

$$r_{s^\frown\eta(\alpha)} \Vdash \text{“}x(\alpha) \in \tau\text{”},$$

and, by choice of A, for $\beta > \alpha$ we can choose $r^{\alpha+1}(\beta) \leq r^\alpha(\beta)$ so that

$$r^{\alpha+1}(\beta) \Vdash \text{“}x(\alpha) \notin \tau\text{”}.$$

If we are in this (“otherwise”) case for all $\alpha < \kappa$, then

$$r = \bigcup\{r_{s^\frown\eta(\alpha)} \mid \alpha < \kappa\}$$

is the desired condition: For $\alpha < \beta$ we have

$$r_{s^\frown\eta(\alpha)} \Vdash \text{“}x(\alpha) \in \tau\text{”},$$
$$r_{s^\frown\eta(\beta)} \leq r^{\alpha+1}(\beta) \Vdash \text{“}x(\alpha) \notin \tau\text{”}.$$

Claim 2: Any condition p can be extended to have the property that for every $s \in \text{split}(p)$, p_s has the property of r in Claim 1, i.e., for each p_s either condition 1 or condition 2 holds. This follows from Proposition 6 and, in particular, the note at the end of its proof.

Claim 3: Given such p, suppose condition 2 holds densely:

$$\{s \in \text{split}(p) \mid (\forall m.a.c.\ A \subseteq \mathbb{Q})(\exists \alpha \in E_s^p)(\exists x \in A)[p_{s^\frown\alpha} \Vdash \text{“}x \in \tau\text{”}]\}$$

is dense in p. We can view the tree p as a κ-closed partial ordering of size κ, with conditions being nodes of p and stronger conditions being extensions. By our supposition, forcing with p adds a $\mathbb{Q}$-generic: If G_p is a generic subset of p, then a $\mathbb{Q}$-generic is generated by

$$\{x \in \mathbb{Q} \mid (\exists \sigma \in G_p)\ [p_\sigma \Vdash_{\mathbb{P}} \text{“}x \in \tau\text{”}]\}.$$

This is a contradiction, since the $\mathbb{Q}$-generic is forced not to be added by a κ-closed forcing of size κ.

Claim 4: Therefore, we can choose $t \in p$ such that

$$(\forall s \supseteq t)\ [s \in \text{split}(p) \implies (\forall \alpha \neq \beta \in E_s^p)\ [p_{s^\frown\alpha} \perp_\tau p_{s^\frown\beta}]].$$

By Lemma 9, $q = p_t$ forces that $g \in M[\tau]$. □

The $\mathbb{P}$-generic g, in contrast, cannot be added by κ-Cohen forcing. This is because κ-Cohen forcing has the κ^+ chain condition (every antichain has size at most κ) but below every $p \in \mathbb{P}$ there is an antichain of size 2^κ. This means that g, even if not of minimal degree over M, has a certain minimality property; g cannot be added by κ-Cohen forcing over M, while every set of smaller M-degree can.

3. Minimality

In the last section, we showed that if F is not κ-saturated, then $\mathbb{P}$ does not add a minimal degree. Throughout this section we will assume that F is κ-saturated, that is, κ cannot be partitioned into κ-many disjoint sets of F-positive measure. We will show that if F is normal, or even simply a p-filter, then $\mathbb{P}$ does add a minimal degree. This extends Brown's result in [1] for the case when F is a normal ultrafilter.

Lemma 11. *Suppose that whenever τ is a term for a subset of M that is not an element of M, and p is a condition with trunk s, then there is a condition $q \leq_0 p$ such that $\varphi(q)$:*

$$(\forall \alpha \neq \beta \in E_s^q)\ [q_{s^\frown\alpha} \perp_\tau q_{s^\frown\beta}] \text{ where } s = \text{trunk}(q).$$

Then $\mathbb{P}$ adds a minimal degree over the ground model M.

