Lecture notes on set theory without choice

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Abstract

The first part of the course will discuss various consequences of AD, the Axiom of Determinacy. AD is inconsistent with AC. On the other hand, it implies many good properties hold for sets o real numbers. For example, AD implies LM (every set of reals is Lebesgue measurable), BP (every set of reals has the property of Baire), and P (every uncountable set of reals contains a homeomorphic copy of the Cantor set). In addition, it is connected with the theory of large cardinals, e.g., AD implies that the first uncountable cardinal, omega-one, is a measurable cardinal. For the second part of the course, we will cover Solovay's model of LM+BP+P. For the last part, we will present some models of set theory in which the axiom of choice fails as badly as we can conceive possible. For example, the Feferman-Levy model in which the real line is the countable union of countable sets.

Contents

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1 Open determinacy

For X a set define
$$X^\omega = \{x \ : \ x : \omega \to X\}$$
where $\omega = \{0, 1, 2, \ldots, \}$. Let
$$X^{<\omega} = \bigcup_{n<\omega} X^n$$
be the finite sequences in X. X is given the discrete topology, i.e., every subset of X is open and X^ω the product topology. A clopen basis for this topology consists of the sets:
$$[s] = \{x \in X^\omega \ : \ s \subseteq x\} \text{ where } s \in X^{<\omega}.$$
Open sets are arbitrary unions of basic clopen sets. Equivalently, $A \subseteq X^\omega$ is open iff for any $x \in A$ there is an $n < \omega$ such that $[x \restriction n] \subseteq A$.

The most popular choices for X are $X = \omega$ or $X = 2 = \{0, 1\}$.

Given $A \subseteq X^\omega$ define the following infinite two person game $G(A)$. There are two players I and II which alternate writing down an element of X^ω:

I	$x(0)$		$x(2)$	
II		$x(1)$		$\ldots$

Player I wins the play of the game x iff $x \in A$.

A strategy for a game such as this is a map $\sigma : X^{<\omega} \to X$. Given two strategies σ and τ then we can define
$$\sigma * \tau = x \in X^\omega$$

as the result of playing the strategy σ for player I against the strategy τ for player II as follows:

$$x(0) = \sigma(\langle\rangle),$$
$$x(2n) = \sigma(\langle x(1), x(3), \ldots, x(2n-1)\rangle)$$

and

$$x(2n+1) = \tau(\langle x(0), x(2), \ldots, x(2n)\rangle).$$

We say that the game $G(A)$ is determined iff either Player I has a winning strategy or Player II has a winning strategy:

$$(\exists\sigma\forall\tau \quad \sigma * \tau \in A) \text{ or } (\exists\tau\forall\sigma \quad \sigma * \tau \in X^\omega \setminus A).$$

Theorem 1.1 *(Gale, Stewart [7]) If $A \subseteq X^\omega$ is open, then $G(A)$ is determined.*

Proof
Suppose the open player (Player I) has no winning strategy. Then the closed player plays so as to never reach a position in the game in which the open player has a winning strategy.
QED

Corollary 1.2 *Closed games are determined.*

Remark. Theorem 1.1 implies the axiom of choice. To see this, suppose we are given a family of nonempty sets Q. Let X be transitive set containing Q. Consider the game on X^ω in which Player II wins iff ($x(1) \in x(0)$ or $x(0) \notin Q$). This game is clopen, Player I cannot have a winning strategy, and a strategy for Player II gives a choice function for Q.

Definition 1.3 *For $T \subseteq X^{<\omega}$ a tree, i.e, $s \subseteq t \in T \to s \in T$, and*

$$A \subseteq [T] =^{def} \{x \in X^\omega \ : \ \forall n < \omega \ \ x \upharpoonright n \in T\}$$

define the game $G(A, T)$ to be the game in which the players are required to stay on the tree T.

Lemma 1.4 *Suppose $T \subseteq X^{<\omega}$ is a tree and $U \subseteq [T]$ is relatively open. Then the game $G(T, U)$ is determined.*[2]

Definition 1.5 *A G_δ set is a set which can be written as a countable intersection of open sets.*

Theorem 1.6 *(P. Wolfe [19]) If $A \subseteq X^\omega$ is a G_δ set, then $G(A)$ is determined.*

Proof
Suppose Player II has no winning strategy. Let T be the set of all $s \in X^{<\omega}$ such that every $r \subseteq s$ is not lost for Player I, i.e., Player II has no winning strategy in the game $G(A)$ starting from the position r.

Let $A = \bigcap_n U_n$ where the U_n are open.

Claim. For any $s \in T$ and $n < \omega$ Player I has a winning strategy in the game $G(U_n, T)$ starting at s.

Otherwise by the Lemma Player II has a winning strategy τ for $G(U_n, T)$ starting at s. But then Player II has a winning strategy starting at s in the game $G(A)$. He[3] just plays τ unless Player I leaves the tree T, but leaving T means that Player II has a winning strategy from the position.

The winning strategy for Player I in $G(A)$ is to first play his winning strategy for $G(T, U_0)$ starting at $s_0 = \langle\rangle$ until a position s_1 is reached with $[s_1] \subseteq U_0$. Then he plays his winning strategy for $G(T, U_1)$ starting at s_1 until

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[2] A good way to visualize this Lemma is to draw the tree T. Color a node of T green if it has the property that all infinite branches thru it are in U. Color the other nodes red. The object of the open set player is go thru a green node. The desire of his opponent is stay on the red nodes.

