ADVANCES IN MATHEMATICS 55, 75–89 (1985)

Minimal Model of "$\aleph_1^L$ Is Countable" and Definable Reals

URI ABRAHAM*

Department of Mathematics, Ben Gurion University, Beer Sheva, Israel

INTRODUCTION

Let N be a countable transitive model of ZFC; we say N is *a minimal model of* "$\aleph_1^L$ *is countable*" iff $\aleph_1^L$ (the ordinal which $L \cap N$ considers to be $\aleph_1$) is a countable ordinal in N, and whenever $K \subseteq N$ is a transitive model of ZFC that contains all the ordinals of N, if $\aleph_1^L$ is countable in K then $K = N$.

The following theorem is due to Prikry (see [S]); we prove it in Section 1.

THEOREM A. *Let M be a countable transitive model of $ZFC + V = L$. There exists a generic extension of M which is a minimal model of "$\aleph_1^L$ is countable."*

In the following M stands for a countable transitive model of $ZFC + V = L$.

In [JS, Sect. 4], Solovay constructs a generic extension $M[a]$ where $a \subseteq \omega$ is a non-constructible Π_2^1 singleton in $M[a]$. In Section 5 there, Jensen strengthens Solovay's construction and finds $a \subseteq \omega$ such that a is a Π_2^1 singleton in the generic extension $M[a]$ and, moreover, *all the constructible reals (i.e., reals in M) are recursive in a.*

In [J1], Jensen finds a generic extension $M[a]$ in which $a \subseteq \omega$ is a non-constructible Π_2^1 singleton and $M[a]$ is a minimal model of $ZFC + V \neq L$. Our result in Section 2 is motivated by trying to find x in the following "equation":

$$\frac{x}{[\mathrm{J1}]} = \frac{[\mathrm{JS, Sect. 5}]}{[\mathrm{JS, Sect. 4}]}.$$

THEOREM B. *Let M be a countable transitive model of $ZFC + V = L$. There exists a generic extension $M[a]$, $a \subseteq \omega$, such that:*

1. *$M[a]$ is a minimal model of "$\aleph_1^L$ is countable."*

2. *a is a Π_2^1 singleton in $M[a]$ and all the constructible reals are recursive in a.*

* Research partially supported by NSF grant (MCS 8103410). At the time of writing the article the author was J. W. Young, Lecturer at Dartmouth College.

0001-8708/85 $7.50

1. A Minimal Model of "$\aleph_1^L$ Is Countable": Proof of Theorem A[1]

1.1. *We begin with some definitions.* Let V be our universe of set theory. We work in V. Assume $2^{\aleph_1} = \aleph_2$ in V. ${}^{\omega}\omega_1$ is the set of all finite sequences of countable ordinals. The letters f, g, h, l denote elements of ${}^{\omega}\omega_1$ which are increasing sequences. n and k are members of ω. $T \subseteq {}^{\omega}\omega_1$ is a *tree* iff T consists of increasing sequences, and $f \in T$ and $g \subseteq f$ imply $g \in T$. $|f|$ is the *length* of the sequence f. T_n is the nth level of T. A *successor* (*in* T) of $f \in T$ is a sequence of the form $f^\frown\langle\alpha\rangle$ which is in T.

Let P be the set of all trees $T \subseteq {}^{\omega}\omega_1$ such that for some $l \in T$ (which we call $s[T]$—the stem of T):

1. For all $f \in T$, $f \subseteq l$ or $l \subseteq f$.
2. If $f \in T$ and $l \subseteq f$ then f has $\aleph_1$ many successors in T.

P is partially ordered by inclusion: $T^* \leqslant T$ means that $T^* \subseteq T$ (T^* *extends* T). $T^* \leqslant T$ implies $s(T^*) \supseteq s(T)$. It is clear that if $\dot{P}$ is a V-generic filter over P then $s = \bigcup\{s(T) | T \in \dot{P}\}$ is an increasing function from ω cofinal in ω_1. Hence $\aleph_1$ is collapsed in $V[\dot{P}]$. Actually $V[s] = V[\dot{P}]$ because if one sets $H = \{T \in P | s$ is a branch of $T\}$ (s is a *branch* of T means that $s \restriction n \in T$ for all $n < \omega$), then $H = \dot{P}$.

The cardinality of P is $2^{\aleph_1} = \aleph_2$, hence cardinals above $\aleph_2$ are not collapsed by P. We have yet to show that $\aleph_2$ is not collapsed and then the minimality of the extension.

1.2. *Further definitions.* For $T \in P$ and $f \in T$ we denote by $T(f)$ the condition of the form $\{g \in T | g$ is compatible with $f\}$. $T(f) \leqslant T$. We say T^* is a *reduction* of T iff $T^* \leqslant T$ and $s(T^*) = s(T)$. We say T^* is a *limited reduction* of T iff T^* is a reduction of T and for any $f \in T^*$ all but countably many successors of f in T are in T^*. Let us define for $n < \omega$, $T^* \leqslant_n T$ iff T^* is a reduction of T and $T^*_k = T_k$ for all $k \leqslant n$.

1.3. Lemma. *If $\langle T^n | n < \omega\rangle$ is a sequence of conditions in P and $T^{n+1} \leqslant_n T^n$ for all n, then there exists $T \in P$ such that $T \leqslant_n T^n$ for all n. (Actually the sequence has a greatest lower bound in P.)*

Proof. Set $T = \bigcup_{n<\omega} T^n_n$.

1.4. Corollary. *Let $\langle D_n | n < \omega\rangle$ be a sequence of dense subsets of P. $T \in P$. Then there exists a reduction T^* of T such that for any $f \in T^*$ and for each $k \leqslant |f|$, if some reduction of $T^*(f)$ is in D_k, then $T^*(f)$ is in D_k.*

[1] The forcing notion P is a modification of Namba [N] and Bukovsky [B] or Laver [L] or Sacks [S].

1.5. *Conclusion.* In $V[\dot{P}]$, $\aleph_2$ remains a cardinal (and becomes the first uncountable cardinal).

