A MINIMAL MODEL FOR ¬CH:
ITERATION OF JENSEN'S REALS

BY

URI ABRAHAM[1]

ABSTRACT. A model of ZFC $+ 2^{\aleph_0} = \aleph_2$ is constructed which is minimal with respect to being a model of ¬CH. Any strictly included submodel of ZF (which contains all the ordinals) satisfies CH. In this model the degrees of constructibility have order type ω_2. A novel method of using the diamond is applied here to construct a countable-support iteration of Jensen's reals: In defining the αth stage of the iteration the diamond "guesses" possible $\beta > \alpha$ stages of the iteration.

Introduction. Let V be a transitive universe (i.e., model of ZFC). We say that V is a *minimal* model for ¬CH (negation of continuum hypothesis) if ¬CH holds in V *and* whenever $V^* \subseteq V$ is a transitive submodel of ZFC + ¬CH which contains all the ordinals of V, then necessarily $V^* = V$.

A minimal model for ¬CH has previously been constructed by Marcia J. Groszek; in fact [G] any countable transitive universe M of CH is generically extended to a minimal (above M) model for ¬CH (i.e., there is no model for ¬CH which is strictly included in between M and the generic extension).

We give here another construction of a minimal model for ¬CH. The main structural difference between our model and Groszek's is that here the degrees of constructibility are linearly ordered in order-type ω_2 while in [G] it is the complexity of the structure of the constructibility degrees which is the key to the minimality of the extension.

So, our paper is devoted to the proof of the following theorem. We use L (the universe of constructible sets) as the ground model.

THEOREM. *There is a constructible poset* $\mathbf{P}$ *such that if* $\dot{\mathbf{P}}$ *is an L-generic filter over* $\mathbf{P}$ *then* $L[\dot{\mathbf{P}}]$ *is a minimal model for* ¬CH *in which the degrees of constructibility have order-type* ω_2.

The proof of this theorem might appear somewhat technical, yet the general ideas are very natural. Therefore, I think the reader will appreciate a description of the proof.

G. Sacks [Sa] considered the poset of all perfect trees; he showed that a generic extension which is obtained via the poset of perfect trees is a minimal extension of the ground model. J. Baumgartner and R. Laver iterated this Sacks' forcing with a

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countable support iteration [**B, L**]. A. Miller, in a by-product of his work [**M**], proved that in the model of [**B, L**] (obtained by iterating ω_2 many times the Sacks forcing) the degrees of constructibility have order-type ω_2. (The degrees of constructibility are the equivalence classes obtained by regarding the partial order "x is constructible from y"—$x \in L[y]$—defined on the subsets of ω.)

This model is the first which comes to mind when the problem of finding a minimal model for ¬CH is considered. If the degrees of constructibility have order-type ω_2, and since $\aleph_2 = \aleph_2^L$ holds in this model, any submodel of ¬CH must contain $\aleph_2$ many reals and hence *all* the reals. Yet, this submodel does not necessarily contain all subsets of ω_1. P. Dordal showed that indeed the model obtained by iterating the Sacks perfect-trees posets is not a minimal model for ¬CH. (See [**G**] for a full discussion and proof.) The reason, in short, is that although all the $\aleph_2$ reals must appear in the intermediate model of ¬CH, the *sequence* of generic reals need not. A possible approach is to use Jensen's method [**J1**] for obtaining a definable real of minimal degree of constructibility.[2] In the next section we describe Jensen's real, and in the subsequent section the iteration of these posets—which is our main point.

Jensen's poset is a *subset* of the collection of all perfect trees. The motivation for constructing such a poset, Jensen says, comes from the construction of a Souslin tree in L. [**J1**] uses the constructible diamond-sequence to thin out a subcollection of the Sacks poset which satisfies the c.a.c. (any antichain is countable). However, the most important property of this subcollection is that it is a rigid poset, and the consequence of this is that the generic object over the Jensen poset is *unique*. So, if we iterate Jensen's posets and if the degrees of constructibility have order-type ω_2 in the resulting model, then any intermediate model of ¬CH must not only contain all the reals but actually also the unique sequence of generic reals and so is the full model. Thus the only problem is to get the right·order-type of constructibility degrees.

When analyzing Miller's proof of the fact that ω_2 is the order-type of the constructibility degrees in the [**B, L**] model of the Sacks iteration, one can see that a crucial point is the closure of the perfect trees under fusions. In fact [**B, L**] formalized the notion of fusion also for the *iteration* of Sacks forcing and it is that notion which is used. But the deflated poset of Jensen is *not* closed under the arbitrary fusion sequence and so Miller's arguments cannot be applied directly. The remedy, of course, is to close the Jensen posets under enough fusion sequences so as to apply the Miller argument, yet to do so sparsely so that a rigid poset will result. But there is a problem here: When constructing the α poset ($\alpha < \omega_2$) in the iteration of Jensen's posets, we have to take into account fusion sequences of iterated conditions which involve posets which are not even yet constructed (they will be constructed at stages β, $\alpha < \beta < \omega_2$). How could we do that? A problem of similar nature appeared before Jensen in [**D, J**] where he iterated Souslin trees ω_2 times and used the diamond and square for that. Here, however, we need a different approach.

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[2] I am indebted to J. Baumgartner and P. Dordal for a discussion of this point.

We use Shelah's idea in his proof [S] of the omitting-type for $L[Q]$. The diamond is used to give the future posets.

In §1 we bring the basic definitions and lemmas needed subsequently; much of the material there is essentially due to Baumgartner-Laver [B, L] and to Miller [M]; the poset $Q(P)$ (when P is the Sacks poset) was considered first by Shelah.

About notations: We use V, W to denote universes of set-theory. V is usually the ground model, and if P is a poset then V^P is the Boolean-valued model of $RO(P)$. P_η will denote an iteration of length η. P_0 is the empty set and V^{P_0} is to be read as V. $\dot{P}$ denotes a generic filter over P, and $V[\dot{P}]$ is the generic extension. For a name $x \in V^P$, $x^{\dot{P}}$ denotes the interpretation of x in $V[\dot{P}]$. Sometimes we mix V^P and $V[\dot{P}]$, and if $a \in V[\dot{P}]$ we may regard it as a name and use it in the forcing language. U, S, T will denote trees; P, Q, R, posets. We write $\varnothing \Vdash^P \varphi$ to mean $(\forall p \in P) \, p \Vdash^P \varphi$.

Let us close the introduction with an open question: Is there in our model a Π^1_2 set of reals which picks just one real from each equivalence class of constructibility? Is it consistent to have such a set in a model of ZFC where the degrees of constructibility have order-type ω_2? (Recall that a Jensen real is a Π^1_2 singleton.)

1. Basic definitions and properties.

Perfect trees. $^\omega 2$ is the collection of all functions from a natural number into $\{0, 1\}$. s and t denote members of $^\omega 2$. $T \subseteq {^\omega 2}$ is a *perfect tree* if

(1) $s \subseteq t \in T \Rightarrow s \in T$, and

(2) T is nonempty and every $s \in T$ splits in T, i.e., there are t_1 and t_2 in T, $s \subseteq t_1 \cap t_2$ but $t_1 \not\subseteq t_2$ and $t_2 \not\subseteq t_1$.

The collection of perfect trees is partially ordered by inclusion. We read $T_1 \subseteq T_2$ as T_1 *is above* T_2.

When are two trees compatible? Given perfect trees T and S, define $T \wedge S$—the meet of T and S—in a Cantor-Bendixon fashion as follows. Begin with $U_0 = T \cap S$ and define trees U_α, $\alpha < \omega_1$, inductively: $U_{\alpha+1}$ is the tree of all $s \in U_\alpha$ which splits in U_α. For limit δ, $U_\delta = \bigcap_{i<\delta} U_i$. Finally, $T \wedge S = U_\alpha$ for the first α such that $U_\alpha = U_{\alpha+1}$.

$T \wedge S$ is either a perfect tree or the empty set. $T \wedge S \neq \varnothing$ just in case T and S are compatible (i.e., $T \cap S$ contains a perfect tree). A consequence of this is that the notion of compatibility is absolute: if $T \wedge S = \varnothing$ in some transitive structure in which the Cantor-Bendixon process can be carried out, then $T \wedge S = \varnothing$ in any extension of that structure and T is incompatible with S.

Also obvious is that $T \wedge S$ is the least upper bound of T and S (in the reversed inclusion partial order): $U \subseteq T \wedge S$ for every perfect tree U such that $U \subseteq T \cap S$. A consequence is that

$$(1.1) \qquad\qquad (T \wedge S) \wedge U = T \wedge (S \wedge U).$$

For a perfect tree T and $s \in {^\omega 2}$ let

$$T_s = \{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}.$$

T_s is a (nonempty) perfect tree just in case $s \in T$.

The following can be easily proved

(1.1a) $(T \wedge S)_s = T_s \wedge S_s,$ $(T_1 \cup T_2) \wedge S = (T_1 \wedge S) \cup (T_2 \wedge S).$

Perfect posets. A collection $\mathbf{P}$ of perfect trees is called a *perfect poset* if

(1) $^\omega 2 \in \mathbf{P}$ and $(^\omega 2)_s \in \mathbf{P}$ for $s \in {}^\omega 2$.

(2) $T \wedge S \in \mathbf{P}$ whenever T and S are compatible and in $\mathbf{P}$. (Hence $T \in \mathbf{P}$ and $s \in T$ imply $T_s \in \mathbf{P}$.)

(3) $T \cup S \in \mathbf{P}$ whenever T and S are in $\mathbf{P}$.

Generic reals. If $\dot{\mathbf{P}}$ is a generic filter over a perfect poset $\mathbf{P}$, then $r = \bigcup \bigcap \dot{\mathbf{P}} \in {}^\omega 2$ is called the *canonical generic* real of $\dot{\mathbf{P}}$.

$\dot{\mathbf{P}}$ can be easily recovered from r and $\mathbf{P}$ as follows. Say τ is a *branch* of T if $\tau \restriction n \in T$ for all $n \in \omega$. Look at the set of all $T \in \mathbf{P}$ such that r is a branch of T, this is $\dot{\mathbf{P}}$.

The posets $\mathbf{Q}(\mathbf{P})$. For any perfect poset $\mathbf{P}$ we associate a new poset, $\mathbf{Q}(\mathbf{P})$, consisting of all pairs (T, n) with $T \in \mathbf{P}$ and $n \in \omega$. The partial order is defined by: $(T', n') \le (T, n)$ (and we say (T, n) *extends* (T', n')) if $T \subseteq T'$, $n' \le n$, and

$$\forall s \in {}^{n'}2(s \in T \Leftrightarrow s \in T').$$

If $\dot{\mathbf{Q}}$ is a generic filter over $\mathbf{Q}(\mathbf{P})$, then (by a density argument)

$$\dot{T} = \bigcup \{T \cap {}^n2 \mid (T, n) \in \dot{\mathbf{Q}}\}$$

is a perfect tree, and

$$(T, n) \le (\dot{T}, n) \quad \text{whenever } (T, n) \in \dot{\mathbf{Q}}.$$

(It is clear that $\dot{T}$ is not a member of $\mathbf{P}$ and $(\dot{T}, n)$ is not in $\mathbf{Q}(\mathbf{P})$, so the above is an acceptable abuse of notation.)

This perfect tree $\dot{T}$ is called *the generic tree of* $\dot{\mathbf{Q}}$ or the tree *derived* from $\dot{\mathbf{Q}}$.