I really enjoy using colored chalk in a lecture. If I had to write one of those "Statement of Teaching Philosophies" that all our job candidates seem to have, I would definitely devote several paragraphs to how using colored chalk has enhanced my enjoyment of giving a lecture.

[3] The politically correct thing is to refer to the players as "she". For example, see the last problem in Komjáth and Totik [11]. Perhaps even more distracting is to sometimes call a random player "he" and other times call him "she". This will guarantee to cause any reader to stop thinking about the math and start contemplating the oppressed role of women in our society.

Perhaps to confirm more closely to reality, we should adopt the rule that the player with a winning strategy is refered to as 'he' and the player that he beats is refered to as 'she'.

he reaches a position s_2 with $[s_2) \subseteq U_1$, etc. Note that T has the property that Player II cannot be the first to leave T.
QED

2 The perfect set game

Definition 2.1 *Given $A \subseteq 2^\omega$ define the game $G_p(A)$ as follows:*

I	$s_0 \in 2^{<\omega}$		$s_1 \in 2^{<\omega}$		$s_2 \in 2^{<\omega}$		$\dots$	
II		$i_0 \in \{0,1\}$		$i_1 \in \{0,1\}$		$i_2 \in \{0,1\}$		$\dots$

Player I wins iff $x \in A$ where $x = s_0{}^\frown\langle i_0\rangle{}^\frown s_1{}^\frown\langle i_0\rangle \cdots$

Definition 2.2 *A tree $T \subseteq 2^{<\omega}$ is perfect iff for any $s \in T$ there exists $t \supseteq s$ such that both $t^\frown\langle 0\rangle$ and $t^\frown\langle 1\rangle$ are in T. A set $P \subseteq 2^\omega$ is perfect iff there exists a perfect tree T such that $P = [T]$.*

Note that perfect sets are those sets which are homeomorphic to 2^ω.

Theorem 2.3 *(M.Davis [6]) For any $A \subseteq 2^\omega$:*

1. *Player I has a winning strategy in $G_p(A)$ iff A contains a perfect set.*
2. *Player II has a winning strategy in $G_p(A)$ iff A is countable.*

Proof

(1)($\leftarrow$)
Player I's strategy is to play along the perfect tree to nodes t with both $t^\frown\langle 0\rangle$ and $t^\frown\langle 1\rangle$ in T.

(1)($\rightarrow$)
Let σ be a winning strategy for Player I. Take the tree generated by positions consistent with σ.

(2)($\leftarrow$)
On his n^{th} move of the game Player II avoids the n^{th} element of A.

(2)($\rightarrow$)
Let τ be a winning strategy for Player II. Let $p = \langle s_0, i_i, s_1, \ldots, s_n, i_n\rangle$ be any position in the game consistent with playing τ (this means that $i_k = \tau(s_0, \ldots, s_k)$ for every $k \le n$).

We think of $x_p \in 2^\omega$ as the element of 2^ω which τ is trying to avoid. It is defined as follows. Let

$$\hat{p} = s_0{}^\frown\langle i_0\rangle{}^\frown s_1{}^\frown\langle i_1\rangle \cdots s_n{}^\frown\langle i_n\rangle$$

and let $l = |\hat{p}|$ be the length of $\hat{p}$ and put

1. $x_p \upharpoonright l = \hat{p}$,
2. $x_p(l) = 1 - \tau(s_0, \ldots, s_n, \langle\rangle)$, and
3. $x_p(k) = 1 - \tau(s_0, \ldots, s_n, t)$ where $t = x_p \upharpoonright [l, k)$.

That is, x_p goes in the opposite direction to what the strategy says.

Claim. $A \subseteq B = \{x_p \;:\; p \text{ is a position consistent with } \tau\}$.

It is enough to prove that if $x \notin B$ then there exists $s_0, i_0, s_1, i_1, \ldots$ consistent with τ such that $x = s_0{}^\frown\langle i_0\rangle{}^\frown s_1{}^\frown\langle i_1\rangle \cdots$. Since τ is a winning strategy for Player II we have that $x \notin A$.

Suppose we have already constructed $p = \langle s_0, i_i, s_1, \ldots, s_n, i_n\rangle$ consistent with τ and $\hat{p} \subseteq x$. Then since $x_p \ne x$ there must be some $k \ge l = |\hat{p}|$ such that $x_p \upharpoonright k = x \upharpoonright k$ and $x_p(k) \ne x(k)$. Take $s_{n+1} = x \upharpoonright [l, k)$. By construction of x_p we have that $x(k) = \tau(s_0, \ldots, s_{n+1})$.

This proves the Claim and since B is countable, A is countable.
QED

3 Axiom of determinacy

Definition 3.1 *AD, axiom of determinacy, is the statement that for every $A \subseteq \omega^\omega$ the game $G(A)$ is determined.*

This axiom was first proposed by Mycielski and Steinhaus [14] in 1962.