Proof. Given a condition T that forces "e: $\aleph_0 \to \aleph_2$" we want to find $T^* \leqslant T$ and an ordinal $\gamma < \aleph_2$ such that $T \Vdash$ "Range$(e) \subseteq \gamma$." Define D_n to be the collection of all conditions in P that for some $\alpha < \aleph_2$ force "$e(n) = \alpha$." D_n is dense above T. Find a reduction T^* of T as in the corollary, and let $\gamma = \text{Sup}\{\alpha \mid \text{For some } f \in T^* \text{ and } n,\ T^*(f) \Vdash \text{“}e(n) = \alpha\text{”}\}$. As T^* is of cardinality $\aleph_1$, $\gamma < \aleph_2$. Now if $T^{**} \leqslant T^*$ and $T^{**} \Vdash$ "$e(n) = \alpha$," then by extending T^{**} further we can assume $|s(T^{**})| \geqslant n$ and so, denoting $(sT^{**}) = g$, we get (by the corollary) that $T^*(g) \Vdash$ "$e(n) = \alpha$." Hence $\alpha < \gamma$.

1.6. *The minimality of the extension.* Remember $s: \aleph_0 \to \aleph_1^V$ is the generic function that collapses $\aleph_1$. Assume $T \Vdash$ "$t: \aleph_0 \to \aleph_1$ is cofinal." We want to find $T^* \leqslant T$ such that $T^* \Vdash$ "s can be reconstructed from T^* and t." This will follow if T^* has the following property:

For every $f \in T^*$ there are $n, k < \omega$ such that $|f| < k$ and:

1. For each $f \subseteq h \in T^*_k$, $T^*(h)$ decides the value of $t(n)$, (i.e., $T^*(h) \Vdash$ "$t(n) = \alpha$" for some $\alpha < \omega_1$).

2. For $h \neq h'$ extending f in T^*_k, the value of $t(n)$ decided by $T^*(h)$ is different from the value decided by $T^*(h')$.

This is enough because if T^* is as above then s can be recovered from t and T^* in $V[t]$. We proceed to obtain such T^*.

1.7. Definition. Let $T \in P$.

a. We say that $f \in T$ *splits* n (with respect to t) iff there is an uncountable set S of successors of f in T, and for each $g \in S$ there is a reduction Rg of $T(g)$ such that:

(1) Rg decides the value of $t(n)$, and

(2) if $g \neq g' \in S$ then the value of $t(n)$ decided by Rg is different from that decided by Rg'.

b. For a tree T with $f = s(T), n, k < \omega$, say that $\varphi(t, T, n, k)$ holds iff $k \geqslant 1$ and there is a reduction R of T such that:

For each $h \in R_{|f|+k}$ $R(h)$ decides the value of $t(n)$, and different such h's decide different values for $t(n)$.

1.8. Lemma. *If for some $n, k < \omega$, $k \geqslant 1$, every $g \in T$ with $|g| = |s(T)| + k - 1$ splits n then $\varphi(t, T, n, k)$ holds. More generally, if for uncountably many successors f of $s(T)$ there are $n, k < \omega$ such that $\varphi(t, T(f), n, k)$ holds, then for some n, $k < \omega$ $\varphi(t, T, n, k)$ holds.*

The proof of the lemma is obvious, (use the fact that given any collection $\langle S_i | i < \aleph_1 \rangle$ of uncountable subsets of $\aleph_1$, there are $S'_i \subset S_i$ uncountable which are pairwise disjoint).

Now it is clear that if for every $T' \leqslant T$ there is some n and k such that $\varphi(t, T', n, k)$ holds, then a reduction T^* of T can be constructed to satisfy the requirement of 1.6.

1.9. *So assume that for some extension of T (which we again call T for convenience) for every n and k, $\varphi(t, T, n, k)$ does not hold.*

We will get a contradiction.

1.10. LEMMA. *For all but countably many successors h of $s(T)$ in T, for every n and k, $\varphi(t, T(h), n, k)$ does not hold.*

The proof is obvious by Lemma 1.8.

Now, by repeating this lemma, we can find a limited reduction U of T such that for *every* $h \in U$ and every n, k, $\varphi(t, T(h), n, k)$ does not hold. In fact, because U is a reduction of T, the following is true:

For every $h \in U$, n, $k < \omega$, $\varphi(t, U(h), n, k)$ does not hold. In particular for any n no $h \in U$ splits n. Hence for any $h \in U$:

1.11. The following set is countable.

$\{\alpha |$ For some n there is a reduction of $U(h)$ that forces "$t(n) = \alpha$"$\}$.

Also for any $h \in U$ the following set is countable:

$\{\alpha |$ For some n and $g \in U$, a successor of h, there is a reduction of $U(g)$ which forces "$t(n) = \alpha$"$\}$.

Using this we can find a limited reduction U' of U such that:

1.12. For every $h \in U'$ with $h \supseteq s(U)$ and for any $\alpha < \omega_1$ and $n < \omega$, if there is a successor g of h in U' and a reduction of $U(g)$ which forces "$t(n) = \alpha$" *then* there are $\aleph_1$ many such successors of h.

But, we claim: there is a reduction of $U(g)$ which forces "$t(n) = \alpha$" iff there is a reduction of $U'(g)$ which forces "$t(n) = \alpha$." And the reason is the following simple lemma.

1.13. LEMMA. *If A, B, $C \in P$, B is a reduction of A and C is a limited reduction of A, then $B \cap C \in P$ is a reduction of C and a limited reduction of B.*

From 1.12 and the claim we just proved follows:

1.14. For any $h \in U'$ with $h \supseteq s(U)$ and for any α, n, if there is a successor g of h and a reduction of $U'(g)$ which forces "$t(n) = \alpha$," then there are $\aleph_1$ many such successors in U'.

1.15. *Conclusion.* For any α, n, if there is an *extension* of U' that forces "$t(n)=\alpha$" then there is a *reduction* of U' which forces "$t(n)=\alpha$."

Proof. Let $S \leqslant U'$ force "$t(n)=\alpha$," and set $s=s(S)$. We prove by induction on $|s|$ that a reduction of U' exists which forces "$t(n)=\alpha$." If $|s|=|s(U')|$ then S *is* a reduction of U'. If $|s|>|s(U')|$ let $h \in U'$ be such that s is a successor of h. As S is a reduction of $U'(s)$, 1.14 implies that there are $\aleph_1$ many successors s_i, $i<\aleph_1$, of h in U' such that some reduction of $U'(s_i)$ forces "$t(n)=\alpha$." Hence we can find a reduction of $U'(h)$ which forces "$t(n)=\alpha$" and apply the induction hypothesis.