Proof. Let τ be any term for a subset of M that is not an element of M. By Proposition 6, the set of conditions p such that $(\forall s \in \text{split}(p))\ [\varphi(p_s)]$ is dense in $\mathbb{P}$. But by Lemma 9, such a condition forces that $g \in M[\tau]$. Therefore, for any $\tau \subseteq M$, either $\tau \in M$ or $g \in M[\tau]$. □

When F is a normal ultrafilter, we can $\leq_0$ extend any condition p to a condition q with the property $\varphi(q)$ of Lemma 11 as follows:

Let α be the smallest element of E_s^p and x be such that $p_{s^\frown\alpha}$ has not decided “$x \in \tau$”. Extend each $p_{s^\frown\delta}$ for $\delta > \alpha$ to decide “$x \in \tau$”; since F is an ultrafilter, we can shrink E_s^p to a measure one set on which each $p_{s^\frown\delta}$ decides “$x \in \tau$” the same way. Extend $p_{s^\frown\alpha}$ to decide “$x \in \tau$” in the opposite way. Now we have a condition in which $p_{s^\frown\alpha} \perp_\tau p_{s^\frown\delta}$, where α is the least element of E_s^p and δ is any larger element.

By applying this same argument, we can take care of each $\alpha \in E_s^p$ in turn, at the cost of shrinking E_s^p each time; this produces a nested sequence of measure one sets. Using the normality of F, we see that the diagonal intersection X of this sequence is a measure one set itself; the condition $\bigcup\{p_{s^\frown\alpha} \mid;\ \alpha \in X\}$ has the properties we want.

Lemma 12 below carries out the first part of this argument in the case that F is not necessarily an ultrafilter but merely κ-saturated. Lemma 13 carries out the second part of the argument in the case that F is not necessarily normal but merely a p-filter.

Lemma 12. *If τ is a term for a subset of M that is not an element of M, p is a condition with trunk s, and $\alpha \in E_s^p$, then there are a measure one set $X \subseteq E_s^p$ with $\alpha \notin X$ and a collection of conditions $\{r_\delta \leq p_{s^\frown\delta} \mid \delta \in E_s^p\}$ such that*

$$(\forall\delta \in X)\ (\forall\sigma \in E_s^p - X)\ [r_\delta \perp_\tau r_\sigma].$$

Proof. We will inductively define

$$\langle q_\gamma, X_\gamma \mid \gamma < \rho\rangle$$

with certain properties.

In particular, the X_γ will be disjoint positive measure sets. At each stage β we will define

$$Y_\beta = \left(E_s^p - \{\alpha\}\right) - \bigcup\{X_\gamma \mid \gamma < \beta\},$$

and as long as Y_β has positive measure, we will choose $X_\beta \subseteq Y_\beta$ to have positive measure; if Y_β has measure zero, we will terminate construction of the sequence, setting $\rho = \beta$. By construction the X_γ are pairwise disjoint positive measure sets, so by assumption on F, we must have $\rho < \kappa$.

We will choose the q_γ to be extensions of $p_{s^\frown\alpha}$, such that

$$\beta > \gamma \implies q_\beta \leq q_\gamma.$$

The condition $r_\alpha \leq p_{s^\frown\alpha}$ will be the limit of the q_γ, which will exist by κ-closure of the forcing.

At each stage β, we will try to make q_β disagree with our candidates for r_δ on some fact about τ; X_β will be the set of δ for which we have succeeded.

Stage β of the construction:

For $\beta = 0$, let $Y_0 = E_s^p - \{\alpha\}$, and let $\overline{q}_0 = p_{s^\frown\alpha}$.

For $\beta > 0$, let $Y_\beta = (E_s^p - \{\alpha\}) - \bigcup\{X_\gamma \mid \gamma < \beta\}$ and $\overline{q}_\beta = \bigcap\{q_\gamma \mid \gamma < \beta\}$.