Corollary 3.2 *$AD \rightarrow P$ where P stands for the perfect set property: for all $A \subseteq 2^\omega$ either A is countable or A contains a perfect subset.*

Corollary 3.3 *$P \to \neg AC$, i.e., the axiom of choice fails. In fact AD, implies we cannot have an ω_1 sequence $\langle x_\alpha \in 2^\omega : \alpha < \omega_1 \rangle$ of distinct elements. Hence AD is inconsistent with the axiom of choice.*

Proposition 3.4 *(Mycielski [15]) $AD \to WCC$, the weak countable choice axiom, i.e., any countable family of nonempty subsets of ω^ω has a choice function.*

Proposition 3.5 *(Levy [12]) $P + WCC$ implies that ω_1 is a strongly inaccessible cardinal of L.*

Proof
WCC implies that ω_1 is regular. It must be a limit cardinal in L, otherwise we would have for some cardinal κ of L that $(\kappa^+)^L = \omega_1$. But then there exists $x \in 2^\omega$ such that $\omega_1 = (\omega_1)^{L[x]}$ and since $L[x]$ is a model of choice there would be an ω_1 sequence in 2^ω.
QED

Proposition 3.6 *(Mycielski [15]) The axiom of determinacy is inconsistent for games on ω_1^ω. It is also inconsistent for games of length ω_1, i.e., payoff sets which are subsets of 2^{ω_1}.*

Proof
Cook up games which if they are determined give an ω_1 distinct sequence in 2^ω.
QED

Since (even clopen) determinacy for games on $(\mathcal{P}(2^\omega))^\omega$ would give a choice function which would well-order 2^ω, we see that AD for games on $(\mathcal{P}(2^\omega))^\omega$ is inconsistent.

Definition 3.7 *$AD_{\mathbb{R}}$ is the statement that all games on X^ω are determined for any set X with $|X| \leq |\mathbb{R}|$.*

Proposition 3.8 *(Mycielski [15]) The axiom of determinacy for games on 2^ω is equivalent to the axiom of determinacy for games on ω^ω*

4 The Banach-Mazur Game

Definition 4.1 *Given $A \subseteq \omega^\omega$ define the Banach-Mazur Game, $G_{BM}(A)$ as follows:*

I	$s_0 \in \omega^{<\omega}$	$s_2 \in 2^{<\omega}$	$s_4 \in 2^{<\omega}$	$\ldots$
II	$s_1 \in \omega^{<\omega}$	$s_3 \in \omega^{<\omega}$	$s_5 \in \omega^{<\omega}$	$\ldots$

With the rule that $s_n \subseteq s_{n+1}$ be proper extension, i.e., $s_n \neq s_{n+1}$. Player II wins iff $x \in A$ where $x = s_0\hat{\ }s_1\hat{\ }s_2\hat{\ }s_3\cdots$.

This game was invented by Mazur in 1935. He asked when is it determined (see Mauldin [13] problem 43).

Definition 4.2 *A set $N \subseteq \omega^\omega$ is nowhere dense iff its closure has no interior. Equivalently, $\forall s \in \omega^{<\omega}\ \exists t \in \omega^{<\omega}\ (s \subseteq t$ and $[t] \cap N = \emptyset)$.*

Since $[t] \cap N = \emptyset$ implies $[t] \cap cl(N) = \emptyset$, it follows that the closure of nowhere dense set (nwd) is closed nowhere dense (cnwd). The complement of a cnwd set is an open dense set.

Definition 4.3 *A set $M \subseteq \omega^\omega$ is meager iff it is the countable union of nowhere dense sets, equivalently M is contained in the countable union of closed nowhere dense sets.*

Definition 4.4 *A set is comeager iff it is the complement of a meager set, equivalently it contains the countable intersection of open dense sets.*

Theorem 4.5 *(Banach, see Oxtoby [18]) For any $A \subseteq \omega^\omega$*

1. *Player II has a winning strategy in $G_{BM}(A)$ iff A is comeager.*
2. *Player I has a winning strategy in $G_{BM}(A)$ iff $A \cap [s]$ is meager for some $s \in \omega^{<\omega}$.*

Proof

(1) $\rightarrow$
If A is comeager, then there exists $\langle U_n : n < \omega\rangle$ open dense sets with $\bigcap_{n<\omega} U_n \subseteq A$. Player II just plays so that $[s_{2n+1}] \subseteq U_n$.

(1) $\leftarrow$
Let τ be a winning strategy for Player II. Inductively construct sets Σ_n such that

1. If $\vec{s} \in \Sigma_n$, then $\vec{s} = \langle s_0, s_1, \ldots, s_{2n+1} \rangle$ is a position in the game consistent with τ.

   For $\vec{s} \in \Sigma_n$, let $last(\vec{s}) = s_{2n+1}$.

2. $\{last(\vec{s}) \ : \ \vec{s} \in \Sigma_n\}$ is a maximal antichain in $\omega^{<\omega}$.

3. The map $\vec{s} \mapsto last(\vec{s})$ is one-to-one on Σ_n.

4. $\Sigma_n = \{\vec{s} \restriction 2n+1 \ : \ \vec{s} \in \Sigma_{n+1}\}$.

To construct these sets, suppose that $\vec{s} = \langle s_0, s_1, \ldots, s_{2n+1} \rangle \in \Sigma_n$. Then note that

$$D = \{t \ : \ \exists r \ \tau(s_0, s_2, \ldots, s_{2n+1}, r) = t\}$$

is dense below s_{2n+1}. Hence it contains an antichain maximal beneath s_{2n+1}. Union up all the sets you obtain and you get Σ_{n+1} as required.

Now define

$$U_n = \bigcup \{ \ [\ last(\vec{s}) \ ] \ : \ \vec{s} \in \Sigma_n\}$$

and note that it is a dense open set. But note that for any $x \in \bigcap_{n<\omega} U_n$ there is a unique sequence $\langle s_n : n < \omega \rangle$ such that

$$x = \bigcup_{n<\omega} s_n \quad \text{and} \quad (\langle s_0, s_1, \ldots, s_{2n+1} \rangle \in \Sigma_n \text{ for every } n).$$

It follows that x is an outcome of the game consistent with the winning strategy τ and hence in the payoff set A. It follows that A contains $\bigcap_{n<\omega} U_n$ and is therefore comeager.