But in 1.11 we proved that there are only countably many values of $t(n)$ which a reduction of U' can force. So $U' \Vdash$ "Range of t is countable," a contradiction to 1.6.

2. Collapsing $\aleph_1$ with a Π^1_2 Singleton: Proof of Theorem B

In this section we let $V=L$ be the ground model. A poset Q which satisfies the $\aleph_2$-a.c. ($\aleph_2$-antichain condition) is defined. Q is a subposet of P of Section 1 and $L[\dot{Q}]$ will also be a minimal model of "$\aleph_1^L$ is countable," but this time $L[\dot{Q}]=L[a]$ for some $a \subseteq \omega$ which is a Π^1_2 singleton and such that all constructible reals are recursive in a. We follow Jensen's construction [J1]. The main three points are: (1) Q satisfies the $\aleph_2$-a.c. and so to say that a filter G is L generic over Q is to say that G intersects all predense subsets of Q of cardinality $\aleph_1^L$, and in $L[G]$ this turns out to be a quantification over reals. (2) But we also must be sure that there is only *one* generic object in $L[G]$ so that a will be a Π^1_2 *singleton.* (3) Finally, Q must be rich enough so that the arguments of Section 1 can be carried on Q instead of P. (Parts (2) and (3) cannot be done in analogy with [J1].)

Let us begin by investigating when two conditions in P are compatible.

2.1. Definition. For T, $T' \in P$ and $f \in T \cap T'$ define a two-player game $G(T,T',f)$. At first move, player I chooses a countable set of successors X_1 of f and player II chooses $f_1 \in T \cap T' - X_1$ a successor of f, if he or she can. And so on, player I blocks $\leqslant \aleph_0$ many successors and player II must move in $T \cap T'$. At the nth move, player I chooses a countable set X_n of successors to f_{n-1} and player II chooses f_n a successor to f_{n-1} in $T \cap T' - X_n$. If player II cannot move at some finite stage he or she loses, otherwise he or she wins. Either player I or player II has a winning strategy in this game. See [Jech, 43.8].

2.2. Lemma and Definition. *T and T' are compatible iff for some $f \in T \cap T'$ player II has a winning strategy in $G(T,T',f)$. In such case, let $T \wedge_f T' = \{g \mid$ some $h \supseteq g$ was chosen by II in some play played according to* *some winning strategy in* $G(T,T',f)\}$. *Then* $T\wedge_f T'$ *is a reduction of* $T(f)$ *and of* $T'(f)$, *called the f-meet of T and T'. In fact, if R is a reduction of* $T(f)$ *and* $T'(f)$ *then R is a reduction of* $T\wedge_f T'$. *(But* $T\wedge_f T'$ *might be incompatible with some extension of* $T(f)$ *and* $T'(f)$.*)*

Proof. If T and T' are compatible, say, $S\leqslant T, T'$, then for $f=s(S)$ an obvious strategy for II in $G(T,T',f)$ is to pick elements in $S\subseteq T\cap T'$; as there are always $\aleph_1$ many successors in S it is possible to avoid those successors blocked by I. It is also clear that if II has a winning strategy in $G(T,T',f)$ then there are uncountably many successors h of f for which II has a winning strategy in $G(T,T',h)$. Hence $T\wedge_f T'$ is a reduction of $T(f)$ and $T'(f)$. The rest of the lemma is clear.

2.3 *Remarks.* The existence of a winning strategy for player I or player II is absolute (for models of ZF^-, say) and so is the definition of $T\wedge_f T'$. Also, because of associativity, we can write $T\wedge_f T'\wedge_f T''$. Finally, if $f\in T\wedge_s T'$, $|f|\geqslant|s|$, then $(T\wedge_s T')(f)=T(f)\wedge_f T'(f)$.

2.4. Definition. We say $Q\subseteq P$ is *closed* iff a–d hold.

a. Q is *closed under restrictions*: $T\in Q$ and $f\in T$ imply $T(f)\in Q$.

b. Q is *closed under meets*: T, $T'\in Q$ *and* $T(f)$, $T'(f)$ have a common reduction imply $T\wedge_f T'\in Q$.

c. Q is *closed under limited reductions*: $T\in Q$ and T' is a limited reduction of T imply that $T'\in Q$.

d. Q is *closed under unions*: If $T\in P$ and for some m for all $h\in T_m$, $T(h)\in Q$, then $T\in Q$.

We also define (and claim):

The *closure of Q under restrictions* is the set $\{T(f)\mid f\in T\in Q\}$.
The *closure of Q under meets* is the set $\{T^1\wedge_f T^2\cdots\wedge_f T^n\mid T^i\in Q\}$.
The *closure of Q under limited reductions* is the set $\{T'\mid T'$ is a limited reduction of some $T\in Q\}$.
The *closure of Q under unions* is the set of all $T\in P$ such that every infinite branch of T has an initial segment h with $T(h)\in Q$.

It is easy to see that the closure of Q under restrictions, meets, limited reductions, and unions is indeed closed under these operations.

2.5. Lemma. a. *If Q is closed under meets then Q is closed under restrictions too.*

b. *If Q is closed under meets then the closure of Q under limited reductions is closed under meets.*

c. *If Q is closed under restrictions (meets) (limited reductions) then* *the closure of Q under unions is closed under restrictions (meets) (limited reductions), respectively.*

Proof. We prove b only. Let $\bar{Q}$ be the closure of Q under limited reductions, and we have to show that $\bar{Q}$ is closed under meets. Let S', $T' \in \bar{Q}$ have a common reduction (in P) and $f = s(T') = s(S')$. Pick $S, T \in Q$ such that S' and T' are limited reductions of S and T, respectively. It is enough to show that $U' = S' \wedge_f T'$ is a limited reduction of $U = S \wedge_f T$ to conclude that $U' \in \bar{Q}$. $U' \subseteq U$ of course.