1.2. LEMMA. *Let* $\mathbf{R}$ *be a perfect poset;* $\dot{\mathbf{Q}}$ *a V-generic filter over* $\mathbf{Q}(\mathbf{R})$; $\dot{T}$ *the generic tree of* $\dot{\mathbf{Q}}$. *Put* $\mathbf{R}^* =$ *the closure under finite unions of*

$$\mathbf{R} \cup \{\dot{T} \wedge S \mid S \in \mathbf{R} \,\&\, \dot{T} \wedge S \ne \varnothing\}.$$

Then, $\mathbf{R}^*$ *is a perfect poset, and for any* $X \in V$, $X \subseteq \mathbf{R}$ *a maximal antichain,* X *is a maximal antichain of* $\mathbf{R}^*$ *too.*

PROOF. Since $\mathbf{R}$ is a perfect poset and since $\mathbf{R}^*$ is closed under finite unions, all that is necessary to conclude that $\mathbf{R}^*$ is a perfect poset is to show the closure of $\mathbf{R}^*$ under (nonempty) meets. So let $S = S_1 \cup \cdots \cup S_n$, $T = T_1 \cup \cdots \cup T_n$ be in $\mathbf{R}^*$ (where S_i, T_i are in $\mathbf{R} \cup \{\dot{T} \wedge S \mid S \in \mathbf{R} \,\&\, \dot{T} \wedge S \ne \varnothing\}$). To show that $S \wedge T \in \mathbf{R}^*$, it is enough to remark that

$$S \wedge T = \bigcup_{1 \le i, j \le n} S_i \wedge T_j \quad \text{and} \quad S_i \wedge T_j \in \mathbf{R} \cup \{\dot{T} \wedge S \mid S \in \mathbf{R}\}.$$

(Use (1.1) and (1.1a) and conclude also that $(\dot{T} \wedge S) \wedge (\dot{T} \wedge S') = \dot{T} \wedge (S \wedge S')$.)

Now if $X \subseteq \mathbf{R}$, in V, is a maximal antichain, and $T \in \mathbf{R}^*$ is arbitrary, we have to prove that T is compatible with some member of X. We can assume ω.l.o.g., $T \in \mathbf{R} \cup \{\dot{T} \wedge S \mid S \in \mathbf{R} \,\&\, \dot{T} \wedge S \ne \varnothing\}$, and, since X is maximal is $\mathbf{R}$, it is enough to deal with $T = \dot{T} \wedge S$ for some $S \in \mathbf{R}$.

Let $(U, n) \in \mathbf{Q}(\mathbf{R})$ be an arbitrary condition which forces $T = \dot{T} \wedge S$. We will find an extension of (U, n) which forces $\dot{T} \wedge S$ to be compatible with some member of X. Since $(U, n) \Vdash T \subseteq U$, it must be that U and S are compatible (in the ground model V, by absoluteness) and $U \wedge S \in \mathbf{R}$. Pick $s \in {}^n2 \cap (U \wedge S)$, let $U' = (U \wedge S)_s$, then $U' \in \mathbf{R}$ and we can find $U'' \in \mathbf{R}$ which is above U' and above a member of X. Now put

$$U^* = U'' \cup \bigcup \{U_t | t \in {}^n2 \cap U \text{ and } t \neq s\},$$

then $U^* \in \mathbf{R}$, $(U, n) \leqslant (U^*, n)$, and $U_s^* = U''$. Since $(U^*, n) \Vdash s \in \dot{T}$ and $\dot{T}_s \subseteq U_s^*$, and since $U_s^* \subseteq S$ is above a member of X, we get that $(U^*, n) \Vdash \dot{T}_s \subseteq \dot{T} \wedge S$ is above a member of X.

In fact, a slightly stronger claim can be proved: Say that $X \subseteq \mathbf{R}$ is a maximal antichain *above* $T \in \mathbf{R}$ iff any $S \subseteq T$ in $\mathbf{R}$ is compatible with some member of X. Then, if $X \in V$ is a maximal antichain above T in $\mathbf{R}$, X remains a maximal antichain above T in $\mathbf{R}^*$. (Look at $X \cup \{S \in \mathbf{R} | S \text{ is incompatible with } T\}$, apply the lemma, and use the absoluteness of incompatibility.)

1.3. *Iteration of perfect posets.* We are interested in iterating ω_2 times perfect posets. $\mathbf{P}_\eta$ will stand for a countable-support iteration of length $\eta \leqslant \omega_2$ of perfect posets. The definition of $\mathbf{P}_\eta$ is by induction on η. The members of $\mathbf{P}_\eta$ are countable functions f with $\mathrm{dom}(f)$ a countable subset of η such that $f \restriction \mu \in \mathbf{P}_\mu$ for all $\mu < \eta$ and $f \restriction \mu \Vdash^{\mathbf{P}_\mu} \text{``} f(\mu) \in \mathbf{R}(\mu)\text{''}$,[3] where $\mathbf{R}(\mu)$ is a name in $V^{\mathbf{P}_\mu}$ of a perfect poset. (But $\mathbf{R}(0)$ *is* a perfect poset and $\mathbf{P}_1$ consists of all functions from 1 into $\mathbf{R}(0)$.) Sometimes we write $f(\mu)$ even when $\mu \notin \mathrm{dom}(f)$ and then we mean $f(\mu)$ to be the name of the full tree—${}^\omega 2$.

The partial order is defined as usual. So, $f \leqslant g$ iff $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $(\forall \alpha \in \mathrm{dom}(f)) \, g \restriction \alpha \Vdash^{\mathbf{P}_\alpha} g(\alpha) \subseteq f(\alpha)$.

If $\dot{\mathbf{P}}_\eta$ is a generic filter over $\mathbf{P}_\eta$ then $\dot{\mathbf{P}}_\mu = \dot{\mathbf{P}}_\eta \cap \mathbf{P}_\mu$ (for $\mu < \eta$) is a generic filter over $\mathbf{P}_\mu$; and $\{f(\mu) | f \in \dot{\mathbf{P}}_\eta\}$, as interpreted in $V[\dot{\mathbf{P}}_\mu]$, form a $V[\dot{\mathbf{P}}_\mu]$ generic filter over $\mathbf{R}(\mu)$. Thus $\dot{\mathbf{P}}_\eta$ gives a sequence $\langle r_\zeta | \zeta \in \eta \rangle$ of generic reals.

1.4. LEMMA. $\dot{\mathbf{P}}_\eta$ *can be recovered from* $\langle r_\zeta | \zeta \in \eta \rangle$ *and* $\mathbf{P}_\eta$.

PROOF. We recover $\dot{\mathbf{P}}_\mu$, $\mu \leqslant \eta$, inductively. $\dot{\mathbf{P}}_1$ is the set of all functions in $\mathbf{P}_1$ such that r_0 is a branch of $f(0)$. Similarly, $\dot{\mathbf{P}}_{\mu+1}$ consists of all $f \in \mathbf{P}_{\mu+1}$ such that $f \restriction \mu \in \dot{\mathbf{P}}_\mu$ and $f(\mu)$ is interpreted in $V[\dot{\mathbf{P}}_\mu]$ as a tree in which r_μ is a branch. In case μ is a limit ordinal

$$\dot{\mathbf{P}}_\mu = \left\{ f \in \mathbf{P}_\mu | f \restriction \tau \in \dot{\mathbf{P}}_\tau \text{ for all } \tau \in \mu \right\}$$

can be easily derived. □

1.5. DEFINITION AND PROPERTIES OF $f | \sigma$. Given $f \in \mathbf{P}_\eta$ and $\sigma: D \to {}^\omega 2$, where D is a finite subset of $\mathrm{dom}(f)$, define $f | \sigma$ as follows: $f | \sigma$ is a function with the same domain as f and

(1) $(f | \sigma)(\mu) = f(\mu)$ for $\mu \notin D$, but

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[3] By a standard trick, we can assume $\varnothing \Vdash \text{``} f(\mu) \in \mathbf{R}(\mu)\text{''}$.

(2) $\varnothing \Vdash^{\mathbf{P}_\mu}(f|\sigma)(\mu) = f(\mu)_{\sigma(\mu)}$, for $\mu \in D$. Recall, $f(\mu)_{\sigma(\mu)}$ is the subtree of $f(\mu)$ of those functions compatible with $\sigma(\mu)$. In case $f|\sigma \in \mathbf{P}_\eta$ (i.e., for every $\mu \in D$, $(f|\sigma)\upharpoonright \mu \Vdash^{\mathbf{P}_\mu}\sigma(\mu) \in f(\mu))$ we say that σ *is consistent with* f.

(1.5a) Put $\mu = \min D$, then σ is consistent with f iff $f\upharpoonright \mu \Vdash \sigma(\mu) \in f(\mu)$, and $\sigma\upharpoonright (D - \{\mu\})$ is consistent with $f|(\sigma\upharpoonright \{\mu\})$.

It is easy to prove that if σ is consistent with f then

(i) $f \le f|\sigma$,

(ii) $\sigma|\eta$ is also consistent with f,

(iii) σ is consistent with $f\upharpoonright \zeta$ whenever $\cup \operatorname{dom}(\sigma) < \zeta$.

In what follows, F denotes a finite function and D_F its domain, $F \colon D_F \to \omega$. We say that σ *is bounded by* F if $\sigma \colon D_F \to {}^\omega 2$ satisfies $\sigma(\mu) \in {}^{F(\mu)}2$ for all $\mu \in D_F$.

If $f \in \mathbf{P}_\eta$ and $D_F \subseteq \operatorname{dom}(f)$, define (f, F) to be *determined* if, for any σ bounded by F, either σ is consistent with f or else there is $\mu \in D_F$ such that $\sigma\upharpoonright \mu$ is consistent with f and $(f\upharpoonright \mu)|(\sigma\upharpoonright \mu) \Vdash \sigma(\mu) \notin f(\mu)$. (The definitions in 1.5 are from [**B, L**, §2].)

If σ is bounded by F and consistent with f we say that σ *is consistent with* (f, F). The next lemma gathers some useful facts. It is similar to Lemma 2.2 in [**B, L**] and its proof is left to the reader.

1.6. LEMMA. (a) *If* (f, F) *is determined then so is* $(f, F\upharpoonright \mu)$.

(b) *If* $f \le g$ *in* $\mathbf{P}_\eta$, (f, F) *is determined and* σ *is consistent with* (g, F), *then* σ *is consistent with* f *too and* $f|\sigma \le g|\sigma$.

(c) *If* $f \le g$ *but* $f(\zeta) = g(\zeta)$ *for* $\zeta \in D_F$ *and if* (f, F) *is determined then so is* (g, F).

(d) *Put* $\mu = \min(D_F)$. (f, F) *is determined iff* $f\upharpoonright \mu$ *knows what is* $f(\mu) \cap {}^{F(\mu)}2$ *and for each* $s \in {}^{F(\mu)}2$ *such that* $f\upharpoonright \mu \Vdash s \in f(\mu)$, (f', F') *is determined where* $f' = f|\{(\mu, s)\}$ *and* $F' = F\upharpoonright (D_F - \mu + 1)$.

(e) *Given a determined* (f, F) *there exists* σ *consistent with* (f, F). *And the set* $\{f|\sigma \mid \sigma \text{ is consistent with } (f, F)\}$ *is a maximal antichain above* f.