The proof for (2) is completely analogous.
QED

Corollary 4.6 *AD implies that*
$\forall A \subseteq \omega^\omega$ the game $G_{BM}(A)$ is determined
and this is equivalent to
$\forall A \subseteq \omega^\omega$ either A is comeager or $\exists s \in \omega^{<\omega}$ such that $A \cap [s]$ is meager

Definition 4.7 *A set B in a topological space X has the property of Baire iff there exists an open set U such that $B \triangle U$ is meager. The symbol $\triangle$ stands for the symmetric difference: $P \triangle Q = (P \setminus Q) \cup (Q \setminus P)$. Sets with the Baire property are also refered to as 'almost open'.*

Corollary 4.8 *$AD \to BP$ where BP is the statement that every subset of ω^ω has the property of Baire.*

5 Dependent Choice DC

Definition 5.1 *The axiom of dependent choice (DC) stands for the proposition that for every set X and $R \subseteq X^2$ a binary relation on X:*

$$(\forall a \in X \; \exists b \in X \;\; aRb) \to (\exists \langle a_n : n < \omega \rangle \in X^\omega \;\; \forall n < \omega \;\; a_n R a_{n+1}).$$

Definition 5.2 *The axiom of countable choice (CC) stands for the statement that every countable family of nonempty sets has a choice function.*

Proposition 5.3 *$DC \to CC$*

Proof
Given $(A_n : n < \omega)$ nonempty sets let X and R be:

$$X = \bigcup_{n<\omega} \prod_{k \leq n} A_k \text{ and } R = \{\langle s, t \rangle \in X^2 \; : \; s \subsetneq t\}$$

QED

Proposition 5.4 *(Blair [1]) The following are equivalent (in ZF):*

1. *DC*
2. *The Baire category theorem for complete metric spaces, i.e., comeager sets are dense.*
3. *A partial order $\mathbb{P}$ is well-founded iff there does not exists a descending sequence, $\langle p_n : n < \omega \rangle \in \mathbb{P}^\omega$ such that $p_{n+1} < p_n$ for every n.*

Proof

(1) $\to$ (2)
Given U_n for $n < \omega$ open dense sets and an arbitrary nonempty open set V_0, construct nonempty open sets V_{n+1} such that

1. $V_{n+1} \subseteq \text{cl}(V_{n+1}) \subseteq U_n \cap V_n$ and

2. the diameter of V_{n+1} is less than $\frac{1}{n+1}$.

Then choose $p_n \in V_n$. The sequence is Cauchy, hence converges to some $p \in X$. Since tails of the sequence are in the closed set $\text{cl}(V_n)$ we have that $p \in \bigcap_n U_n$.

(1) $\to$ (3)

Given poset $\mathbb{P}$ and nonempty subset $X \subseteq \mathbb{P}$ with no minimal element define $R \subseteq X^2$ by aRb iff $b < a$.

(3) $\to$ (1)

Let $\mathbb{P} = X^{<\omega}$ ordered by $p \leq q$ iff $q \subseteq p$. Then the subset $A \subseteq \mathbb{P}$ defined by:

$$A = \{(a_i : i < n) \; : \; n < \omega, \text{ and } \forall i < n-1 \;\; a_i R a_{i+1}\}$$

has no minimal element.

(2) $\to$ (1)

Given X and R the space X^ω where X has the discrete topology has the complete metric d defined by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n+1} & \text{if } x \upharpoonright n = y \upharpoonright n \text{ and } x(n) \neq y(n) \end{cases}$$

Let

$$U_n = \{x \in X^\omega \; : \; \exists m \;\; x(n) \; R \; x(m)\}.$$

Then U_n is open dense. If $x \in \bigcap_{n<\omega} U_n$, then $\forall n \; \exists m \;\; x(n) \; R \; x(m)$. Since the range of x is well-ordered we can construct a sequence as required without using choice.

QED

Kechris [10] has shown that $Con(ZF + AD) \to Con(ZF + AD + DC)$. Hence if we could derive a contradiction by using $AD + DC$ we could already get one from AD alone.

6 Superperfect sets

Definition 6.1 *A tree $T \subseteq \omega^{<\omega}$ is superperfect iff for every $s \in T$ there exists $t \supseteq s$ such $\exists^\infty n \;\; t\hat{\ }\langle n\rangle \in T$. A set $P \subseteq \omega^\omega$ is superperfect iff there exists a superperfect tree T such that $P = [T]$.*

Proposition 6.2 *A set $P \subseteq \omega^\omega$ is superperfect iff it is closed and homeomorphic to ω^ω.*

Definition 6.3 *For $f, g \in \omega^\omega$*

$$f \leq^* g \text{ iff } \forall^\infty n \in \omega \;\; f(n) \leq g(n).$$

Proposition 6.4 *For any $f \in \omega^\omega$ the set*

$$\{g \in \omega^\omega \;:\; g \leq^* f\}$$

is σ-compact, ie. the countable union of compact sets. Also if $F \subseteq \omega^\omega$ is σ-compact, then there exists f such that $g \leq^* f$ for all $g \in F$.

Definition 6.5 *Given $A \subseteq \omega^\omega$ the superperfect game $G_{sup}(A)$ is defined as follows:*

I	$s_0 \in \omega^{<\omega}$	$s_1 \in \omega^{<\omega}$ $s_1(0) > n_0$	$s_2 \in \omega^{<\omega}$ $s_2(0) > n_1$	$\ldots$
II	$n_0 \in \omega$	$n_1 \in \omega$	$n_2 \in \omega$	$\ldots$

Player I wins iff $x \in A$ where $x = s_0 \hat{} s_1 \hat{} s_2 \cdots$.