For any $h \in U'$ we have to show that there are only countably many successors of h in U which are not in U'. Suppose that the number of these successors is uncountable, then there is a reduction R of $U(h)$ which is incompatible with U'. But $U(h)$ is a reduction of $S(h)$ and so is R. $S'(h)$ is a limited reduction of $S(h)$. By 1.13 $R \cap S'(h)$ is a reduction of $S'(h)$ and a limited reduction of R. Similarly $R \cap S'(h) \cap T'(h)$ is a reduction of $T'(h)$, hence a reduction of $S'(h) \wedge_h T'(h)$ (which extends U'), a contradiction.

Define the *closure* of Q as the closure under unions of the closure under limited reductions of the closure under meets of Q.

2.6. Definitions. a. For $D \subseteq K \subseteq P$, $k < \omega$, define $\psi_K(D, T, k)$ iff there is a reduction T' of T such that for any $h \in T'_k$ $T'(h) \in K$ and extends a member of D. (We do not ask $T' \in K$.)

b. For $D \subseteq K \subseteq P$ say that *D is strongly dense in K* iff for every $T \in K$, $\psi_K(D, T, k)$ holds for some k.

2.7. Lemma. a. *Assume $D \subseteq K \subseteq P$ and D is dense in K, then D is dense in the closure of K under unions.*

b. *Assume $D \subseteq K \subseteq P$ and this time D is strongly dense in K, then D is predense in the closure of K under limited reductions.*

Proof of b. Let $\bar{K}$ be the closure of K under limited reductions. Given $T' \in \bar{K}$ we have to show T' is compatible in $\bar{K}$ with some member of D. Pick $T \in K$ such that T' is a limited reduction of T. For some k $\psi_K(D, T, k)$ holds. This implies that there is $h \in T'$, $|h| = k$, such that a reduction $R \in K$ of $T(h)$ extends a member of D. But as $h \in T'$, $T'(h)$ is a limited reduction of $T(h)$, Lemma 1.13 says $R \cap T'(h)$ is a limited reduction of R, hence $R \cap T'(h) \in \bar{K}$ shows that T' is compatible with some tree in D.

The other direction is also true:

2.8. Lemma. *Assume that $K \subseteq P$ is closed under limited reductions. Let $D \subseteq K$ be dense. Then D is strongly dense in K.*

Proof. Assume $T \in K$ and $\psi_K(D, T, k)$ never holds. Then the set $X = \{h \in T \mid h$ is a successor of $s(T)$ and, for some k, $\psi_K(D, T(h), k)$ holds$\}$ is countable. So, for every successor of $s(T)$ in T, h, out of X, $\psi_K(D, T(h), k)$ holds for no k. Repeating this step, one can obtain a limited reduction T' of T such that for every $h \in T'$ $\psi_K(D, T(h), k)$ never holds. Hence of course $\psi_K(D, T'(h), k)$ never holds, as $T'(h)$ is a reduction of $T(h)$. $T' \in K$ by the assumption of the lemma. But D is dense in K, pick $S \leqslant T'$, $S \in D$. $h = s(S)$ shows that $\psi_K(D, T'(h), |h|)$ holds, a contradiction.

2.9. We want to show now that some of the arguments that were made in Section 1 for the poset P can be adopted for any closed poset. Let $K \subseteq P$ be a closed subposet and let t be a name in K forcing of a function $\aleph_0 \to \aleph_1$. Change the definition a little in 1.7 and say for $S \in K$ and $n, k < \omega$ that $\varphi_K(t, S, n, k)$ holds iff $k \geqslant 1$ and:

(i) For every $h \in S$ with $|h| = |s(S)| + k$, $S(h)$ decides (in K forcing) the value of $t \restriction n$.

(ii) If $h \neq h'$ are as above then the value of $t \restriction n$ decided by $S(h)$ differs from that decided by $S(h')$.

2.10. LEMMA. *Assume $S \Vdash^K$ "$t\colon \aleph_0 \to \aleph_1$ is cofinal". Then $\varphi_K(t, S', n, k)$ for some n, k, and S' which is a reduction of S in K.*

The proof follows Lemmas 1.8, 1.10, and 1.15. We leave it to the reader to check where the assumption that K is closed is used. (We cannot claim, however, that the requirement of 1.6 is satisfied.)

2.11. *Properties of the poset Q.* We shall define by induction on $\alpha < \omega_2$ posets $Q_\alpha \subseteq P$ of cardinality $\aleph_1$ such that $\alpha < \beta \to Q_\alpha \subset Q_\beta$ and $Q_\delta = \bigcup_{\alpha < \delta} Q_\alpha$ for limit $\delta < \omega_2$. Finally, we will set $Q = \bigcup_{\alpha < \omega_2} Q_\alpha$ as the desired poset. Also defined inductively is an increasing sequence of ordinals $\langle \gamma_\alpha < \omega_2 \mid \alpha < \omega_2 \rangle$ described now. Let $\mathscr{T}$ be the theory of ZF^- (ZF without the power set axiom) $+$ $V = L$ $+$ "all ordinals are of cardinality $\leqslant \aleph_1$."

γ_α is the first ordinal of cofinality $\aleph_1$ such that L_{γ_α} is a model of $\mathscr{T}$ and $\aleph_1 \cup \{\gamma_i \mid i < \alpha\} \in L_{\gamma_\alpha}$.

This gives of course the diamond sequence of Jensen [J2]. The requirement that γ_α is of cofinality $\aleph_1$ is made to assure that every countable subset of L_{γ_α} is a member of L_{γ_α}. Let us just state what it means that the γ_α's give a diamond sequence: For all $X \subseteq L_{\omega_2}$ there is a closed unbounded subset of ω_2, C, such that for $\delta \in C$, $X \cap L_\delta \in L_{\gamma_\delta}$.

The construction of the Q_α is done in such a way that:

a. $Q_\alpha \in L_{\gamma_\alpha}$ and even $\langle Q_i \mid i \leqslant \alpha \rangle$ is definable in L_{γ_α}.

b. Q_α is closed under restrictions and meets, and if α is successor then Q_α is closed under limited reductions and unions in L_{γ_α}.

c. If $D \subseteq Q_\alpha$, $D \in L_{\gamma_\alpha}$ is strongly dense in Q_α, then D is predense in $Q_{\alpha+1}$.