1.7. DEFINITION OF UNION IN $\mathbf{P}_\eta$. Let $f_1, \ldots, f_n \in \mathbf{P}_\eta$ be given. Suppose for some $\mu < \eta$ $f_i \upharpoonright \mu = f_j \upharpoonright \mu$ for $i, j \le n$. Moreover, for distinct $s_1, \ldots, s_n \in {}^k 2$,

$$f_i \upharpoonright \mu \Vdash f_i(\mu) \cap {}^k 2 = \{s_i\}.$$

We define $f = \bigcup_{1 \le i \le n} f_i$ to be the condition in $\mathbf{P}_\eta$, with $\operatorname{dom}(f) = \bigcup_{1 \le i \le n}\operatorname{dom}(f_i)$, determined by the following conditions:

(1) $f\upharpoonright \mu = f_i \upharpoonright \mu$,

(2) $\varnothing \Vdash f(\mu) = \bigcup_{1 \le i \le n} f_i(\mu)$,

(3) if the generic real for $\mathbf{R}(\mu)$ extends s_i then $f(\zeta) = f_i(\zeta)$ (for $\zeta \ge \mu$).

A direct check shows that $f \le f_i$, $f|\{(\mu, s_i)\} = f_i$, and if $\forall i$, $g \le f_i$ then $g \le f$. Also, $\{f_i \mid 1 \le i \le n\}$ is a maximal antichain above f: if $f \le f'$ then f' is compatible with some f_i.

For $F \colon D \to \omega$, $F' \colon D' \to \omega$, we say $F \le F'$ if $D \subseteq D'$ and $F(\mu) \le F'(\mu)$ for all $\mu \in D$.

1.8. DEFINITION OF $\mathbf{Q}(\mathbf{P}_\eta)$. $\mathbf{Q}(\mathbf{P}_\eta)$ is the collection of all pairs (f, F) where $f \in \mathbf{P}_\eta$ and $F \colon D_F \to \omega$, $D_F \subset \operatorname{dom}(f)$ is finite.

$\mathbf{Q}(\mathbf{P}_\eta)$ is partially ordered as follows: $(f, F) \leqslant (f', F')$ iff $f \leqslant f'$ in $\mathbf{P}_\eta$, $F \leqslant F'$, and for every $\mu \in D_F$

$$f' \upharpoonright \mu \Vdash f(\mu) \cap {}^{F(\mu)}2 = f'(\mu) \cap {}^{F(\mu)}2.$$

We adopt the convention that $F(\mu) = 0$ in case $\mu \notin D_F$, and $f(\mu) = {}^\omega 2$ if $\mu \notin \mathrm{dom}(f)$. Clearly, if $F \leqslant F'$ then $(f, F) \leqslant (f, F')$.

1.9. LEMMA. (a) *If* $(f, F) \leqslant (g, F)$ *and* σ *is bounded by* F, *then* σ *is consistent with* g *if* σ *is consistent with* f. (*If we assume, moreover, that* (f, F) *is determined, then, together with Lemma* 1.6(b) *we get that* σ *is consistent with* g *iff* σ *is consistent with* f.)

(b) *For any* $(f, F) \in \mathbf{Q}(\mathbf{P}_\eta)$ *there is* $g \in \mathbf{P}_\eta$ *such that* $(f, F) \leqslant (g, F)$ *and* (g, F) *is determined*.

PROOF OF (a). The proof goes by induction on $|D_F|$ (the cardinality of D_F). Suppose $D_F = \{\mu\}$, $(f, F) \leqslant (g, F)$ and σ (bounded by F) is consistent with f, then $f \upharpoonright \mu \Vdash \sigma(\mu) \in f(\mu)$. But (since $(f, F) \leqslant (g, F)$) also $g \upharpoonright \mu \Vdash f(\mu) \cap {}^{F(\mu)}2 = g(\mu) \cap {}^{F(\mu)}2$, hence $g \upharpoonright \mu \Vdash \sigma(\mu) \in g(\mu)$.

Suppose now $|D_F| > 1$ and put $\mu = \min D_F$. Assume the premise of the lemma and that σ is consistent with f, we want to prove that σ is consistent with g. For that we shall use equivalence 1.5(a). The previous paragraph shows that $g \upharpoonright \mu \Vdash \sigma(\mu) \in g(\mu)$. It remains to prove that $\sigma \upharpoonright (D_F - \{\mu\}) = \sigma'$ is consistant with $g | (\sigma \upharpoonright \{\mu\}) = g'$. Since σ is consistent with f, 1.5(a) implies that σ' is consistent with $f | (\sigma \upharpoonright \{\mu\}) = f'$. Put $F' = F - \{(\mu, F(\mu))\}$. If we prove $(f', F') \leqslant (g', F')$, then the induction hypothesis implies that σ' is consistent with g'.

First, $f' \leqslant g'$ is an easy consequence of $f \leqslant g$ and the fact that $\sigma \upharpoonright \{\mu\}$ is consistent with f and with g. Then, since $g \leqslant g'$ (by 1.5(i)), and since $(f, F) \leqslant (g, F)$, and since $f'(\tilde{\mu}) = f(\tilde{\mu})$ and $g'(\tilde{\mu}) = g(\tilde{\mu})$ for $\mu < \tilde{\mu}$, $g' \Vdash f'(\tilde{\mu}) \cap {}^{F(\tilde{\mu})}2 = g'(\tilde{\mu}) \cap {}^{F(\tilde{\mu})}2$, for $\tilde{\mu} \in D_F - \{\mu\}$. It follows that $(f', F') \leqslant (g', F')$.

PROOF OF (b). Again, by induction on $|D_F|$. When $|D_F| = 1$, say $D_F = \{\mu\}$, simply extend $f \upharpoonright \mu$ and find $g \in \mathbf{P}_\eta$, $f \leqslant g$, such that $g \upharpoonright \mu$ describes $f(\mu) \cap {}^{F(\mu)}2$ and $g(\mu) = f(\mu)$. When $|D_F| > 1$, put $\mu = \min D_F$ and $F' = F \upharpoonright (D_F - \{\mu\})$. Extending $f \upharpoonright \mu$, we can assume that $f \upharpoonright \mu$ describes $f(\mu) \cap {}^{F(\mu)}2$. Let $s_1, \ldots, s_n \in {}^{F(\mu)}2$ be all those that $f \upharpoonright \mu$ knows to be in $f(\mu) \cap {}^{F(\mu)}2$. We shall define inductively $f_0, \ldots, f_n \in \mathbf{P}_\eta$ such that $i < j \to f_i \upharpoonright \mu \leqslant f_j \upharpoonright \mu$.

To begin with, $f_0 = f$. If f_i is defined, put $f_i' = f_i \upharpoonright \mu \cup ((f | \{\mu, s_{i+1}\}) \upharpoonright \eta - \mu)$, then extend f_i', using the induction hypothesis, and find $f_{i+1} \geqslant f_i'$ such that $(f_i', F') \leqslant (f_{i+1}, F')$ and (f_{i+1}, F') is determined.

Now, when f_n is defined (assuming ω.l.o.g. that $f_i \upharpoonright \mu = f_j \upharpoonright \mu$) we set $g = \bigcup_{1 \leqslant i \leqslant n} f_i$. $f \leqslant g$ since $(\forall 1 \leqslant i \leqslant n)$ $f \leqslant f_i$ (see 1.7). (g, F) is determined since $g \upharpoonright \mu \Vdash g(\mu) \cap {}^{F(\mu)}2 = \{s_1, \ldots, s_n\}$, and $g | \{(\mu, s_i)\} = f_i$ and (f_i, F') is determined. $(f, F) \leqslant (g, F)$ is also obvious. $\square$

1.10. LEMMA. *Let* $D \subseteq \mathbf{P}_\eta$ *be dense. For any determined* $(f, F) \in \mathbf{Q}(\mathbf{P}_\eta)$ *there is* $(g, F) \geqslant (f, F)$ *satisfying this:*

$$g | \sigma \in D \text{ whenever } \sigma \text{ is consistent with } (g, F).$$

PROOF. Again by induction on $|F|$; or use Lemma 3.1 of [M].

The next lemma is similar to Lemma 5 of [M] but the proof is complicated by the fact that an arbitrary perfect poset is not closed under fusions. (And so, when we are in the middle of an iteration we do not know that what remains to be forced is again a countable-support iteration.[4])

Let us call sets of the form $\{u \in {}^\omega 2 \mid s \subseteq u\}$ for some $s \in {}^\omega 2$ *basic open sets* (of height $\|s\|$).

1.11. LEMMA. *Suppose $f \in \mathbf{P}_\eta$, τ is a name, and $f \Vdash \tau \in {}^\omega 2 \wedge \tau \notin V^{\mathbf{P}_\zeta}$ for all $\zeta < \eta$. Assume (f, F) is determined. Put $\Sigma = \{\sigma \mid \sigma$ is consistent with $(f, F)\}$. Then there are disjoint basic open sets C_σ, for $\sigma \in \Sigma$, and there is $g \in \mathbf{P}_\eta$ with $(f, F) \leqslant (g, F)$ such that*

$$g \mid \sigma \Vdash \tau \in C_\sigma, \quad for \ \sigma \in \Sigma.$$

So $g \Vdash \tau \in \bigcup \{C_\sigma \mid \sigma \in \Sigma\}$. (We can clearly assume that all the C_σ have the same height.)

PROOF. Observe first that if $f \leqslant h \in \mathbf{P}_\eta$, $\mu < \eta$, then there are h_1, h_2 extending h, $h_1 \upharpoonright \mu = h_2 \upharpoonright \mu$, giving incompatible information about $\tau \upharpoonright m$ for some $m \in \omega$ (i.e., $h_i \Vdash \tau \upharpoonright m = s_i$, $i = 1, 2$, and $s_1 \neq s_2$). Otherwise, we would get $h \Vdash \tau \in V^{\mathbf{P}_\mu}$. Continuing, for any given n, there are extensions $h_1, \ldots, h_n$ of h giving incompatible information on $\tau \upharpoonright m$ (for some m) such that $h_1 \upharpoonright \mu = h_2 \upharpoonright \mu = \cdots = h_n \upharpoonright \mu$.

The proof of the lemma proceeds by induction on $|F|$. For $|F| = 1$ put $D_F = \{\mu\}$. Let $s_1, \ldots, s_k$ be all those functions forced by $f \upharpoonright \mu$ to be in $f(\mu) \cap {}^{F(\mu)}2$. Set $f^1 = f \mid \{(\mu, s_1)\}$; use the observation above to find $f_1^1, \ldots, f_k^1$ extending f^1 giving incompatible information on $\tau \upharpoonright m$ such that $f_1^1 \upharpoonright \mu = \cdots = f_k^1 \upharpoonright \mu$. Repeating this process $k - 1$ more times—for $s_2, \ldots, s_k$—and extending a little bit more, we can find f_j^i $(1 \leqslant i, j \leqslant k)$ with $f_j^i \upharpoonright \mu = f_{j'}^{i'} \upharpoonright \mu$, f_j^i extends $f \mid \{(\mu, s_i)\}$; and for some m for each $1 \leqslant i \leqslant k$, f_j^i $(j = 1, \ldots, k)$ gives incompatible information on $\tau \upharpoonright m$. Now choose $j(i) \leqslant k$ for each $i \leqslant k$ so that $f_{j(i)}^i$ $(i = 1, \ldots, k)$ gives incompatible information on $\tau \upharpoonright m$. Finally let g be the union of $f_{j(1)}^1, \ldots, f_{j(k)}^k$. (See 1.7 for definition and properties of unions.)