If Y_β has positive measure, then proceed as follows. Because τ is forced not to be in the ground model, there is some x for which $\overline{q}_\beta$ does not decide "$x \in \tau$". Define

$$Z_0 = \{\delta \in Y_\beta \mid p_{s^\frown\delta} \not\Vdash \text{“}x \notin \tau\text{”}\},$$
$$Z_1 = \{\delta \in Y_\beta \mid p_{s^\frown\delta} \not\Vdash \text{“}x \in \tau\text{”}\}.$$

Since $Z_0 \cup Z_1 = Y_\beta$, one of Z_0 and Z_1 has positive measure; suppose it is Z_0. (If not, then Z_1 has positive measure, and we proceed symmetrically.) Let $X_\beta = Z_0$. For $\delta \in X_\beta$, choose

$$(r_\delta \leq p_{s^\frown\delta})\ [r_\delta \Vdash \text{“}x \in \tau\text{”}].$$

Choose

$$(q_\beta \leq \overline{q}_\beta)\ [q_\beta \Vdash \text{“}x \notin \tau\text{”}],$$

so that for $\delta \in X_\beta$ we have $r_\delta \perp_\tau q_\beta$. Notice that if $\sigma \in Y_\beta - X_\beta$, then $\sigma \in Z_1$, so $p_{s^\frown\sigma} \Vdash x \notin \tau$ and we have $r_\delta \perp_\tau p_{s^\frown\sigma}$ as well.

This completes stage β, provided that Y_β has positive measure.

As noted above, the X_β are disjoint positive measure sets. Since κ cannot be partitioned into κ-many disjoint positive measure sets, this construction must halt at some stage before κ; for some $\beta < \kappa$ we have that Y_β has measure zero.

When this happens, terminate the construction of the sequence, setting $\rho = \beta$, and proceed as follows.

Since $E_s^p - \{\alpha\}$ has measure one and Y_β has measure zero,

$$(E_s^p - \{\alpha\}) - Y_\beta = \bigcup\{X_\gamma \mid \gamma < \beta\}$$

has measure one. Let

$$X = \bigcup\{X_\gamma \mid \gamma < \beta\},$$
$$r_\alpha = \overline{q}_\beta = \bigcap\{q_\gamma \mid \gamma < \beta\}.$$

Choose any $\delta \in X$. By construction, there is some γ for which $\delta \in X_\gamma$.

Since $r_\alpha \le q_\gamma$, we have $r_\delta \perp_\tau r_\alpha$.

If $\sigma \in E_s^p - X$ and $\sigma \neq \alpha$, then σ is in every $Y_\gamma - X_\gamma$, and therefore we have $r_\delta \perp_\tau p_{s^\frown\sigma}$, and so we can set $r_\sigma = p_{s^\frown\sigma}$ to complete the proof. $\square$

Lemma 13. *Suppose that F is a p-filter, τ is a term for a subset of M that is not in M, and p is a condition with trunk s. Then there is a condition $q \le_0 p$ with the property:*

$$(\forall \alpha \neq \beta \in E_s^q)\ [q_{s^\frown\alpha} \perp_\tau q_{s^\frown\beta}].$$

Proof. We will use Lemma 12 to build a nested sequence of measure one sets X_β with empty intersection, and conditions $\overline{q}_\delta \le p_{s^\frown\delta}$, such that whenever $\delta \in X_\beta$ and $\gamma \notin X_\beta$, then $\overline{q}_\delta$ and $\overline{q}_\gamma$ force incompatible facts about τ. Then we will use the p-filter property of F to find a measure one set X that intersects each $X_{\beta+1} - X_\beta$ in a set Y_β of size less than κ. If δ and γ are in different Y_β, then $\overline{q}_\delta$ and $\overline{q}_\gamma$ force incompatible facts about τ. Finally, we will use the small size of the Y_β and the κ-closure of the forcing to further extend the $\overline{q}_\delta$ to q_δ such that if δ and γ are in the same Y_β, then q_δ and q_γ also force incompatible facts about τ. Then $q = \bigcup\{q_\delta \mid \delta \in X\}$ has the right properties.