2.12. LEMMA. a. *If* $D \in L_{\gamma_\alpha}, D \subseteq Q_\alpha$ *is strongly dense in* Q_α, $\alpha < \beta$, *then* D *is predense in* Q_β.

b. Q *satisfies the* $\aleph_2$*-a.c.*

Proof. a. For $\beta \geqslant \alpha$ set $D_\beta = \{T \in Q_\beta \mid T$ extends some tree in $D\}$. Prove inductively that D_β is strongly dense in Q_β as follows. If D_β is strongly dense in Q_β then, by c, $D_{\beta+1}$ is dense in $Q_{\beta+1}$. $Q_{\beta+1}$ is closed in $L_{\gamma_{\beta+1}}$ under limited reductions, (by b), and as Lemma 2.8 holds in $L_{\gamma_{\beta+1}}$, $D_{\beta+1}$ is strongly dense in $Q_{\beta+1}$. Now if β is a limit ordinal, $Q_\beta = \bigcup_{i<\beta} Q_i$ and $D_\beta = \bigcup_{i<\beta} D_i$, so that from the induction's assumption follows that D_β is strongly dense in Q_β.

b. To show the $\aleph_2$-a.c. pick an arbitrary $D \subseteq Q$, dense. We will show that for some $\alpha < \omega_2, D \cap Q_\alpha$ is predense in Q. As $|Q_\alpha| = \aleph_1$ this implies that every antichain in Q is of cardinality $\leqslant \aleph_1$. Observe first that Q is closed (Definition 2.4), so if $D \subseteq Q$ is dense, D is actually strongly dense (2.8). Pick $\alpha < \omega_2$ such that $D \cap Q_\alpha$ is strongly dense in $Q_\alpha, Q \cap L_\alpha = Q_\alpha$ and $D \cap L_\alpha \in L_{\gamma_\alpha}$. So $D \cap L_\alpha = D \cap Q \cap L_\alpha = D \cap Q_\alpha \in L_{\gamma_\alpha}$ is strongly dense in Q_α, and part a of the Lemma is used to conclude that $D \cap Q_\alpha$ is predense in Q.

2.13. DEFINITION OF THE Q_α. Q_0 is the closure in L_{γ_0} of the full tree in $\{{}^{\omega}\omega_1\}$ consisting of all increasing functions. If δ is limit then $Q_\delta = \bigcup_{i<\delta} Q_i$ as already said. $Q_\delta \in L_{\gamma_\delta}$ because of the uniform definition of the Q_i's. Next, the definition of $Q_{\alpha+1}$ depends on whether α is limit ordinal or successor. If α is limit, then $Q_{\alpha+1}$ is simply the closure of Q_α in $L_{\gamma_{\alpha+1}}$. Q_α is already closed under restrictions and meets so in $L_{\gamma_{\alpha+1}}$ the closure under unions of the closure under limited reductions of Q is closed in $L_{\gamma_{\alpha+1}}$ (Lemma 2.5). And by Lemma 2.7, c of 2.11 holds true.

Case α *is a successor ordinal.* We define first a poset $C_\alpha \in L_{\gamma_\alpha}$ which is σ-closed, then find in $L_{\gamma_{\alpha+1}}$ an L_{γ_α} generic filter over C_α, $\dot{C}_\alpha$. Then a tree T^* is defined from $\dot{C}_\alpha$ and we let $Q_{\alpha+1}$ be the closure of $Q_\alpha \cup \{T^*\}$ in $L_{\gamma_{\alpha+1}}$. As α is successor, Q_α is closed in L_{γ_α}.

Choose some $T^\alpha \in Q_\alpha$ and some $t_\alpha \in L_{\gamma_\alpha}$ which is a name in Q_α forcing such that in L_{γ_α}, $T^\alpha \Vdash^{Q_\alpha}$ "$t_\alpha : \aleph_0 \to \aleph_1$ is cofinal." Actually T^α and t_α are not chosen arbitrarily, but depend on α in a uniform way (so that a of 2.11 is satisfied) and such that for any T, t as above for some successor ordinal α, $T^\alpha = T$ and $t_\alpha = t$. Observe that if $T \Vdash^Q$ "$t: \aleph_0 \to \aleph_1$ is cofinal," then as Q satisfies the $\aleph_2$-a.c., t can be taken to be hereditarily of cardinality $\leqslant \aleph_1$, and from some α on $T \Vdash^{Q_\alpha}$ "$t: \aleph_0 \to \aleph_1$ is cofinal." (Two conditions in Q_α are compatible in Q_α iff they are compatible.)

2.14. DEFINITION. In L_{γ_α}, C_α is the collection of all (F, E) such that $E \subseteq T^\alpha$ is countable and F is a function with countable domain denoted by

$D(F)$ such that: $D(F) \subseteq T^\alpha$ is a subtree. The range of F is a subset of Q_α. And

1. $D(F) \cap E = \emptyset$, $s(T^\alpha) \in D(F)$.
2. $F(f)$ is a reduction of $T^\alpha(f)$ for $f \in D(f)$.
3. $D(F)(f) \subseteq F(f)$ for $f \in D(F)$ and if $f \subseteq g \in D(F)$ then $F(g) \subseteq F(f)$. (See Definition 1.2 to understand $D(F)(f)$.)
4. For every $f \in D(F)$ there are n, k such that $\varphi_{Q_\alpha}(t_\alpha, F(f), n, k)$. (See definition in 2.9.)

The partial order of C_α is defined by: (F', E') extends (F, E) iff $F' \supseteq F$, $E' \supseteq E$. The following lemma is clear.

2.15. LEMMA. *C_α is σ-closed. (In L or in L_{γ_α}—it does not matter.)*

As L_{γ_α} has cardinality $\aleph_1$ in $L_{\gamma_{\alpha+1}}$ and all countable subsets of L_{γ_α} are members of L_{γ_α}, there is in $L_{\gamma_{\alpha+1}}$ an L_{γ_α} generic filter over C_α. Let $\dot{C}_\alpha$ be the minimal such filter (in the constructible well-order). Set $T^* = \bigcup \{D(F) \mid (F, E) \in \dot{C}_\alpha \text{ for some } E\}$.

2.16. LEMMA. *$T^* \in P$ is a reduction of T^α and if $(F, E) \in \dot{C}_\alpha$ and $f \in D(F)$ then $T^*(f)$ is a reduction of $F(f)$.*

Proof. T^* is a union of subtrees of T^α and a simple density argument shows that it is a reduction of T^α. Similarly the rest of the lemma is done, using 3 of 2.14.