Next, assume $|F| > 1$ and $\mu = \min D_F$. Let $s_1, \ldots, s_k$ be the members of $f(\mu) \cap {}^{F(\mu)}2$ (that is, those forced by $f \upharpoonright \mu$ to be there). Let $F' = F \upharpoonright (D_F - \{\mu\})$. Begin with $f_1' = f \mid \{(\mu, s_1)\}$, (f_1', F') is determined (1.6(d)) and $(f, F') \leqslant (f_1', F')$. Let $\nu = \min D_{F'}$. Extending $f_1' \upharpoonright \nu$ and calling this extension $f_1' \upharpoonright \nu$ again, we can find $l > F(\nu)$ such that $f_1' \upharpoonright \nu$ completely describes $f(\nu) \cap {}^l 2$ and, moreover, each $s \in f(\nu) \cap {}^{F(\nu)}2$ has $\geqslant \|F'\| \cdot k$ many different extensions in $f(\nu) \cap {}^l 2$ (where $\|F'\|$ is the number of possible σ's bounded by F').

Now let $F^* \geqslant F'$ with $\mathrm{dom}(F^*) = \mathrm{dom}(F')$ be defined by $F^*(\nu) = l$ and equal to F' at other arguments. Clearly, $(f_1', F^*) \geqslant (F_1', F')$ is determined too. The induction assumption can be applied to yield $(f_1, F^*) \geqslant (f_1', F^*)$ and $m(1) \in \omega$ such that for different σ (consistent with (f_1, F^*)) $f_1 \mid \sigma$ gives incompatible information on $\tau \upharpoonright m(1)$.

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[4]However, in §2 we are using only c.a.c. posets. Hence, for the proof of our theorem, Lemma 5 of [M] is perfectly sufficient.

Next, we repeat this procedure for s_2 etc., and find $f_1, f_2, \ldots, f_k$ with $f_i \upharpoonright \mu \leq f_j \upharpoonright \mu$ for $i < j$, such that f_i relates to s_i the same way f_1 relates to s_1. Then, extending a little more we can assume $f_i \upharpoonright \mu = f_j \upharpoonright \mu$, and the following hold:

(1) f_i extends $f \mid \{(\mu, s_i)\}$.

(2) $(f, F') \leq (f_i, F_i^*)$, (f_i, F_i^*) is determined and $\mathrm{dom}(F_i^*) = \mathrm{dom}(F')$, $F_i^*(\nu) = l(i)$ but F_i^* is equal to F' on other arguments, and

(3) $f_i \upharpoonright \nu$ forces that each $s \in f(\nu) \cap {}^{F(\nu)}2$ has $\geq \|F'\| \cdot k$ many different extensions in $f(\nu) \cap {}^{l(i)}2$.

(4) For different σ's (consistent with (f_i, F_i^*)) $f_i \mid \sigma$ gives incompatible information on $\tau \upharpoonright m(i)$.

Extending f_i furthermore (Lemma 1.10) calling again this extension by f_i, we can assume $m(i) = m$ (does not depend on i).

In the second stage of the proof of the lemma, we will find $g_i \geq f_i$, such that the following will hold:

(α) $(f, F') \leq (g_i, F')$, but $g_i \upharpoonright \nu$ forces that every member of $g_i(\nu) \cap {}^{F(\nu)}2$ has only *one* extension in $g_i(\nu) \cap {}^{l(i)}2$.

(β) If $i \neq j$ then $g_i \mid \sigma$ and $g_j \mid \sigma'$ give incompatible information on $\tau \upharpoonright m$ whatever σ and σ' (bounded by F' and consistent with g_i and g_j respectively) are.

The construction of the g_i is inductive. Suppose now it is the turn of g_i to be defined. Call σ (consistent with (f_i, F_i^*)) *bad* if the value of $\tau \upharpoonright m$ decided by $f_i \mid \sigma$ has already been given by $g_j \mid \sigma'$ for some $j < i$ and σ' (consistent with (g_j, F')). There are less than $k \cdot \|F'\|$ possible bad σ's. Hence for each member of $f_i(\nu) \cap {}^{F(\nu)}2$ (i.e., forced by $f_i \upharpoonright \nu$ to be there—which is the same as being forced by $f \upharpoonright \nu$ to be there, since $(f, F') \leq (f_i, F')$) we can find an extension in $f_i(\nu) \cap {}^{l(i)}2$ which is not $\sigma(\nu)$ for a bad σ.

Define $g_i \geq f_i$ such that $(f, F') \leq (g_i, F')$ and no member of $g_i(\nu) \cap {}^{l(i)}2$ is $\sigma(\nu)$ for a bad σ, and every member of $g_i(\nu) \cap {}^{F(\nu)}2$ has only one extension in $g_i(\nu) \cap {}^{l(i)}2$. It is clear that the g_i's satisfy (α) and (β). Finally let $g = \bigcup_{1 \leq i \leq k} g_i$ then g is as required. $\square$

1.12. *Projections of* $\mathbf{Q}(\mathbf{P}_\eta)$. The map $(f, F) \mapsto (f \upharpoonright \mu, F \upharpoonright \mu)$ is a projection of $\mathbf{Q}(\mathbf{P}_\eta)$ onto $\mathbf{Q}(\mathbf{P}_\mu)$. Hence, if G is a generic filter over $\mathbf{Q}(\mathbf{P}_\eta)$ then $\{(f \upharpoonright \mu, F \upharpoonright \mu) \mid (f, F) \in G\}$ is generic over $\mathbf{Q}(\mathbf{P}_\mu)$ ($\mu < \eta$).

1.13. *M-generic conditions and filters.* $H(\aleph_3)$ is the collection of all sets with transitive closure of cardinality less than $\aleph_3$. $\mathbf{P}_\eta$ denotes an iteration of length $\eta < \omega_2$ of perfect posets. It will turn out that $\mathbf{P}_\eta \in H(\aleph_3)$. $V = L$ is our ground model (although we shall not use this fact in this section).

In what follows, M is a countable elementary substructure of $H(\aleph_3)$ and $\mathbf{P} \in M$ is a poset. Let us review some of the notions we shall need from Shelah's theory of proper forcing.

Say that $f \in \mathbf{P}$ is an *M-generic condition* (*over* $\mathbf{P}$) if for every $D \in M$, a dense subset of $\mathbf{P}$, and for every $f' > f$ (in $\mathbf{P}$) there are $f^* \geq f'$ and $d \in D \cap M$ with $d \leq f^*$.

Say that $G \subseteq \mathbf{P}$ is an *M-generic filter* (*over* $\mathbf{P}$) if $G \cap M$ is a filter over $\mathbf{P} \cap M$ and $G \cap D \cap M \neq \varnothing$ whenever $D \in M$ is dense in $\mathbf{P}$.

As will be clear later on, $\mathbf{P}_\eta$ will be an iteration of perfect posets of cardinality $\aleph_1$ (GCH is assumed in V) each satisfies the c.a.c. (countable antichain condition), the iteration is taken with countable support (as explained in 1.3); it follows then (from a general theorem of Shelah about proper forcing [S1]) that $\mathbf{P}_\eta$ satisfies the $\aleph_2$-a.c. In fact, if $\eta < \omega_2$, then this theorem provides a dense subset of $\mathbf{P}_\eta$ of cardinality $\aleph_1$. Let us describe this dense subset. (Laver's argument [L] is the prototype, but we cannot apply it literally since our posets are not closed under arbitrary fusions.)

Define when $f \in \mathbf{P}_\alpha$ is *essentially countable* by induction on α. Any $f \in \mathbf{P}_1$ is essentially countable (f then is a function on 1 such that $f(0)$ is a perfect tree). For a successor ordinal: $f \in \mathbf{P}_{\alpha+1}$ is essentially countable when $f \restriction \alpha$ is and when $f(\alpha) = a$ is a name of a perfect tree of the following kind: $a = \langle E_t \mid t \in {}^\omega 2 \rangle$ where $E_t \subseteq \mathbf{P}_\alpha$ is always a countable collection of essentially countable conditions. (The interpretation of a is the collection of all $t \in {}^\omega 2$ such that $E_t \cap \dot{\mathbf{P}}_\alpha \neq \varnothing$.) For limit δ, $f \in \mathbf{P}_\delta$ is essentially countable if, for all $\alpha < \delta$, $f \restriction \alpha$ is.

The proof of the $\aleph_2$-a.c. proceeds by showing that the essentially countable conditions in $\mathbf{P}_\eta$ form a dense subset. Therefore we stipulate that $\mathbf{P}_\eta$ consists only of essentially countable conditions. It easily follows now that $\mathbf{P}_\eta \in H(\aleph_3)$ ($\eta < \omega_2$).

Let $\dot{\mathbf{P}}_\eta$ be a V-generic filter over $\mathbf{P}_\eta$. $M[\dot{\mathbf{P}}_\eta]$ denotes the collection formed by interpreting in $V[\dot{\mathbf{P}}_\eta]$ all the names which are in M. Similarly we understand $H(\aleph_3)[\dot{\mathbf{P}}_\eta]$ ($H(\aleph_3)$ is in V). It follows that $H(\aleph_3)[\dot{\mathbf{P}}_\eta] = (H(\aleph_3))^{V[\dot{\mathbf{P}}_\eta]}$. (Use the $\aleph_2$-a.c. of $\mathbf{P}_\eta$ and the fact that the cardinality of $\mathbf{P}_\eta$ is $\leq \aleph_2$.)

Suppose $\mathbf{P}_\eta \in M$ and f is an M-generic condition (over $\mathbf{P}_\eta$), and $f \in \dot{\mathbf{P}}_\eta$, then $\dot{\mathbf{P}}_\eta$ is an M-generic filter and $M[\dot{\mathbf{P}}_\eta]$ is an elementary substructure of $H(\aleph_3)^{V[\dot{\mathbf{P}}_\eta]}$. (See [S1].)

Denote by $\overline{M}$ the transitive structure isomorphic to M, and by $\pi: M \to \overline{M}$ the Mostowski collapse.

Put $G = M \cap \dot{\mathbf{P}}_\eta$. Then $\pi''G$ is an $\overline{M}$-generic filter over $\pi(\mathbf{P}_\eta)$ and $\overline{M}[\pi''G]$ is a transitive generic extension of $\overline{M}$. π can be extended to collapse $M[\dot{\mathbf{P}}_\eta]$ onto $\overline{M}[\pi''G]$, and we continue to denote by π this extension.

The next lemma shows the role of the poset $\mathbf{Q}(\mathbf{P}_\eta)$ (defined in 1.8) is to provide M-generic conditions over $\mathbf{P}_\eta$.

1.14. LEMMA. *If $G \subseteq M \cap \mathbf{Q}(\mathbf{P}_\eta)$ is an M-generic filter over $\mathbf{Q}(\mathbf{P}_\eta)$ and $f \in \mathbf{P}_\eta$ satisfies $e \leq f$ whenever $(e, F) \in G$, then f is an M-generic condition over $\mathbf{P}_\eta$.*

PROOF. Let $D \in M$ be a dense subset of $\mathbf{P}_\eta$. Since Lemmas 1.9 and 1.10 hold in M and since G is M-generic, there is a determined $(g, F) \in G$ such that $g \mid \sigma \in D$ whenever σ is consistent with (g, F).