Theorem 6.6 *(Kechris [9]) Suppose $A \subseteq \omega^\omega$ then*

1. *Player I has a winning strategy in $G_{sup}(A)$ iff A contains a superperfect set.*
2. *Player II has a winning strategy in $G_{sup}(A)$ iff there exists $f \in \omega^\omega$ such that $g \leq^* f$ for every $g \in A$.*

Proof
The proof of (1) and (2) $\leftarrow$ are left to the reader.

(2) $\rightarrow$

Let τ be a winning strategy for Player II. Let $p = \langle s_0, s_1, \ldots, s_n \rangle$ be a position in the game consistent with τ. (This simply means that $s_{k+1}(0) > \tau(s_0, \ldots, s_k)$ for every $k < n$.)

Define a map $f_p : \omega^{<\omega} \to \omega$ as follows:

$$f_p(t) = \tau(s_0, s_1, \ldots, s_n, s)$$

if $(s_0, s_1, \ldots, s_n, s)$ is consistent with τ and $t = s_0 {}^\frown s_1 \ldots {}^\frown s_n {}^\frown s$. Otherwise $f_p(t) = 0$.

Since there are only countably many p there exists a $h : \omega^{<\omega} \to \omega$ such that

$$\forall^\infty s \in \omega^{<\omega} \;\; f_p(s) < h(s)$$

for every position p consistent with τ.

Claim. $A \subseteq \{g \in \omega^\omega \; : \; \forall^\infty n \;\; g(n) \leq h(g \restriction n)\}$.

Suppose

$$\exists^\infty n \;\; g(n) > h(g \restriction n).$$

Then we will construct an infinite sequence $s_0, s_1, \ldots$ which is consistent with τ such that $g = s_0 {}^\frown s_1 {}^\frown s_2 \cdots$. But since τ is a winning strategy this implies $g \notin A$.

Suppose we have already constructed $p = (s_0, s_1, \ldots, s_n)$ consistent with τ so that

1. $s_0 {}^\frown s_1 {}^\frown s_2 \cdots {}^\frown s_n \subseteq g$ and
2. $g(l) > \tau(s_0, s_1, \ldots, s_n)$ where l is the lenght of $s_0 {}^\frown s_1 {}^\frown s_2 \cdots {}^\frown s_n$.

Then

$$(\forall^\infty m \;\; h(g \restriction m) > f_p(g \restriction m)) \text{ and } (\exists^\infty m \;\; g(m) > h(g \restriction m)).$$

Hence we can find $m > l$ with $g(m) > f_p(g \restriction m)$ and take $s_{n+1} = g \restriction [l, m)$ as required.
QED

7 Measurability and AD

Lebesgue measure μ is a countably additive measure on the Borel subsets of the real line. It satisfies

- $\mu : \text{Borel}(\mathbb{R}) \to [0, \infty]$,
- $\mu([a,b]) = b - a$ for every interval $[a,b]$, and
- it is countably additive, meaning if $(B_n : n < \omega)$ are pairwise disjoint Borel sets, then
$$\mu(\bigcup_{n<\omega} B_n) = \sum_{n<\omega} \mu(B_n).$$

The outer measure μ^* is defined by

$$\mu^*(X) = \inf\{\sum_{n=0}^{\infty} \mu(I_n) \ : \ I_n \text{ are intervals such that } X \subseteq \bigcup_{n<\omega} I_n\}.$$

For Borel sets μ and μ^* agree. A set X has measure zero iff $\mu^*(X) = 0$. A set X is measurable iff there is a Borel set B such that $B \triangle X$ has measure zero.

Proposition 7.1 *If $X \subseteq \mathbb{R}$ is not measurable, then there exists a Borel set $B \subseteq X$ such that if $Y = X \setminus B$, then Y has inner meaure zero, i.e, any measurable $Z \subseteq Y$ has measure zero but the outer measure of Y is positive, $\mu^*(Y) > 0$.*

Theorem 7.2 *(Mycielski-Świerczkowski [16]) Let LM stands for the proposition that every set of reals is Lebesgue measurable. Then $AD \to LM$.*

Before proving this theorem we will need two technical lemmas.

Lemma 7.3 *Suppose $X \subseteq \mathbb{R}$ has measure zero and we are given $(\epsilon_n > 0 : n < \omega)$. Then there exists $(C_n : n < \omega)$ such that*

1. *each C_n is a finite union of closed intervals,*
2. *$\mu(C_n) \leq \epsilon_n$ for each n, and*
3. *$X \subseteq \bigcup_{n<\omega} C_n$.*

Proof
Without loss we may assume that $\sum_{n<\omega} \epsilon_n < \infty$. Since X has measure zero we may find closed intervals I_n such that

$$X \subseteq \bigcup_{n<\omega} I_n \text{ and } \sum_{n<\omega} \mu(I_n) = \sum_{n<\omega} \epsilon_n.$$

Actually we might first get $\leq$ above but we could always expand I_0 to make it an equality. Now define $\delta_n = \mu(I_n)$. We can break up the I_n using the following algorithm: Find n so that

$$\delta_0 + \cdots + \delta_{n-1} < \epsilon_0 \leq \delta_0 + \cdots + \delta_{n-1} + \delta_n.$$