Now define $K = K_\alpha = \{S \wedge_h T^* \mid S \in Q_\alpha$ has a common reduction with $T^*(h)\} \cup Q_\alpha$. K is closed under restrictions because $(S \wedge_h T^*)(f) = S(f) \wedge_f T^*$. K is closed under meets because $(S \wedge_h T^*) \wedge_f (S' \wedge_{h'} T^*) = (S \wedge_f S') \wedge_f T^*$, and Q_α is closed under meets.

2.17. LEMMA. *If $D \in L_{\gamma_\alpha}$, $D \subseteq Q_\alpha$ is strongly dense in Q_α, then D is strongly dense in K.*

Proof. Let $S \wedge_h T^* \in K$, and a strongly dense $D \subseteq Q_\alpha$ be given.

What follows is a density argument in L_{γ_α}. Let $(F, E) \in C_\alpha$ be such that $(F, E) \Vdash$ "S and $T^*(h)$ have a common reduction" we wish to find k and an extension of (F, E) forcing $\psi_K(D, S \wedge_h T^*, k)$. By extending (F, E), we can assume that $h \in D(F)$. (F, E) forces that $S \wedge_h T^*$ is a reduction of S and $F(h)$, so by absoluteness, S and $F(h)$ have a common reduction in L_{γ_α}. And $S \wedge_h F(h) = W \in Q_\alpha$ and $s(W) = h$. D is strongly dense, hence $\psi_{Q_\alpha}(D, W, k)$ holds for some k. We *claim* that (F, E) forces $\psi_K(D, S \wedge_h T^*, k)$. The reason follows. Because $\psi_{Q_\alpha}(D, W, k)$, there is a reduction R of W such that for all $g \in R_k$, $R(g) \in Q_\alpha$ extends a member of D. If we prove that $(F, E) \Vdash$ "$R \wedge_h T^*$ is a reduction of R," then, as R is a reduction of S, $(F, E) \Vdash$ "$R \wedge_h T^*$ is a reduction of $S \wedge_h T^*$," and this clearly implies our claim above. Now to prove indeed that $(F,E) \Vdash$ "$R \wedge_h T^*$ is a reduction of R" use the following lemma.

2.18. LEMMA. *If $(F,E) \in C_\alpha$, $f \in D(F)$, $f \supseteq s(T^*)$, and $R \in Q_\alpha$ is a reduction of $F(f)$, then $(F,E) \Vdash$ "$R \wedge_f T^*$ is a reduction of R, and moreover, there are uncountably many successors g of f in R such that $T^*(g)$ is a reduction of $R(g)$."*

Proof. Assume to the contrary that only countably many such successors g exist. There is an extension of (F,E) (which we suppose to be (F,E) itself) and a countable set $H \in L_{\gamma_\alpha}$ of successors of f in R, such that $(F,E \Vdash$"For no successor $g \notin H$ of f is $T^*(g)$ a reduction of $R(g)$." Pick $g \in R$ a successor of f, $g \notin H \cup D(F) \cup E$. Define a condition $(F',E) \in C_\alpha$ extending (F,E) by setting $D(F') = D(F) \cup \{g\}$ and $F'(g)$ is defined such that (1) $F'(g)$ is a reduction of $R(g)$ (hence also of $F(f)(g)$); (2) $\varphi_{Q_\alpha}(t_\alpha, F'(g), n, p)$ for some n, p. This is possible by Lemma 2.10. $(F',E) \in C_\alpha$. From Lemma 2.16 it follows that $(F',E) \Vdash$ "$T^*(g)$ is a reduction of $F'(g)$," but $F'(g)$ is a reduction of $R(g)$, a contradiction, proving the lemma and 2.17.

Finally, $Q_{\alpha+1}$ is defined in $L_{\gamma_{\alpha+1}}$ to be the closure of K_α under limited reductions and unions. Lemmas 2.17 and 2.7 imply that c of 2.11 holds.

This ends the construction of the Q_α's, but we want more information on T^* constructed at stage $\alpha + 1$, α a successor ordinal, before proving that Q is as required.

Recall that $T^\alpha \Vdash^{Q_\alpha}$ "$t_\alpha : \aleph_0 \to \aleph_1$ is cofinal." t_α can naturally be regarded as a name in $Q_{\alpha+1}$. Because every dense subset of Q_α in L_{γ_α} is strongly dense, every dense subset of Q_α in L_{γ_α} is predense in $Q_{\alpha+1}$, and hence every maximal antichain of Q_α in L_{γ_α} is a maximal antichain in $Q_{\alpha+1}$ too. This implies that $T^\alpha \Vdash^{Q_{\alpha+1}}$ "$t_\alpha : \aleph_0 \to \aleph_1$ is cofinal." Now look at $T^* \leqslant T^\alpha$. For each $f \in T^*$ define $t_f \in {}^{\omega}\omega_1$ to be the maximal sequence such that for some n $T^*(f) \Vdash^{Q_{\alpha+1}}$ "$t_\alpha \upharpoonright n = t_f$." Clearly $f \subseteq f' \Rightarrow t_f \subseteq t_{f'}$. The function $f \mapsto t_f$ is in some sense one to one, as the next lemma and lemma 2.20 say.

2.19. LEMMA. *For every $g \in T^*$ there are $m > |g|$ and n such that if h, $h' \in T^*(g)_m$, $h \neq h'$, then $t_h \upharpoonright n$ and $t_{h'} \upharpoonright n$ are incompatible.*

Proof. If $g \in T^*$, pick $(F,E) \in \dot{C}_\alpha$ such that $g \in D(F)$. By 4 in 2.14, $\varphi_{Q_\alpha}(t_\alpha, F(g), n, k)$ for some n, k. But as any maximal antichain of Q_α in L_{γ_α} remains maximal in $Q_{\alpha+1}$, one gets that $\varphi_{Q_{\alpha+1}}(t_\alpha, F(g), n, k)$ holds too. $T^*(g)$ is a reduction of $F(g)$ (Lemma 2.16) hence $\varphi_{Q_{\alpha+1}}(t_\alpha, T^*(g), n, k)$.