Given $f' \geq f$ (w.l.o.g. (f', F) is determined), use 1.6(e) to find σ consistent with (f', F). But $g \leq f'$ by assumption, so σ is consistent with g and $g \mid \sigma \leq f' \mid \sigma$ (by 1.6(b)). $f' \leq f' \mid \sigma$ (1.5(i)), so an extension of f' is found above $g \mid \sigma \in D \cap M$.

1.15. DEFINITION OF g AND $\mathbf{H}$. In the ground model V, M is a countable elementary substructure of $H(\aleph_3)$ and $\mathbf{P}_\eta \in M$. So there clearly is $G \subseteq \mathbf{Q}(\mathbf{P}_\eta) \cap M$ which is an M-generic filter over $\mathbf{Q}(\mathbf{P}_\eta)$. For any such G we will define now $g \in \mathbf{P}_\eta$ and a sequence of names $\mathbf{H}$ (g and $\mathbf{H}$ depend on $M, \mathbf{P}_\eta$ and G). $\mathrm{dom}(g) = \mathrm{dom}(\mathbf{H}) = M \cap \eta$, and, for $\mu \in M \cap \eta$, $\mathbf{H}(\mu)$ and $g(\mu)$ are names in $\mathbf{P}_\mu$ forcing defined as

follows. Actually, we put ourselves in $V[\dot{\mathbf{P}}_\mu]$ (where $\dot{\mathbf{P}}_\mu$ is V-generic over $\mathbf{P}_\mu$) and describe the interpretations of $g(\mu)$ and $\mathbf{H}(\mu)$; this will convince the reader that the names $\mathbf{H}(\mu)$ and $g(\mu)$ can be defined in V.

Collect all $(f, F) \in G$ such that $f \restriction \mu \in \dot{\mathbf{P}}_\mu$; for each such (f, F) look at the interpretation of $f(\mu)$, $f(\mu)^{\dot{\mathbf{P}}_\mu}$, and form the pair $((f(\mu))^{\dot{\mathbf{P}}_\mu}, F(\mu))$ ($F(\mu)$ is 0 if $\mu \notin \operatorname{dom}(F)$ and $f(\mu)^{\dot{\mathbf{P}}_\mu}$ is $^\omega 2$ if $\mu \notin \operatorname{dom}(f)$). The collection of all these pairs is $(\mathbf{H}(\mu))^{\dot{\mathbf{P}}_\mu}$, a subset (possibly empty) of $\mathbf{Q}(\mathbf{R}(\mu))$. (Where, remember, $\mathbf{R}(\mu)$ is in $V[\dot{\mathbf{P}}_\mu]$ the perfect poset which is iterated in the next stage, $\mathbf{P}_\mu * \mathbf{R}(\mu)$ is $\mathbf{P}_{\mu+1}$.)

Suppose $U = \bigcup \{T \cap {}^n 2 \mid (T, n) \in (\mathbf{H}(\mu))^{\dot{\mathbf{P}}_\mu}\}$ is a perfect tree and even a member of $\mathbf{R}(\mu)$, then we set

$$g(\mu)^{\dot{\mathbf{P}}_\mu} = U; \quad \text{otherwise, } g(\mu)^{\dot{\mathbf{P}}_\mu} = {}^\omega 2.$$

1.16. LEMMA. (*in the notation of 1.15*) *Suppose $\mu \in M$, $\mu \in \eta$, and $f \in \mathbf{P}_\mu$ is such that $h \restriction \mu \leqslant f$ whenever $(h, F) \in G$. Then*

$$f \Vdash \mathbf{H}(\mu) \text{ is an } M\big[\dot{\mathbf{P}}_\mu\big]\text{-generic filter over } \mathbf{Q}(\mathbf{R}(\mu)).$$

PROOF. Note that $\dot{\mathbf{P}}_\mu$ is the name of the generic filter. The definition of $\mathbf{H}$ implies that $f \Vdash \mathbf{H}(\mu) = \{(h(\mu), F(\mu)) \mid (h, F) \in G\}$. Since G is a filter, it is not too difficult to check that $f \Vdash \mathbf{H}(\mu)$ is a filter over $\mathbf{Q}(\mathbf{R}(\mu)) \cap M[\dot{\mathbf{P}}_\mu]$. Why is this filter $M[\dot{\mathbf{P}}_\mu]$-generic?

Observe first that (since $M \prec H(\aleph_3)$ and as the forcing relation can be defined in M) $M[\dot{\mathbf{P}}_\mu]$ is forced (by f) to be an elementary substructure of $H(\aleph_3)^{V[\dot{\mathbf{P}}_\mu]}$. (See [S1].)

Let $f < f_1 \in \mathbf{P}_\mu$ and $\mathbf{D} \in M$ a name (in $V^{\mathbf{P}_\mu}$) be given such that $f_1 \Vdash \mathbf{D}$ is dense open in $\mathbf{Q}(\mathbf{R}(\mu))$. Our aim is to prove $f_1 \Vdash \mathbf{D} \cap \mathbf{H}(\mu) \neq \varnothing$, and then, by a density argument for $\mathbf{P}_\mu$, the desired property of $\mathbf{H}(\mu)$ follows.

Find a name $\mathbf{D}' \in M$ such that for any $p \in \mathbf{P}_\mu$, $p \Vdash \mathbf{D}'$ is dense open in $\mathbf{Q}(\mathbf{R}(\mu))$, and if $p \Vdash \mathbf{D}$ is dense open, then $p \Vdash \mathbf{D} = \mathbf{D}'$.

Define now (in M)

$$E = \big\{(h, F) \in \mathbf{Q}(\mathbf{P}_{\mu+1}) \mid h \restriction \mu \Vdash^{\mathbf{P}_\mu} (h(\mu), F(\mu)) \in \mathbf{D}'\big\}.$$

We claim that E is dense in $\mathbf{Q}(\mathbf{P}_{\mu+1})$. Indeed, given arbitrary $(h, F) \in \mathbf{Q}(\mathbf{P}_{\mu+1})$ (by Lemma 1.9(b) we assume it is determined), we have $h \restriction \mu \Vdash (h(\mu), F(\mu))$ has an extension in $\mathbf{D}'$. So we can find a name $(\mathbf{a}, \mathbf{n})$ such that $h \restriction \mu \Vdash (h(\mu), F(\mu)) \leqslant (\mathbf{a}, \mathbf{n}) \in \mathbf{D}'$. Using Lemma 1.10 now, there is $(g, F \restriction \mu) \geqslant (h \restriction \mu, F \restriction \mu)$ in $\mathbf{Q}(\mathbf{P}_\mu)$ such that for every σ consistent with $(g, F \restriction \mu)$ there is $n(\sigma) \in \omega$ and $g \mid \sigma \Vdash (h(\mu), F(\mu)) \leqslant (\mathbf{a}, n(\sigma)) \in \mathbf{D}'$.

Let $n \geqslant$ any possible $n(\sigma)$. Then $g \mid \sigma \Vdash (h(\mu), F(\mu)) \leqslant (\mathbf{a}, n(\sigma)) \leqslant (\mathbf{a}, n) \in \mathbf{D}'$. 1.6(e) implies that $g \Vdash (h(\mu), F(\mu)) \leqslant (\mathbf{a}, n) \in \mathbf{D}'$. So

$$\big(g \cup \{(\mu, \mathbf{a})\}, (F \restriction \mu) \cup \{(\mu, n)\}\big) \in E$$

and extends (h, F).

So E is dense. And since $G \cap \mathbf{Q}(\mathbf{P}_{\mu+1})$ is M-generic (1.12) we can find $(h, F) \in E \cap G$. By the premise of the lemma $h \restriction \mu \leqslant f$, and $f \leqslant f_1$, so $f_1 \Vdash (h(\mu), F(\mu)) \in \mathbf{D}'$. Yet $f_1 \Vdash \mathbf{D}' = \mathbf{D}$, so $f_1 \Vdash (h(\mu), F(\mu)) \in \mathbf{H}(\mu) \cap \mathbf{D}$. □

2. Description of the iteration of Jensen's reals. L is the ground model, and the following constructions are done there. Let $\langle S_\alpha | \alpha \in \omega_1 \rangle$ be a $\diamondsuit$ sequence. For definiteness and absoluteness reasons we shall take the canonical diamond sequence (see [**J2**]) in which $S_\alpha \subseteq \alpha$ is the first (in the well-order of L) "counterexample" to the sequence $\langle S_i | i < \alpha \rangle$ when α is limit countable ordinal, and $S_{\alpha+1} = \varnothing$. It follows that the definition of S_α is absolute for transitive structures in which this definition can be carried out.

For every $\omega_1 \leqslant \zeta < \omega_2$ let θ_ζ be the first constructible bijection of ω_1 onto ζ.

Suppose $\mathbf{P}_\mu$ has been defined, we want to describe the next step in the iteration. Describing this step in terms of actual generic extensions, assume $\dot{\mathbf{P}}_\mu$ is an L-generic filter over $\mathbf{P}_\mu$ and $\langle r_\zeta | \zeta \in \mu \rangle$ is the resulting L-generic sequence of reals over $\mathbf{P}_\mu$; we have to define in $L[\langle r_\zeta | \zeta \in \mu \rangle] = L[\dot{\mathbf{P}}_\mu]$ a perfect poset $\mathbf{R} = \mathbf{R}(\mu)$ (the Jensen poset) and then, if $\underset{\sim}{\mathbf{R}}$ is the name of that poset, set $\mathbf{P}_{\mu+1} = \mathbf{P}_\mu * \underset{\sim}{\mathbf{R}}$. (Or $\mu = 0$ and we want actually to construct the poset $\mathbf{R}(0)$.)

Let $A = A(\mu) \subseteq \omega_1$ encode the generic *sequence* $\langle r_\zeta | \zeta \in \mu \rangle$ in some canonical straightforward way. For example, if $\mu \geqslant \omega_1$, define a relation Z on ω_1 by $(i, j) \in Z$ iff $\theta_\mu(i) < \theta_\mu(j)$; also put $Y = \{(\zeta, k) | k \in r_{\theta_\mu(\zeta)}\}$. Then ask A to encode Z and Y, using the canonical correspondence between ω_1 and $\omega_1 \times \omega_1$. So,

$$L[A] = L[\langle r_\zeta | \zeta \in \mu \rangle] = L[\dot{\mathbf{P}}_\mu] \quad (\text{or } A = \varnothing \text{ in case } \mu = 0).$$

In $L[A]$ we define inductively an increasing and continuous sequence $\langle \mathbf{R}_i = \mathbf{R}_i(\mu) | i \in \omega_1 \rangle$ of countable perfect posets; then we will set $\mathbf{R}(\mu) = \mathbf{R} = \bigcup_{i<\omega_1} \mathbf{R}_i$.

2.1. To begin with, $\mathbf{R}_0$ is the closure under finite unions of $\{({}^\omega 2)_s | s \in {}^{<\omega} 2\}$; and for limit δ, $\mathbf{R}_\delta = \bigcup_{i<\delta} \mathbf{R}_i$. Suppose $\mathbf{R}_i$ is defined, the construction of $\mathbf{R}_{i+1}$ is described below.

Set $\mathbf{R}_{i+1} = \mathbf{R}_i$ unless the following happens.