Break up $\delta_n = \delta_n^0 + \delta_n^1$ into nonnegative reals so that

$$\delta_0 + \cdots + \delta_{n-1} + \delta_n^0 = \epsilon_0.$$

Break I_n into the overlapping closed intervals I_n^0 and I_n^1 where I_n^0 is lefthand piece of I_n of length δ_n^0 and I_n^1 is righthand piece of I_n of length δ_n^1. Set

$$C_0 = I_0 \cup I_1 \cup I_2 \cup \ldots I_{n-1} \cup I_n^0.$$

Then proceed with the intervals $I_n^1, I_{n+1}, \ldots$ and the equal sums

$$\delta_n^1 + \delta_{n+1} + \delta_{n+2} + \cdots = \epsilon_1 + \epsilon_2 + \cdots$$

and use the same procedure to find C_1, etc.
QED

Lemma 7.4 *Suppose $X \subseteq \mathbb{R}$ has measure zero and we are given $(\epsilon_n > 0 : n < \omega)$. Then there exists $(C_n : n < \omega)$ such that*

1. *each C_n is a finite union of open intervals with rational endpoints,*
2. *$\mu(C_n) < \epsilon_n$ for each n, and*
3. *$X \subseteq \bigcup_{n<\omega} C_n$.*

Proof
This is an immediate corollary of the preceeding lemma. Apply it to the sequence $(\frac{\epsilon_n}{2} : n < \omega)$. Then fattening up each closed interval by just a little bit.
QED

Lemma 7.5 *If $f : \omega^\omega \to A \subseteq \mathbb{R}$ is onto and continuous, then A is measurable.*

Proof
This will be proved latter. It follows from the fact that analytic sets are measurable.
QED

Now we are ready to prove the Theorem. By the proposition if LM is false then we can find a set of reals X such that the inner measure of X is zero, but the outer measure of X is positive. Without loss of generality we may assume that $X \subseteq [0,1]$. Choose $(\epsilon_n > 0 : n < \omega)$ so that

$$\sum_{n<\omega} 2^n \epsilon_n < \frac{1}{2}\mu^*(X).$$

And define $\mathcal{C}_n$ to be the set of C such that C is a finite union of open intervals with rational endpoints and $\mu(C) < \epsilon_n$.

Consider the following game G_{meas}:

I	I_0		I_1		I_2		$\ldots$	
II		C_0		C_1		C_2		$\ldots$

We require that $C_n \in \mathcal{C}_n$ and $I_n = [\frac{k}{2^n}, \frac{k+1}{2^n}]$ for some integer k,

Player II wins iff

$$(X \cap \bigcap_{n<\omega} I_n) \subseteq \bigcup_{n<\omega} C_n.$$

Claim. Player I cannot have a winning strategy in G_{meas}.

Proof
Suppose σ were such a winning strategy. Clearly the intervals played by σ must be nested. Then σ determines a continuous mapping

$$f : \prod_{n<\omega} \mathcal{C}_n \to [0,1]$$

by the rule that $f(C_n : n < \omega) = x$ iff $\{x\} = \bigcap_{n<\omega} I_n$ where $\sigma(C_n : n < m) = I_m$. But the image A of f is measurable by Lemma 7.5 and since we are assuming σ is a winning strategy $A \subseteq X$. Since X has inner measure zero, A has measure zero. But by Lemma 7.4, A can be covered by $(C_n : n < \omega) \in \prod_{n<\omega} \mathcal{C}_n$. But if Player II plays this sequence then σ losses this play of the game.

QED

Hence, if the game G_{meas} is determined, then Player II has a winning strategy τ. But note that there are only 2^n possible plays of I_n for Player I. This means that we can find $\mathcal{D}_n \subseteq \mathcal{C}_n$ of size 2^n which includes all possible n^{th} plays of τ. Since Player I can squeeze down on any $x \in X$ it follows that

$$X \subseteq \bigcup_{n<\omega} \cup \mathcal{D}_n \text{ and } \mu(\bigcup_{n<\omega} \cup \mathcal{D}_n) \leq \sum_{n<\omega} 2^n \epsilon_n \leq (\frac{1}{2})\mu^*(X)$$

which is a contradiction.
QED

8 Souslin Operation

Definition 8.1 *The Souslin operation is defined as follows: Given a family of sets* $(A_s : s \in \omega^{<\omega})$ *define the set* A *as*

$$A = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} A_{f\upharpoonright n}.$$

This operation is also known as "Operation $\mathcal{A}$*".*

Definition 8.2 *Given a topological space* X *define* $\mathcal{A}(X)$ *to be all sets of the form* $\bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} C_{f\upharpoonright n}$ *for* C_s*'s closed in* X*.*

Proposition 8.3 *The family* $\mathcal{A}(X)$ *is closed under countable unions and countable intersections.*

Proof
Given

$$A_m = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} C^m_{f\upharpoonright n}$$

then

$$\bigcup_{m<\omega} A_m = \bigcup_{g\in\omega^\omega} \bigcap_{n<\omega} D_{g\upharpoonright n}$$

where

$$D_{g\upharpoonright n} = C^{g(0)}_{g\upharpoonright(0,n)}.$$

For intersection note that

$$\forall n < \omega \; \exists f \in \omega^\omega \;\; P(n, f) \text{ iff } \exists (f_n \in \omega^\omega : n < \omega) \; \forall n < \omega \;\; P(n, f_n)$$

by weak countable choice, WCC. Hence

$$\bigcap_{m<\omega} A_m = \bigcup_{g \in \omega^{\omega\times\omega}} \bigcap_{n<\omega} D_{g\restriction(n\times n)}$$

where

$$D_{g\restriction(n\times n)} = \bigcap \{ C^k_{g_k\restriction n} \; : \; k < n \}$$

where $g_k \in \omega^\omega$ is defined by $g_k(i) = g(k, i)$. To make this look more like the Souslin operation, let $\{p_n : n < \omega\}$ be a listing of $\omega \times \omega$ which satisfies the property [4] that for every $n < \omega$:

$$\{p_i : i < n^2\} = n \times n.$$

Then

$$\bigcap_{m<\omega} A_m = \bigcup_{g \in \omega^{\omega\times\omega}} \bigcap_{n<\omega} D_{g\restriction\{p_i : i<n\}}$$

where we put $D_{g\restriction\{p_i : i<k\}} = X$ for any k with $n^2 < k < (n+1)^2$.
QED