Define $X_0 = E_s^p$ and, for $\delta \in X_0$, $q^0_{s^\frown\delta} = p_{s^\frown\delta}$.

If λ is a limit ordinal less than κ, define $X_\lambda = \bigcap\{X_\beta \mid \beta < \lambda\}$ and, for $\delta \in X_\lambda$, $q^\lambda_{s^\frown\delta} = \bigcap\{q^\beta_{s^\frown\delta} \mid \beta < \lambda\}$.

Given X_β and $q^\beta_{s^\frown\delta}$ for $\delta \in X_\beta$, apply Lemma 12 to $q^\beta = \bigcup\{q^\beta_{s^\frown\delta} \mid \delta \in X_\beta\}$ and $\alpha_\beta = \min(X_\beta)$ to get a measure one set $X_{\beta+1} \subseteq X_\beta$ with $\alpha_\beta \notin X_{\beta+1}$ and conditions $q^{\beta+1}_{s^\frown\delta} \le q^\beta_{s^\frown\delta}$ for $\delta \in X_\beta$ such that if $\delta \in X_{\beta+1}$ and $\gamma \in X_\beta - X_{\beta+1}$,

$$q^{\beta+1}_{s^\frown\gamma} \perp_\tau q^{\beta+1}_{s^\frown\delta}.$$

Since the X_β form a continuous nested sequence with empty intersection (this last because $\min(X_\beta) \notin X_{\beta+1}$), for each $\delta \in E_s^p$ there is a unique ordinal β such that $\delta \in X_\beta - X_{\beta+1}$. Since F is a p-filter, we can find a measure one set $X \subseteq X_0$ almost contained in each X_β; that is, $Y_\beta = X \cap (X_\beta - X_{\beta+1})$ partitions X into sets of size less than κ. Note that if $\beta < \alpha$, $\delta \in Y_\beta \subseteq X_\beta - X_{\beta+1}$, and $\gamma \in Y_\alpha \subseteq X_{\beta+1}$, we have

$$q^{\alpha+1}_{s^\frown\gamma} \le q^{\beta+1}_{s^\frown\gamma} \perp_\tau q^{\beta+1}_{s^\frown\delta}.$$

That is, if δ and γ are respectively in Y_β and Y_α with $\beta \neq \alpha$, then $q^{\beta+1}_{s^\frown\delta}$ and $q^{\alpha+1}_{s^\frown\gamma}$ force incompatible facts about τ.

Now, for each Y_β, extend the conditions $q^{\beta+1}_{s^\frown\delta}$, for all $\delta \in Y_\beta$ to conditions $q_{s^\frown\delta}$ with the property that for all $\delta \neq \gamma$ in Y_β,

$$q_{s^\frown\delta} \perp_\tau q_{s^\frown\gamma}.$$

It is easy to do this for a single pair of conditions: Since τ is forced not to be in M, there is some x such that $q^{\beta+1}_{s^\frown\delta}$ does not decide "$x \in \tau$"; so extend $q^{\beta+1}_{s^\frown\gamma}$ to decide "$x \in \tau$", and then extend $q^{\beta+1}_{s^\frown\delta}$ to decide "$x \in \tau$" in the opposite way. But since Y_β has size less than κ and $\mathbb{P}$ is κ-closed, in less than κ-many successive extensions we can take care of all pairs in Y_β.

Finally, set

$$q = \bigcup\{q_{s^\frown\delta} \mid \delta \in X\}.$$

By construction, for all $\delta \neq \gamma$ in $E_s^q = X$, whether or not δ and γ are in the same Y_β we have that $q_{s^\frown\delta}$ and $q_{s^\frown\gamma}$ force incompatible facts about τ. $\square$

Theorem 14. *If F (a nonprincipal κ-complete filter over κ) is a p-filter and κ-saturated, then $\mathbb{P}$ adds a generic of minimal degree over the ground model.*

Proof. This follows immediately from Lemmas 13 and 11. $\square$

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