(1) S_i encodes (in some canonical obvious way) three objects: a relation $E_i \subseteq i \times i$ and two ordinals a_i, b_i smaller than i. ($E_i \times \{a_i\} \times \{b_i\}$ which is a subset of $i \times i \times i \times i$ is encoded by $S_i \subseteq i$.) Moreover, E_i is well founded and (i, E_i) is a model of ZF^- (set theory without the power-set axiom). Put $(\overline{M}, \in)$ to be the transitive structure isomorphic with (i, E_i). We also ask that $i \in \overline{M}$ is "the first uncountable cardinal" there and that the isomorphism of i onto $\overline{M}$ takes $a_i \in i$ to $a_i^{\overline{M}} = \overline{\mathbf{P}}_{\overline{\eta}} \in \overline{M}$ a poset which is, in $\overline{M}$, an iteration of length $\overline{\eta}$ of perfect posets. And $b_i \in i$ is taken by that isomorphism to $b_i^{\overline{M}}$ which is a function in $\overline{M}$.

(2)   The decoding of $A \cap i$ gives a sequence $\langle s_\zeta | \zeta \in \overline{\mu} \rangle$ of reals of length $\overline{\mu}$ and $\overline{\mu} \in \overline{M}$. This sequence is $\overline{M}$-generic over $\mathbf{P}_{\overline{\mu}}$, where $\overline{\mu} \leqslant \overline{\eta}$ and $\overline{\mathbf{P}}_{\overline{\mu}} = \{g \upharpoonright \overline{\mu} | g \in \overline{\mathbf{P}}_{\overline{\eta}}\}$. (Or $\overline{\mu} = 0$.)

(3)   When forming the extension $\overline{N} = \overline{M}[\langle s_\zeta | \zeta \in \overline{\mu} \rangle]$ we get $\mathbf{R}_i \in \overline{N}$. (In case $\overline{\mu} = 0$, $\overline{N} = \overline{M}$.) For $X \in \overline{M}$, $X^{\overline{N}}$ denotes the interpretation of the name X in the generic extension $\overline{N}$.

In case (1)–(3) hold, we look at $Q(\mathbf{R}_i)$ and pick in $L[A]$ some $\bar{N}$-generic filter, $\dot{Q}_i$, over $Q(\mathbf{R}_i)$ such that $(b_i^M(\bar{\mu}))^N \in \dot{Q}_i$ (if that is a condition in $Q(\mathbf{R}_i)$). In fact, great care is given to the choice of $\dot{Q}_i$ (in 2.5), but let us postpone that for awhile and see what can be deduced so far. Let $\dot{T}$ be the generic tree of $\dot{Q}_i$. Finally let $\mathbf{R}_{i+1}$ be the closure under finite unions of $\mathbf{R}_i \cup \{\dot{T} \wedge S \mid S \in \mathbf{R}_i \& \dot{T} \wedge S \neq \varnothing\}$. $\mathbf{R}_{i+1}$ is a perfect poset (see Lemma 1.2).

Let us denote the transitive model $\bar{M}$ by $\bar{M}(S_i)$ and the generic extension $\bar{N}$ by $\bar{M}(S_i)[A \cap i]$. Observe that $i < j \to \bar{M}(S_i) \subseteq \bar{M}(S_j)$. (As i is countable in $M(S_j)$ the set S_i can be defined there.)

2.2. LEMMA. *If* $X \in \bar{M}(S_i)[A \cap i][\dot{T}]$ *is a maximal antichain in* R_{i+1}, *then* X *is a maximal antichain in* R_j *for* $j \geqslant i + 1$.

PROOF. By induction on j. If j is limit the argument is trivial since $\mathbf{R}_j = \bigcup_{l<j}\mathbf{R}_l$. Suppose $j = k + 1$, where $k > i$ and X is maximal in $\mathbf{R}_k$. If $\mathbf{R}_{k+1} \neq \mathbf{R}_k$,

$$\bar{M}(S_k)[A \cap k]$$

can be constructed and $\mathbf{R}_k$ is found there, so $\dot{T}$ is there and $X \in \bar{M}(S_k)[A \cap k]$. Lemma 1.2 gives the desired conclusion when V there is replaced by $\bar{M}(S_k)[A \cap k]$. Using the absoluteness of incompatibility (§1) we get that if X in the lemma is maximal antichain above $U \in \mathbf{R}_{i+1}$, then it stays maximal above U in $\mathbf{R}_j, j \geqslant i + 1$. □

2.3. LEMMA. *In* $L[A] = L[\dot{\mathbf{P}}_\mu]$, $\mathbf{R}$ *satisfies the c.a.c.*

PROOF. The proof is by induction on μ. Assuming each $\mathbf{R}(\mu')$, $\mu' < \mu$, satisfies the c.a.c. in $L[\dot{\mathbf{P}}_{\mu'}]$, $\mathbf{P}_\mu$ is an iteration of proper posets each of cardinaltiy $\aleph_1$; hence $\mathbf{P}_\mu$ satisifes the $\aleph_2$-a.c. and is proper (see 1.13).

Let $X \in L[\langle r_\zeta \mid \zeta \in \mu \rangle]$ be a maximal antichain of $\mathbf{R}$. We show X is countable. Let $\mathbf{X}$ be the name of X in $\mathbf{P}_\mu$ forcing, and $\underset{\sim}{\mathbf{R}}$ the name of $\mathbf{R}$. Find in L an elementary substructure $K \prec H(\aleph_3)$ of cardinality $\aleph_1$ such that

$$\aleph_1 + 1 \subseteq K \quad \text{and} \quad \mu, \mathbf{P}_\mu, \underset{\sim}{\mathbf{R}}, \mathbf{X} \in K.$$

Put $K = \bigcup_{\alpha \in \omega_1} M_\alpha$ a union of a continuous and increasing chain of countable elementary substructures of K (such that $\mu, \mathbf{P}_\mu, \mathbf{R}, \mathbf{X} \in M_0$). Since K has cardinality $\aleph_1$, there is in L a well-founded relation E on ω_1 such that (ω_1, E) is isomorphic to $(K, \in)$. By a standard coding (see 2.1(1)), we find constructible $E' \subseteq \omega_1$ which encodes three objects: the relation E and two countable ordinals, one representing $\mathbf{P}_\mu$ (in (ω_1, E)) and the other an arbitrary condition on $Q(\mathbf{P}_\mu)$ (for this argument it does not matter which). The set $\{\alpha \mid E' \cap \alpha = S_\alpha\}$ is stationary; and remains so in $L[\dot{\mathbf{P}}_\mu]$, since $\mathbf{P}_\mu$ is proper (see [S1]).

The following three sets are closed unbounded in ω_1.

$\{\alpha \mid \alpha = \aleph_1 \cap M_\alpha\}$,

$\{\alpha \mid (M_\alpha, \in)$ is isomorphic to $(\alpha, E \cap \alpha \times \alpha)$ and $E \cap \alpha \times \alpha$ is encoded by $E' \cap \alpha\}$,

$\{\alpha \mid \dot{\mathbf{P}}_\mu$ is M_α-generic over $\mathbf{P}_\mu\}$.

This last set is closed unbounded since $\mathbf{P}_\mu$ satisfies the $\aleph_2$-a.c. and $\dot{\mathbf{P}}_\mu$ is, hence, K-generic over $\mathbf{P}_\mu$.

Pick i in the intersection of those three closed unbounded sets such that $E' \cap i = S_i$. Then S_i encodes a transitive structure $\overline{M}$ which is isomorphic to M_i. Let π: $M_i \to \overline{M}$ be the isomorphism. $\pi(\aleph_1) = i$. $\pi(\mathbf{P}_\mu) = \overline{\mathbf{P}}_{\tilde\mu}$ is in $\overline{M}$ an iteration of perfect posets (and $\bar\mu = \pi(\mu)$). $\overline{\mathbf{P}}_{\tilde\mu}$ and some member of $\mathbf{Q}(\overline{\mathbf{P}}_{\tilde\mu})$ are also decoded through S_i. $M_i[\dot{\mathbf{P}}_\mu] \prec H(\aleph_3)^{L[\dot{\mathbf{P}}_\mu]}$. $\dot{\mathbf{P}}_\mu$ is M_i-generic over $\mathbf{P}_\mu$, so $\pi''\dot{\mathbf{P}}_\mu$ is $\overline{M}$-generic over $\overline{\mathbf{P}}_{\tilde\mu}$ and $N = \overline{M}[\pi''\dot{\mathbf{P}}_\mu]$ is the collapse of $M_i[\dot{\mathbf{P}}_\mu]$. We continue to use π to denote this collapse function of $M_i[\dot{\mathbf{P}}_\mu]$ onto $\overline{N}$. The sequence $\langle r_\zeta \,|\, \zeta \in \mu \rangle$ and its code A are in $M_i[\dot{\mathbf{P}}_\mu]$. So $\pi(\langle r_\zeta \,|\, \zeta \in \mu \rangle) = \langle s_\zeta \,|\, \zeta \in \bar\mu \rangle$, encoded by $\pi(A) = A \cap i$, is $\overline{M}$-generic. Also, $\langle s_\zeta \,|\, \zeta \in \bar\mu \rangle$ is the sequence of generic reals given by $\pi''\dot{\mathbf{P}}_\mu$, hence $\overline{N} = \overline{M}[A \cap i] = \overline{M}[\langle s_\zeta \,|\, \zeta \in \bar\mu \rangle]$.

Now, $\mathbf{R} \in M_i[\dot{\mathbf{P}}_\mu]$ (since $\mathbf{R} \in M_i$) and $\mathbf{R} = \bigcup_{j<\omega_1} \mathbf{R}_j$, hence $\pi(\mathbf{R}) = \bigcup_{j<i} \pi(\mathbf{R}_j)$ (since $i = \pi(\aleph_1)$), but $\pi(\mathbf{R}_j) = \mathbf{R}_j$ (since $\mathbf{R}_j$ is hereditarily countable) so that finally $\pi(\mathbf{R}) = \bigcup_{j<i} \mathbf{R}_j = \mathbf{R}_i \in \overline{N}$. Similarly $X \in M_i[\dot{\mathbf{P}}_\mu]$, and $\pi(X) = X \cap \mathbf{R}_i \in \overline{N} = M(S_i)[A \cap i]$ is a maximal antichain of $\mathbf{R}_i$.

All conditions (1)–(3) in the definition of $\mathbf{R}_{i+1}$ are fulfilled, Lemmas 1.2 and 2.2 can be applied to derive that $\pi(X)$ is maximal in $\mathbf{R}$. Hence $\pi(X) = X$. And $\pi(X)$ is countable.

2.4. LEMMA (UNIQUENESS OF THE GENERIC OBJECT). *Let $\dot{\mathbf{R}}$ be an $L[\dot{\mathbf{P}}_\mu]$-generic filter over $\mathbf{R}$. Then $\dot{\mathbf{R}}$ is the unique $L[\dot{\mathbf{P}}_\mu]$-generic filter over $\mathbf{R}$ in $L[\dot{\mathbf{P}}_\mu, \dot{\mathbf{R}}]$.*

PROOF. The canonical generic real of $\dot{\mathbf{R}}$ was defined at the beginning of §1. A function $\tau \in {}^\omega 2$ is called $L[\dot{\mathbf{P}}_\mu]$-*generic over* $\mathbf{R}$ (*generic* for short) if $\{U \in \mathbf{R} \,|\, \tau$ is a branch of $U\}$ is an $L[\dot{\mathbf{P}}_\mu]$-generic filter over $\mathbf{R}$. By the duality between generic reals and generic filters, it is enough to prove that the canonical generic real is the unique generic function in $L[\dot{\mathbf{P}}_\mu, \dot{\mathbf{R}}]$.