Corollary 8.4 *If X is a metric space, then $\mathcal{A}(X)$ contains all Borel subsets of X.*

Proof
Clearly all closed sets are in $\mathcal{A}(X)$ and since in a metric space open sets are the countable union of closed sets, the result follows.
QED

Examples of spaces in which not every open set is in $\mathcal{A}(X)$ would be $X = \omega_1$ or $X = \omega_1 + 1$ with the order topology.

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[4] We pause to here to let the reader verify that it is possible to have such a listing of the pairs $\cdots\cdots\cdots$ This morning I had biscuits and gravy for breakfast. The Saw-Mill gravy was made by browning flour in the drippings of Jimmy Dean's hot sausage and then adding milk. It was poured over hot freshly baked Pillsbury Grande biscuits. Manna from Heaven! $\cdots\cdots\cdots$ OK lets go on.

Proposition 8.5 *For any topological space X and $A \subseteq X$ the following are equivalent:*

1. $A \in \mathcal{A}(X)$
2. *there exist a closed set $C \subseteq \omega^\omega \times X$ such that*

$$A = proj_X(C) =^{def} \{x \in X \ : \ \exists f \in \omega^\omega \ (f,x) \in C\}.$$

Proof
(1) → (2)

Suppose

$$A = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} C_{f\restriction n}$$

where the C_s's are closed. Define

$$C = \bigcap_{n<\omega} \bigcup_{s\in\omega^n} ([s] \times C_s).$$

Then C is closed in $\omega^\omega \times X$.

Suppose $x \in proj_X(C)$. Then for some f we have $(f,x) \in C$. But then it is easy to check that $x \in \bigcap_{n<\omega} C_{f\restriction n}$. Suppose $x \in A$. Then there exists $f \in \omega^\omega$ such that $x \in C_{f\restriction n}$ for every n. But then $(f,x) \in [f \restriction n] \times C_{f\restriction n}$ so $(f,x) \in C$ and $x \in proj_X(C)$.

(2) → (1)

Suppose $A = proj_X(C)$ for $C \subseteq \omega^\omega \times X$ closed. For $s \in \omega^{<\omega}$ define

$$Q_s = \{x \in X \ : \ ([s] \times \{x\}) \cap C \neq \emptyset\} \text{ and } C_s = \mathrm{cl}(Q_s).$$

Suppose $x \in proj_X(C)$. Then there exist $f \in \omega^\omega$ such that $(x,f) \in C$. But then it follows that $x \in Q_{f\restriction n} \subseteq C_{f\restriction n}$ for every n. For the opposite inclusion suppose that

$$x \in \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} C_{f\restriction n}$$

and choose f so that $x \in C_{f\restriction n}$ for every n. Suppose for contradiction that $(f,x) \notin C$. Since C is closed there exists an open U and an $n < \omega$ with

$x \in U$ and $U \times [f \restriction n]$ disjoint from C. But this means that U is disjoint from $Q_{f\restriction n}$ and hence $x \notin C_{f\restriction n}$. Hence $x \in proj_X(C)$.
QED

Another proof that the family of sets $\mathcal{A}(X)$ is closed under countable intersections:

Given $A_m = proj_X(C_m)$ for $m < \omega$ let

$$C = \{(f, x) \in \omega^{\omega\times\omega} \times X \ : \ \forall m < \omega \ \ (f_m, x) \in C_m\}.$$

Then

$$\bigcap_{m<\omega} A_m = proj_X(C).$$

Corollary 8.6 *For X a metric space, if $B \subseteq \omega^\omega \times X$ is Borel, then $A = proj_X(B)$ is in $\mathcal{A}(X)$.*

Proof
Since B is in $\mathcal{A}(\omega^\omega \times X)$ there exists a closed set $C \subseteq \omega^\omega \times (\omega^\omega \times X)$ with $B = proj_{\omega^\omega \times X}(C)$. But then $A = proj_X(C)$.
QED

Definition 8.7 *The class of $\mathbf{\Sigma^1_1}$ sets are those sets A which are in $\mathcal{A}(X)$ for some Polish space X, i.e., X is a separable completely metrizable space. These sets are also known as analytic sets.*

Corollary 8.8 *A set $A \subseteq \omega^\omega$ is $\mathbf{\Sigma^1_1}$ iff there exists a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ such that*

$$A = p[T] =^{def} \{x \ : \ \exists y \in \omega^\omega \ \ \forall n \ \ (x \restriction n, y \restriction n) \in T\}.$$

Proof
Given a closed set $C \subseteq \omega^\omega \times \omega^\omega$ whose projection onto the first coordinate gives A define T by

$$T = \{(s, t) \ : \ ([s] \times [t]) \cap C \neq \emptyset\}.$$

QED

9 Analytic sets and the perfect set property

Theorem 9.1 *(Souslin 1917) Suppose $A \subseteq 2^\omega$ is $\mathbf{\Sigma^1_1}$. Then either A is countable or A contains a perfect subset.*

Proof
Suppose $T \subseteq 2^{<\omega} \times \omega^{<\omega}$ is a tree such that $A = p[T]$. Suppose that A is uncountable and define

$$\hat{T} = \{(s,t) \in T \;:\; p[T_{(s,t)}] \text{ is uncountable } \}$$

where $T_{(s,t)}$ is the subtree of T of nodes comparable to (s,t).