2.20. LEMMA. *For any infinite branch b of T^* and any m, there is an initial segment g of b such that $|t_g| \geqslant m$.*

Proof. This is a simple application of the previous lemma. Observe the absolute character of this lemma. Even if new branches to T^* are added it is still true: (because what it says is that a certain subset of T^* is conversely well founded). Another immediate application of 2.19 is:

2.21. LEMMA. *There is a set $X = X(T^*) \subseteq T^*$ such that:*

1. *$s(T^*) \in X$ and for each $h \in X$ there is $k > |h|$ such that for all $f \in T^*_k$ if $f \supseteq h$ then $f \in X$.*
2. *If $h, h' \in X$ are incompatible, then t_h and $t_{h'}$ are incompatible.*
3. *X is dense in T^* (this follows from 1).*
4. *If $g \in X, h \in T^*$ and $t_h \supseteq t_g$ then $h \supseteq g$.*

Let $\dot{Q}$ be an L-generic filter over $Q = \bigcup_{\alpha < \omega_2} Q_\alpha$. Let $s = \bigcup \{s(T) \mid T \in \dot{Q}\}$, then s is an ω-sequence of cofinal $\aleph_1^L$. For any $t: \aleph_0 \to \aleph_1^L$ *in* $L[\dot{Q}]$ define $G(t) = \{T \mid t \text{ is a branch of } T\}$, and say that t is *generic* iff $G(t)$ is L-generic over Q. As $G(s) = \dot{Q}$, s is of course generic.

2.22. LEMMA. *$L[\dot{Q}]$ is a minimal model of "$\aleph_1^L$ is countable."*

Proof. Given $T \Vdash^Q$ "$t: \aleph_0 \to \aleph_1^L$ is a cofinal map," we have to find $T^* \Vdash$ "$s \in L[t]$." Q satisfies the $\aleph_2$-a.c. so t can be assumed to be of cardinality hereditarily $\leqslant \aleph_1$ and so the pair (T, t) was considered at some successor stage α and $T^* \in Q_{\alpha+1}$ was constructed then. But as the function $h \to t_h$ is one to one on $X(T^*)$, it follows that s can be recovered from $X(T^*)$ and t. ($X(T^*) \in L_{\gamma_{\alpha+1}}$.)

Now we come to the uniqueness of the generic filter.

2.23. LEMMA. *In $L[\dot{Q}]$, s is the only generic sequence.*

Proof. Let $t \in L[\dot{Q}]$ be a generic sequence different from s. Pick $p < \omega$ such that $t \upharpoonright p \neq s \upharpoonright p$. Choose $T \in \dot{Q}$ such that: $|s(T)| > p$, T decides the value of $t \upharpoonright p$ and $T \Vdash^Q$ "$t \upharpoonright p \neq s \upharpoonright p$ and t is a generic sequence." We will get a contradiction. Find a successor ordinal α such that $t \in L_{\gamma_\alpha}$ (as a name), $T \in Q_\alpha$, and (T, t) was considered at that stage. Look at T^*, the "generic" tree that was constructed in $L_{\gamma_{\alpha+1}}$, and at the function $h \mapsto t_h$ defined on T^*. Call B the tree obtained from the t_h's, i.e., $B = \{g \mid g \subseteq t_n \text{ for some } h \in T^*\}$. In $L_{\gamma_{\alpha+1}}$ define the following subset of K_α.

$D = \{S \in K_\alpha \mid S \cap B \text{ is bounded}\}$ ($S \cap B$ is *bounded* iff for some n for all $f \in S \cap B$, $|f| \leqslant n$), D is clearly open. Set $e = t_{s(T^*)}$.

2.24. LEMMA. *D is strongly dense in K_α.*

Before proving the lemma let us see how a contradiction is derived and thence the conclusion of Lemma 2.23. If D is indeed strongly dense in K_α, then D is predense in $Q_{\alpha+1}$ and therefore in Q (by 2.7, 2.8, and 2.12a). Any generic sequence must be a branch in some tree in D. There exists, therefore, an extension T^{*1} of T^* in Q and a tree $S \in D$ such that $T^{*1} \Vdash$ "t is a branch of S." Because $S \in D$, $S \cap B$ is contained in the first n, say, levels of S. Find $f \in S_{n+1}$ and $T^{*2} \leqslant T^{*1}$ forcing "$f \subseteq t$." If we show $f \in B$ then the contradiction is clear. By Lemma 2.20, there is $h \in T^{*2}$ such that $|t_h| \geqslant n+1$. But as $T^{*2}(h)$ extends both $T^*(h)$ and T^{*2}, it must be the case that $t_h \supseteq f$, hence $f \in B$.

Proof of Lemma 2.24. Let $R \in K_\alpha$ and $e \subseteq s(R)$ (w.l.o.g.). We must show $\psi_{K_\alpha}(D, R, k)$ holds for some k. R cannot be of the form $S \wedge_h T^*$ because $T^* \leqslant T$ and e is incompatible with $s(T)$. So $R \in Q_\alpha$. Let $f = s(R)$. Let e', $e \subseteq e' \subseteq f$ be the largest sequence of the form t_g for some $g \in X(T^*)$ (see 2.21). It follows from 2.21-4 that if $g' \in T^*$ is incompatible with g then $t_{g'} \not\supseteq t_g$. Now we turn to L_{γ_α}. Let $(F, E) \in C_\alpha$ be any condition forcing the above information, such that $g \in D(F)$. There are $k \geqslant 1$ and n such that $\varphi_{Q_\alpha}(t, F(g), n, k)$ (by 2.14-4).

Claim. $(F, E) \Vdash \psi_{K_\alpha}(D, R, n)$.