Suppose $S \in \mathbf{R}$, $\tau \in L[\dot{\mathbf{P}}_\mu]$ is a name in $\mathbf{R}$-forcing and in $L[\dot{\mathbf{P}}_\mu]$

$$S \Vdash^{\mathbf{R}} \tau \in {}^\omega 2 \text{ is a function different from the canonical generic function.}$$

We would like to get an extension of S in $\mathbf{R}$ which forces "τ is not generic".

By extending S and using the supposition, we can assume there are $n \in \omega$ and $e \in {}^n 2$ such that $e \notin S$ but $S \Vdash \tau \restriction n = e$.

As in the proof of Lemma 2.3, there is $i \in \omega_1$ such that S_i encodes a transitive structure $\overline{M}(S_i)$—a model of ZF^-—in which i is the first uncountable cardinal. And such that the following holds: $A \cap i$ encodes a generic sequence and

$$\overline{N} = \overline{M}(S_i)[A \cap i]$$

is a generic extension via $\overline{\mathbf{P}}_{\tilde\mu}$. $\overline{N}$ is the transitive collapse of $M[\dot{\mathbf{P}}_\mu]$—an elementary substructure of $H(\aleph_3)^{L[\dot{\mathbf{P}}_\mu]}$—and $\mathbf{R}, S, \tau \in M[\dot{\mathbf{P}}_\mu]$. Moreover

$$\left(b_i^{\overline{M}(S_i)}(\bar\mu) \right)^{\overline{N}} = (S, 0).$$

τ is a name in $\mathbf{R}$-forcing of a real; so, since $\mathbf{R}$ satisfies the c.a.c., τ is hereditarily countable and $\pi(\tau) = \tau$. Also $\pi(S) = S$, and $\pi(\mathbf{R}) = \mathbf{R}_i$. It follows that, in $\overline{N}$,

$$S \Vdash^{\pi(\mathbf{R})} \tau \in {}^\omega 2 \text{ is a function different from the canonical generic function.}$$

Also, $S \Vdash^{\pi(\mathbf{R})} \tau \restriction n = e$.

Now, in $\overline{N}$, for each $U \in \pi(\mathbf{R}) = \mathbf{R}_i$ construct the following set $D(U) \subseteq \mathbf{Q}(\mathbf{R}_i)$. $D(U) = \{(T, m) \in \mathbf{Q}(\mathbf{R}_i) \mid T \subseteq S$ and for some $U' \subseteq U$, $U' \in \mathbf{R}_i$, $T \Vdash^{\mathbf{R}_i} \tau$ is not a branch of $U'\}$.

Claim. $D(U) \in \overline{N}$ is dense above $(S, 0)$ in $\mathbf{Q}(\mathbf{R}_i)$ for any $U \in \mathbf{R}_i$.

PROOF. Let $U \in \mathbf{R}_i$ be given. $D(U) \in \overline{N}$ is clear since the definition is done in $\overline{N}$. For any $(T, m) \in \mathbf{Q}(\mathbf{R}_i)$, $T \subseteq S$, look for $k \in \omega$ such that $U \cap {}^k 2$ contains more than 2^m members. Then extend T to T' such that $T \cap {}^m 2 = T' \cap {}^m 2$ and

$$(\forall s \in T \cap {}^m 2)(\exists e(s) \in {}^k 2) T'_s \Vdash \tau \restriction k = e(s).$$

Pick $t \in U \cap {}^k 2$, $t \neq e(s)$ for all s, and put $U' = U_t$. Then $T' \Vdash \tau$ is not a branch of U'. And $(T, m) \leqslant (T', m)$. $\square$

Recall how $\mathbf{R}_{i+1}$ was defined: a tree $\dot{T}$ was derived from an $\overline{N}$-generic filter $\dot{Q}_i$ over $\mathbf{Q}(\mathbf{R}_i)$. We choose S_i in such a way that S_i, through $b_i^{\overline{M}(S_i)}$, points at $(S, 0)$, so that $(S, 0) \in \dot{Q}_i$.

The $D(U)$ are dense sets in $\mathbf{Q}(\mathbf{R}_i)$ above $(S, 0)$. Hence for any $U \in \mathbf{R}_i$ there is $(T, m) \in \dot{Q}_i \cap D(U)$; it follows that there is $U' \subseteq U$, in $\mathbf{R}_i$, such that

$T \Vdash^{\mathbf{R}_i} \tau$ is not a branch of U'. (The forcing is in $\overline{N}$.) By Lemma 1.2 any maximal antichain of $\mathbf{R}_i$ in $\overline{N}$ remains a maximal antichain of $\mathbf{R}_{i+1}$; hence τ is a name of a branch also in the $\mathbf{R}_{i+1}$-forcing.

But since $\dot{T} \subseteq T$ it follows from Lemma 1.2 that

$$(*) \qquad\qquad \dot{T} \Vdash^{\mathbf{R}_{i+1}} \tau \text{ is not a branch of } U' \text{ (in } \overline{N}[\dot{T}]).$$

So in $\overline{N}[\dot{T}]$, every $U \in R_i$ has an extension U' in $\mathbf{R}_i$ such that $(*)$ holds. But this is also true for any $U \in \mathbf{R}_{i+1}$: Because if $U \in \mathbf{R}_{i+1} - \mathbf{R}_i$, say $U = \dot{T} \wedge S^*$ for some $S^* \in \mathbf{R}_i$ (w.l.o.g. U has this form), then of course U cannot have τ as a branch. (Recall $S \Vdash \tau \restriction n = e$, so $\dot{T} \Vdash \tau \restriction n = e$, but $e \notin S$ and a fortiori $e \notin \dot{T} \wedge S^*$.) Hence there is in $\overline{N}[\dot{T}]$ a dense set of $U' \in \mathbf{R}_{i+1}$ for which $(*)$ holds. So by Lemma 2.2 any member of $\mathbf{R}$ is compatible with some such U', Hence $(*)$ holds in $\mathbf{R}$ for a dense set of U', hence $\dot{T} \Vdash \tau$ is not generic. $\square$

2.5. Let us come back to the definition of $\mathbf{R}$. We want to describe in more detail how the filter $\dot{Q}_i$ is actually chosen and then to use its special properties. Come back to 2.1 (the section where $\mathbf{R}_{i+1}$ is defined) and assume (1)–(3) hold. Let $\overline{G} \in L$ be the first (in L canonical ordering) $\overline{M} = \overline{M}(S_i)$-generic filter over $\mathbf{Q}(\overline{\mathbf{P}}_{\bar{\eta}})$ with $b_i^{\overline{M}} \in \overline{G}$ (if $b_i^{\overline{M}} \in \mathbf{Q}(\overline{\mathbf{P}}_{\bar{\eta}})$). Assume $\bar{\mu} < \bar{\eta}$ (if $\bar{\mu} = \bar{\eta}$, $\dot{Q}_i$ can be arbitrary). Let

$$\overline{H}(\bar{\mu}) = \left\{ \left(\left(f(\bar{\mu}) \right)^{\overline{N}}, F(\bar{\mu}) \right) \mid \text{where } (f, F) \in \overline{G} \right\}.$$

In case $\overline{H}(\bar{\mu})$ is $\overline{N}$-generic over $\mathbf{Q}(\mathbf{R}_i)$, define $\dot{Q}_i = \overline{H}(\bar{\mu})$; otherwise $\dot{Q}_i$ can be an arbitrary $\overline{N}$-generic filter (in $L[A]$).

This ends the description of the iteration: $\mathbf{P}_{\omega_2}$ is our final poset. In the next section we show that in the generic extension $L[\dot{\mathbf{P}}_{\omega_2}]$ the degrees of constructibility have order-type ω_2. Then we conclude the theorem. Yet a major technical piece is

missing; this is provided by the following:

2.6. MAIN LEMMA: THE FUSION LEMMA. *In L, for any $a \in H(\aleph_3)$, $\eta \in \omega_2$, and any $g_0 \in \mathbf{P}_\eta$ there is a countable elementary substructure $M \prec H(\aleph_3)$ with $a \in M$, and a condition $g \in \mathbf{P}_\eta$ extending g_0 such that the following is true.*

$$\{(f, F) \in M \mid (f, F) \leqslant (g, F)\} \text{ is an } M\text{-generic filter over } \mathbf{Q}(\mathbf{P}_\eta).$$

(*It is clear that any two conditions in this set are compatible.*)

PROOF. As in the proof of 2.3, pick in L some elementary substructure $K \prec H(\aleph_3)$ with $\mathrm{card}(K) = \aleph_1$, $\omega_1 + 1 \subseteq K$, and $g_0, \eta, \mathbf{P}_\eta, a \in K$. Encode the model K, the poset $\mathbf{P}_\eta$, and the condition g_0 by a subset E of ω_1. Put $K = \bigcup_{\alpha \in \omega_1} M_\alpha$ a continuous and increasing chain of countable elementary substructures of K ($\eta, \mathbf{P}_\eta, a \in M_0$). As in 2.3, find $i \in \omega_1$ such that $\overline{M} = \overline{M}(S_i)$ is isomorphic to M_i, $\pi: M_i \to \overline{M}$ is the collapsing function, $\pi(\aleph_1) = i$, and S_i also points to $\overline{\mathbf{P}}_{\bar\eta} = \pi(\mathbf{P}_\eta)$ and to $(\bar{g}, \varnothing)$ where $\bar{g}_0 = \pi(g_0)$. (In the notation of 2.1, $a_i^{\overline{M}} = \pi(\mathbf{P}_\eta)$ and $b_i^{\overline{M}} = (\bar{g}_0, \varnothing)$.) Let $\overline{G}$ be the first constructible $\overline{M}$-generic filter over $\mathbf{Q}(\overline{\mathbf{P}}_{\bar\eta})$ containing $(\bar{g}, \varnothing)$.

$$G = \{\pi^{-1}(g) \mid g \in \overline{G}\}$$

is an M_i-generic filter over $\mathbf{Q}(\mathbf{P}_\eta)$ containing $(g_0, \varnothing)$. In 1.15 a condition $g \in \mathbf{P}_\eta$ and a function $\mathbf{H}$ were constructed with M_i and G as parameters. The next lemma clearly shows that M_i is the required model and g is the required condition.

2.7. LEMMA. *If $(f, F) \in G$ then $(f, F) \leqslant (g, F)$.*

PROOF. We show by induction on $\zeta \in M_i$ that for any $(f, F) \in G$

(2.8) $(f {\restriction} \zeta, F {\restriction} \zeta) \leqslant (g {\restriction} \zeta, F {\restriction} \zeta)$.

For $\zeta = 0$ or limit ordinal there is no problem. So say $\zeta = \mu + 1$. It is enough to show $g {\restriction} \mu \Vdash (f(\mu), F(\mu)) \leqslant (g(\mu), F(\mu))$, for any $(f, F) \in G$.

The inductive assumption (2.8) and 1.12 and Lemma 1.14 imply that $g {\restriction} \mu$ is an M_i-generic condition over $\mathbf{P}_\mu$.

The argument proceeds in terms of actual generic extensions; let $\dot{\mathbf{P}}_\mu$ be an L-generic filter over $\mathbf{P}_\mu$ with $g {\restriction} \mu \in \dot{\mathbf{P}}_\mu$. We intend to show that for any $(f, F) \in G$ the interpretation of $g(\mu)$ (which is denoted $g(\mu)^{\dot{\mathbf{P}}_\mu}$) is a subtree of $f(\mu)^{\dot{\mathbf{P}}_\mu}$, and both trees have the same intersection with $^{F(\mu)}2$.