Lemma 9.2 *For any $(s,t) \in \hat{T}$ there exists $n < \omega$ such that $(s, t\hat{\ }\langle n\rangle) \in \hat{T}$.*

Proof
Note that

$$p[T_{(s,t)}] = \{x \in [s] \;:\; \exists y \in [t] \;\; (x,y) \in [T]\} = \bigcup_{n<\omega} p[T_{(s,t\hat{\ }\langle n\rangle)}].$$

QED

Lemma 9.3 *For any $(s,t) \in \hat{T}$ there exists r with $s \subseteq r \in 2^{<\omega}$ such that both $(r\hat{\ }\langle 0\rangle, t)$ and $(r\hat{\ }\langle 1\rangle, t)$ are in $\hat{T}$.*

Proof
Let

$$Q = \{r \in 2^{<\omega} \;:\; s \subseteq r \text{ and } (r,t) \in \hat{T}\}.$$

If the lemma is false then all of the elements of Q would have to lie along a single branch $x \in 2^\omega$. But then

$$p[T_{(s,t)}] \subseteq \{x\} \cup \bigcup \{p[T_{(r,t)}] \;:\; s \subseteq r \text{ and } r \notin Q\}$$

and this is a countable union of countable sets.
QED

Using the two lemmas we can now construct our perfect set and prove Theorem 9.1. Inductively choose (s_σ, t_σ) for $\sigma \in 2^{<\omega}$ so that:

1. $(s_\sigma, t_\sigma) \in \hat{T}$,
2. $|s_\sigma| = |t_\sigma|$,
3. $s_{\sigma \frown \langle 0 \rangle}$ and $s_{\sigma \frown \langle 1 \rangle}$ are incomparble, and
4. $s_\sigma \subseteq s_{\sigma \frown \langle i \rangle}$ and $t_\sigma \subseteq t_{\sigma \frown \langle i \rangle}$ for $i = 0, 1$.

This is easy to do since given (s_σ, t_σ) we first apply Lemma 9.3 to find $s_{\sigma \frown \langle 0 \rangle}$ and $s_{\sigma \frown \langle 1 \rangle}$ and then use Lemma 9.2 repeatedly to find $t_{\sigma \frown \langle i \rangle}$ of the same length as $s_{\sigma \frown \langle i \rangle}$. The maps $f : 2^\omega \to 2^\omega$ and $g : 2^\omega \to \omega^\omega$ defined by

$$f(x) = \bigcup_{n<\omega} s_{x \upharpoonright n} \text{ and } g(x) = \bigcup_{n<\omega} t_{x \upharpoonright n}$$

are continuous and f is one-to-one. Since $(f(x), g(x)) \in [T]$ for every x it follows that the range of f is a perfect subset of A.
QED

Souslin's Theorem is true for every $\mathbf{\Sigma^1_1}$ set, i.e., it holds for any Polish space in place of 2^ω.

Theorem 9.4 *(Kechris) Suppose $A \subseteq \omega^\omega$ is $\mathbf{\Sigma^1_1}$. Then either A contains a superperfect set or $\exists g \in \omega^\omega \;\; \forall f \in A \;\; f \leq^* g$.*

Proof

10 Miscellania

Does a game with a computable payoff set always have a computable winning strategy?

Example 10.1 *(Galvin) There exists a computable set $A \subseteq \omega^3$ such that the three move game below has no computable winning strategy:*

I	$x \in \omega$		$s \in \omega$
II		$y \in \omega$	

Player I wins iff $(x, y, s) \in A$.

Proof
Let B be a simple set with computable enumeration $\bigcup_s B_s$. Put

$$(x, y, s) \in A \text{ iff } y \leq x \text{ or } y \in A_s.$$

Player II has a winning strategy, he just plays an element of the complement of A bigger than x. But he cannot have a computable winning strategy since A is simple.
QED

Example 10.2 *(Kleene) There is a computable closed set $C \subseteq \omega^\omega$ for which there is no hyperarithmetic winning strategy.*

Proof
By computable closed set we mean Π^0_1 class, equivalently there exists a computable tree $T \subseteq \omega^{<\omega}$ such that $C = [T]$. Kleene has shown that there exists a nonempty Π^0_1 class B which contains no hyperarithmetic reals. If $C = B \times \omega^\omega$, then clearly Player I has a winning strategy, just play an element of B and ignore Player II. But he cannot have an hyperarithmetic winning strategy, since the play against the constant zero function would give an hyperarithmetic element of B.
QED

For an interesting paradoxical game, see Zwicker [21]. Zwicker calls a game G almost finite iff it always eventually ends. In the Hypergame Player I plays an almost finite game G, then Player II plays the first move in G and then they continue in G until it ends. Is the Hypergame itself an almost finite game?

Proposition 10.3 *The Hypergame is almost finite.*

Proof
After the play of the almost finite game G the players play the game G which since it is almost finite, must end after finitely many more plays.
QED

Proposition 10.4 *The Hypergame is not almost finite.*

Proof
If the Hypergame H is almost finite, then player I can play it on the first move. Since they are now playing the Hypergame, Player II can play it as his first move, etc. Hence the following infinite play is legal:

I	H		H		H		...
II		H		H		H	...

QED

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