Proof. Assume not, and without loss of generality assume $(F, E) \Vdash \neg \psi_{K_\alpha}(D, R, n)$. Then (F, E) forces that there is $f' \supseteq f$, $f' \in R_{n-1}$ and there is a countable set Y such that for $h \in R_n - Y$ a successor of f', there is no reduction of $R(h)$ in D. So we can find an extension of (F, E), which again we assume to be (F, E) itself, and f', Y as above, such that $(F, E) \Vdash$ "No reduction of $R(h)$ is in D if $h \in R_n - Y$ is a successor of f'." For every successor of f' $h \in R_n - Y$ there is one and only one $g' \in F(g)$ such that $g' \supseteq g$, $|g'| = |g| + k$ and $F(g)(g') \Vdash^{Q_\alpha}$ "$t \supseteq h$." And the reason is this: the uniqueness is a result of $\varphi_{Q_\alpha}(t, F(g), n, k)$. And if $h \in R_n - Y$ a successor of f' is given, then $(F, E) \Vdash$ "No reduction of $R(h)$ is in D," so some extension of (F, E) forces that $h \subseteq t_{g'}$ for some $g' \in T^*$. By a previous remark g' must be compatible with g, and so if $g'' \in F(g)_{|g|+k}$ is compatible with g' then $F(g)(g'')$ must force, in Q_α, that $t \restriction n = h$. Now, as there are uncountably many successors of f' in R, we can pick such a successor $h \in R_n - Y$ such that the only $g' \in F(g)$ as above is not in $D(F)$. Put $E' = E \cup \{g'\}$, then (F, E') extends (F, E) and $(F, E') \Vdash^{C_\alpha}$ "$h \notin B$." So, $(F, E') \Vdash$ "$R(h) \in D$," a contradiction.

2.25. Now that we have seen that $L[\dot{Q}]$ is a minimal collapse of $\aleph_1^L$, via an $\aleph_2$-a.c. Q in which $\dot{Q}$ is the unique generic filter, we have to define the set of integers a that will prove Theorem B.

Given any $x \subseteq \omega$, a sequence of sets $x_0, x_1, \ldots$ is defined by $x_i = \{n \mid \langle i, n \rangle \in x\}$. Conversely any such sequence can be coded by one set x. We write for short: $x = \langle x_i \mid i < \omega \rangle$. Let $a_1, \ldots, a_n, \ldots$ be the generic sequence of ordinals provided by $\dot{Q}$. For each $i \geqslant 1$ let $a_i \subseteq \omega$ be a natural coding of a well-ordering of ω of order type α_i which is the minimal such coding in L. Set $a_0 = \langle b, c \rangle$ where b is a well-ordering of ω of order type ω_1^L which is defined uniquely and naturally from $\langle a_i | 1 \leqslant i < \omega \rangle$, and $c = \langle c_n | n < \omega \rangle$ where c_n is the ξth constructible real (in the L well-order) where ξ is the place of n in the well-order provided by b. Finally, $a = \langle a_i | i < \omega \rangle$ is as required.

To show that a is a Π_2^1 singleton we have to describe an appropriate Π_2^1 formula $\varphi(x)$. In the next paragraph we describe what that formula says, and in the one after it we argue that this can be said in a Π_2^1 manner.

$\varphi(x)$ is equivalent to: $x = \langle x_i | i < \omega \rangle$ and for $i \geqslant 1$ x_i codes a well-ordering of ω such that if α_i is the order type of that well-order then the sequence α_i, $i \geqslant 1$, is L-generic over Q. Moreover x_i, $i \geqslant 1$, is constructible and no constructible subset of ω which precedes x_i in the constructibility well-order can code a well-order of order type α_i. $x_0 = (y, z)$ where y naturally encodes $\sup\{\alpha_i | 1 \leqslant i < \omega\}$ and $z = \langle z_i | i < \omega \rangle$ is such that z_i is the ξth constructible subset of ω where ξ is the order type of i in the y ordering.

It is clear that $\varphi(a)$ and that $\varphi(d) \rightarrow a = d$. All that remains is to show $\varphi(x)$ can be put to be Π_2^1. Let $\mathscr{E}'$ be the theory $\mathscr{E}$ of 2.11 strengthened by the sentence which says that for cofinally many ordinals α, L_α is a model of $\mathscr{E}$. To say that $x_i, i \geqslant 1$, codes a well-order is Π_1^1. To say that y encodes the supremum of α_i is simpler. Now the following is Π_2^1. For any $H \subseteq \omega$ which encodes a structure and for any $H' \subseteq \omega$ which encodes a truth assignment for the structure H which shows that H is a model of $\mathscr{E}'$. Either H is not well founded, or the ordinals in H can be embedded into y, or there is $G: \omega \rightarrow \omega$ such that for each $1 \leqslant i < \omega$ $G(i)$ is seen by H as a subset of the natural numbers and is equal to x_i and it is said (by H') about $G(i)$ that (it is constructible and) no preceding subset of natural numbers can encode the order type encoded by $G(i)$. And if we construct Q in H then it turns out that $\{G(i) | i \leqslant i < \omega\}$ is an H generic sequence over Q (i.e., for every successor α and $D \in L_{\gamma_\alpha}$ which is dense in Q_α there is a tree in D for which the $G(i)$'s form a branch.) And $z = \langle z_i | i < \omega \rangle$ is a sequence of subsets of ω which enumerate all the $\aleph_1^L$ constructible reals in the order given by y.

References

[B] L. Bukovsky, Changing cofinality of $\aleph_2$, *in* "Set Theory and Hierarchy Theory," Lecture Notes in Mathematics No. 537, pp. 37–49, Springer-Verlag, Berlin/New York, 1976.

[D] K. J. Devlin, "Aspects of Constructibility," Lecture Notes in Mathematics No. 354, Springer-Verlag, Berlin/New York, 1973.

[Jech] T. Jech, "Set Theory," Academic Press, New York, 1978 (Vol. 79 in "Pure and Applied Mathematics").

[J1] R. B. JENSEN, Definable sets of minimal degree, *in* "Math. Logic and Foundations of Set Theory" (Y. Bar-Hillell, Ed.), pp. 122–128, North-Holland, Amsterdam, 1970.

[J2] R. B. JENSEN, The fine structure of the constructible hierarchy, *A.M.L.* **4** (1972), 229–308.

[JS] R. B. JENSEN AND R. M. SOLOVAY, Some application of almost disjoint sets, *in* "Math Logic and Foundations of Set Theory" (Y. Bar-Hillel, Ed.), pp. 84–104, North-Holland, Amsterdam, 1970.

[L] R. LAVER, On the consistency of Borel's conjecture, *Acta Math.* **137** (1976), 151–169.

[N] K. NAMBA, Independence proof of (ω, ω_α)-distributive law in complete Boolean algebra, *Comm. Math. Univ. St. Pauli* **19** (1970), 1–12.

[S] G. E. SACKS, Countable admissible ordinals and hyperdegrees, *Adv. in Math.* **19** (1976), 213–262.