Let $A \subseteq \omega_1$ be the canonical encoding of the generic sequence of reals provided by $\dot{\mathbf{P}}_\mu$. $\dot{\mathbf{P}}_\mu$ is M_i-generic over $\mathbf{P}_\mu$, hence $\pi''(\dot{\mathbf{P}}_\mu \cap M_i)$ is $\overline{M}$-generic over $\pi(\mathbf{P}_\mu) = \overline{\mathbf{P}}_{\bar\mu}$. π can be extended to collapse $M_i[\dot{\mathbf{P}}_\mu]$ ($\prec H(\aleph_3)^{L[\dot{\mathbf{P}}_\mu]}$) onto $\overline{N} = \overline{M}[\pi''\dot{\mathbf{P}}_\mu]$. $A \in M_i[\dot{\mathbf{P}}_\mu]$ and $\pi(A) = A \cap i$ encodes $\langle s_\zeta \mid \zeta \in \bar\mu \rangle$—the $\overline{M}$-generic sequence of generic reals given by $\pi(\dot{\mathbf{P}}_\mu)$. $\pi(\mathbf{R}(\mu)) = \mathbf{R}_i(\mu) = \mathbf{R}_i$. It follows that (1)–(3) of 2.1 hold and $\overline{N} = \overline{M}(S_i)[A \cap i]$.

Recall how $\mathbf{H}(\mu)$ is defined in $L[\dot{\mathbf{P}}_\mu]$ (1.15): All $(f, F) \in G$ such that $f {\restriction} \mu \in \dot{\mathbf{P}}_\mu$ were collected—but by induction these are *all* $(f, F) \in G$, since $g {\restriction} \mu \in \dot{\mathbf{P}}_\mu$—then $\mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu}$ is formed by the pairs $(f(\mu)^{\dot{\mathbf{P}}_\mu}, F(\mu))$ thus obtained. $g(\mu)^{\dot{\mathbf{P}}_\mu}$ in turn, is the tree derived from $\mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu}$—if that tree is in $\mathbf{R}(\mu)$—and is $^\omega 2$ otherwise. We will show in

the following paragraph that the first possibility occurs, this obviously implies

$$\left(f(\mu)^{\dot{\mathbf{P}}_\mu}, F(\mu) \right) \leqslant \left(g(\mu)^{\dot{\mathbf{P}}_\mu}, F(\mu) \right)$$

whenever $(f, F) \in G$ (Lemma 1.16, and the display prior to Lemma 1.2.)

The induction hypothesis and Lemma 1.16 give that $\mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu}$ is $M_i[\dot{\mathbf{P}}_\mu]$-generic over $Q(\mathbf{R}(\mu))$. Since π is the identity on the hereditarily countable sets, $\pi''\mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu} = \mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu}$. Hence $\mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu}$ is $\overline{N}$-generic over $Q(\mathbf{R}_i)$.

But $\pi''\mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu} = \{(f(\bar\mu)^{\overline{N}}, F(\bar\mu)) | (f, F) \in \overline{G}\}$ follows from the elementarity of π, and this set, called $\overline{\mathbf{H}}(\bar\mu)$ in 2.5, is the $\overline{N}$-generic filter $\dot{\mathbf{Q}}_i$ used to construct $\mathbf{R}_{i+1}$. This is why the tree derived from $\mathbf{H}(\mu)^{\dot{\mathbf{P}}_\mu}$ is in $\mathbf{R}_{i+1} \subset \mathbf{R}(\mu)$.

3. The degrees of constructibility have order-type ω_2. Let $\dot{\mathbf{P}}_{\omega_2}$ be an L-generic filter over $\mathbf{P}_{\omega_2}$, and for $\alpha < \omega_2$ let $\dot{\mathbf{P}}_\alpha$ be the projection of $\dot{\mathbf{P}}_{\omega_2}$ in $\mathbf{P}_\alpha$ (i.e., $\mathbf{P}_\alpha = \{f \restriction \alpha \,|\, f \in \dot{\mathbf{P}}_{\omega_2}\}$).

For every real $r \subseteq \omega$ in $L[\dot{\mathbf{P}}_{\omega_2}]$ let $\eta(r) < \omega_2$ be the least ordinal η such that $r \in L[\dot{\mathbf{P}}_\eta]$. The headline of this section is consequence of the next lemma.

3.1. LEMMA. *For r and r' reals in $L[\dot{\mathbf{P}}_{\omega_2}]$*

$$r' \in L[r] \text{ if and only if } \eta(r') \leqslant \eta(r).$$

PROOF. The nontrivial direction is the right to left implication. So assume $\eta(r') \leqslant \eta(r)$ and let us prove that $r' \in L[r]$.

Put $\eta = \eta(r)$ and $\eta' = \eta(r')$. Let $\mathbf{r}$ and $\mathbf{r}'$ be names of r and r' in $\mathbf{P}_\eta$. Pick $g_0 \in \dot{\mathbf{P}}_\eta$ such that $g_0 \Vdash (\forall \zeta \in \eta)\mathbf{r} \notin L[\dot{\mathbf{P}}_\zeta]$.

Lemma 2.6 has the form "for any $g_0 \in \mathbf{P}_\eta$ and $a \in H(\aleph_3)$ there is $g \in \mathbf{P}_\eta$, $g_0 \leqslant g$, and M such that...". This form indicates that the set of all such g's is dense in $\mathbf{P}_\eta$. Hence there is in L a countable $M \prec H(\aleph_3)$ such that $g_0, \mathbf{r}, \mathbf{r}', \eta, \eta' \in M$, and there is $g \in \dot{\mathbf{P}}_\eta$, $g_0 \leqslant g$, such that

(3.2) $\{(f, F) \in \mathbf{P}_\eta \cap M | (f, F) \leqslant (g, F)\}$ is an M-generic filter over $Q(\mathbf{P}_\eta)$.

g is an M-generic condition over $\mathbf{P}_\eta$ (Lemma 1.14), hence $\dot{\mathbf{P}}_\eta$ is an M-generic filter over $\mathbf{P}_\eta$. Let $\overline{M}$ be the collapse of M, then $M[\dot{\mathbf{P}}_\eta]$ is collapsed to $\overline{N} = \overline{M}[\pi(\dot{\mathbf{P}}_\eta)]$ (π denotes the Mostowski collapsing isomorphism). Let $\langle r_i | i \in \eta \rangle$ be the sequence of generic reals obtained from $\dot{\mathbf{P}}_\eta$. $\langle r_i | i \in \eta \rangle \in M[\dot{\mathbf{P}}_\eta]$; put $\langle s_i | i \in \pi(\eta) = \bar\eta \rangle = \pi(\langle r_i | i \in \eta \rangle) \in \overline{N}$.

The proof that $r' \in L[r]$ consists of two parts:
(1) showing that $r' \in L[\langle s_i | i \in \bar\eta \rangle]$ and then,
(2) $\langle s_i | i \in \bar\eta \rangle \in L[r]$.
(1) is evident, as $r' \in M[\dot{\mathbf{P}}_\eta]$ and

$$\pi(r') = r' \in \overline{M}\left[\pi(\dot{\mathbf{P}}_\eta)\right] = \overline{M}\left[\langle s_i | i \in \bar\eta \rangle\right], \quad \text{and} \quad \overline{M} \in L.$$

(2) Or actually $\langle r_i | i \in \eta \cap M \rangle \in L[r]$ is concluded below. For each $\mu \in \eta \cap M$ and $n \in \omega$ define in L a subset $D(\mu, n)$ of $Q(\mathbf{P}_\eta)$ as follows.

> $(f, F) \in D(\mu, n)$ iff $(f, F) \in Q(\mathbf{P}_\eta)$ is incompatible with $(g_0, \varnothing)$ or (f, F) is determined $\mu \in \text{dom}(F)$ and $n \leqslant F(\mu)$ and there is a collection of pairwise disjoint basic open sets $\{C_\sigma | \sigma \text{ consistent with } (f, F)\}$ such that for each such σ, $f | \sigma \Vdash^{\mathbf{P}_\eta} \mathbf{r} \in C_\sigma$.

Remark that the function which associates to each $(f, F) \in D(\mu, n)$ compatible with $(g_0, \varnothing)$ the correspondence $\sigma \mapsto C_\sigma$, is constructible.

Lemmas 1.9 and 1.11 imply that $D(\mu, n)$ is dense in $\mathbf{Q}(\mathbf{P}_\eta)$. In fact $D(\mu, n) \in M$. Hence, by (3.2) there is $(f, F) \in D(\mu, n)$ such that $(f, F) \leq (g, F)$.

Let (f, F) be any member of $D(\mu, n)$ such that $(f, F) \leq (g, F)$. Since $(g_0, \varnothing) \leq (g, \varnothing)$ it follows that $(g_0, \varnothing)$ and (f, F) are compatible in $\mathbf{Q}(\mathbf{P}_\eta)$ (by (g, F)). Hence, (f, F) is determined and there are disjoint basic sets C_σ, for σ consistent with (f, F), such that $f \mid \sigma \Vdash \mathbf{r} \in C_\sigma$. But σ is consistent with (f, F) iff σ is consistent with (g, F) and in that case $f \mid \sigma \leq g \mid \sigma$ (see Lemmas 1.6b and 1.9.) Hence $g \mid \sigma \Vdash \mathbf{r} \in C_\sigma$, for each such σ. It follows that for each such σ

$$g \Vdash^{\dot{\mathbf{P}}_\eta} \mathbf{r} \in C_\sigma \leftrightarrow \sigma = \langle \mathbf{r}_i \restriction F(i) \mid i \in \mathrm{dom}(F) \rangle.$$

But $g \in \dot{\mathbf{P}}_\eta$, so

$$r \in C_\sigma \leftrightarrow \sigma = \langle r_i \restriction F(i) \mid i \in \mathrm{Dom}(F) \rangle.$$

This holds for any $\mu \in \eta \cap M$ and $n \in \omega$ and $(f, F) \in D(\mu, n)$ with $(f, F) \leq (g, F)$. The sequence $\langle D(\mu, n) \mid \mu \in \eta \cap M, n \in \omega \rangle$ is in L. So $\langle r_i \mid i \in \eta \cap M \rangle \in L[r]$: To know what is $r_\mu \restriction n$, pick any $(f, F) \in D(\mu, n)$ with $(f, F) \leq (g, F)$, then find the only σ consistent with (f, F) such that $r \in C_\sigma$—we have $r_\mu \restriction n = \sigma(\mu) \restriction n$.

4. Proof of the theorem: minimality of the extension. Let W be a transitive model of set theory included in $L[\mathbf{P}_{\omega_2}]$ and including L. Suppose that the negation of the continuum hypothesis holds in W; we have to prove $W = L[\dot{\mathbf{P}}_{\omega_2}]$. Since W contains $\aleph_2$ many reals, and since the degrees of constructibility are well ordered of order-type ω_2, W contains *all* the reals of $L[\dot{\mathbf{P}}_{\omega_2}]$. The desired property of W follows once we show $\langle r_i \mid i \in \omega_2 \rangle \in W$. This is an easy inductive consequence of Lemma 2.4: r_j must be in the jth degree of constructibility and it is the only real in that equivalence class which is $L[\langle r_i \mid i < j \rangle]$-generic over $\mathbf{R}(j)$.

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DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NEW HAMPSHIRE 03755

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY, BEER SHEVA, ISRAEL (Current address)