M873 Topics in Foundations
A.Miller
Spring 2005

I plan to present the Jackson-Mauldin solution to the Steinhaus problem. In the 1950s Steinhaus asked if there exists a subset S of the plane which meets every isometric copy of the integer lattice, $\mathbb{Z}\times\mathbb{Z}$, in exactly one point. I also plan to cover other results which involve the set theoretical properties of the plane. Davies has shown that the plane can be partitioned into countably many pieces none of which contains two pairs of points the same distance apart iff the continuum hypothesis is true. On the other hand Schmerl has shown that without any extra set theoretical hypotheses the plane can always be partitioned into countably many pieces so that no piece contains the vertices of an isoceles triangle.

Theorem 1 *(R.O.Davies [11], also in [28]) CH is equivalent to:*
There exists $S_k \subseteq \mathbb{R}^2$ such that

$$\mathbb{R}^2 = \bigcup_{k<\omega} S_k$$

and for every k if $x, y, u, v \in S_k$ and $x \neq y$ then

$$d(x, y) = d(u, v) \text{ iff } \{x, y\} = \{u, v\}$$

i.e., distances are unique in each S_k.

Theorem 2 *(Kunen [31]) CH implies that for each positive integer n there exists S_k such that $\mathbb{R}^n = \bigcup_{k<\omega} S_k$ and distances are unique in each S_k.*

Question 3 *(Kunen [31]) For the l_p norm $1 < p < \infty$ does CH imply that $\mathbb{R}^n$ can be partitioned into countably many sets in which distances are unique?*

Remarks [31]. It is false for $p = 1$ or $p = \infty$ in the plane. It is also false for infinite dimensional Hilbert space l_2 by the following proposition.

Proposition 4 *(Posa see [28]) There exists an uncountable $\mathcal{A} \subseteq l_2$ such that $||a - b||$ is rational for any $a, b \in \mathcal{A}$.*

Question 5 *(Erdos) Does there exists a dense $D \subseteq \mathbb{R}^2$ such that the distance between any two points of D is rational?*

Lemma 6 *If V is a vector space over a countable field F, then there are S_k*

$$V \setminus \{\underline{0}\} = \bigcup_{k<\omega} S_k$$

such that each S_k contains no nontrivial solution to

$$\alpha x + \beta y = z$$

for $\alpha, \beta \in F$. Trivial means

$$\alpha x + (1-\alpha)x = x.$$

Theorem 7 *(Rado, in [28]) $\mathbb{R}^n$ can be partitioned into countably many sets*

$$\mathbb{R}^n = \bigcup_{k<\omega} S_k$$

none of which contains the vertices of a degenerate isoceles triangle (or equivalently a 3-term arithmetic progression):

$$\frac{p+r}{2} = q.$$

Theorem 8 *(Ceder [5] also in [28]) $\mathbb{R}^2$ can be partitioned into countably many sets none of which contains the vertices of an equilateral triangle.*

Theorem 9 *(Ceder [5]) If $\mathbb{R}^n = \bigcup_{k<\omega} S_k$ and each S_k is Borel, then there exists $k < \omega$ such that for every finite $F \subseteq \mathbb{R}^n$ there exists $p \in \mathbb{R}^n$ and $\alpha > 0$ such that $p + \alpha F \subseteq S_k$.*

Question 10 *(Erdos see [42]) If x_n is a sequence decreasing to 0, does there exist a set of positive measure C which avoids every similar copy of it? What about $x_n = \frac{1}{2^n}$?*

Theorem 11 *(Gallai see [17]) Suppose $\mathbb{R}^n = \bigcup_{k<N} S_k$ where N finite. Then for every finite $F \subseteq \mathbb{R}^n$ there exists $k < N$ and $p \in \mathbb{R}^n$ and $\alpha > 0$ such that $p + \alpha F \subseteq S_k$*

The Hales-Jewett Theorem is also in [17] as well as [19]. Shelah's proof appears in the second edition 1990 of [17] and there is an even better bound proved by Gowers [16] [32]. For a version of Hales-Jewett with an infinite alphabet, due to Miller and Prikry, see Carlson [6] Theorem 15.

Lemma 12 *(Tarski) The theory RCF (real-closed fields) eliminates quantifiers.*

For a proof see Marker [33] or Shoenfield [40].

Lemma 13 *(Implicit function theorem for real-analytic, see [34] 1.3.10)*
Suppose $\Omega_1 \subseteq \mathbb{R}^n, \Omega_2 \subseteq \mathbb{R}$ *are open,* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *is real-analytic,* $(a, b) \in \Omega_1 \times \Omega_2$,

$$f(a, b) = 0 \text{ and } \frac{\partial f}{\partial x_{n+1}}(a, b) \neq 0.$$

Then there exists open $U_1 \subseteq \Omega_1$ *and* $U_2 \subseteq \Omega_2$ *with* $(a, b) \in U_1 \times U_2$ *such that for every* $x \in U_1$ *there exists a unique* $y = y(x) \in U_2$ *such that* $f(x, y) = 0$. *Furthermore the function* $x \mapsto y(x)$ *is real-analytic.*

Theorem 14 *(Schmerl [35])* $\mathbb{R}^n$ *can be partitioned into countably many sets none of which contains the vertices of an isoceles triangle.*

The same partition has the property that the distance between distinct points in any piece is irrational. This yields a result of Komjath [27]. It also has the property that for the plane, no piece contains the vertices of a triangle with rational area, a result proved by Kunen (unpublished) assuming CH and generalized to $\mathbb{R}^n$ by Komjath [24],[25].

Exercise 15 *Let* $g : (\mathbb{R}^n)^3 \to \mathbb{R}$ *be defined by* $g(a, b, c)$ = *the area of the triangle with vertices* a, b, c. *Then* g *is definable in* $(\mathbb{R}, +, \cdot)$. *Prove or disprove: It is real-analytic.*

Theorem 16 *(Erdos-Kakutani [12]) CH is equivalent to* $\mathbb{R} \setminus \{0\} = \cup_{n<\omega} H_n$ *where each* H_n *is a Hamel base.*

Remark. Assuming CH we can make the H_n pairwise disjoint Hamel bases.

Theorem 17 *(Beslagic [4]) $\neg$CH is equivalent to: for every partition $\mathcal{P}$ of the $\mathbb{R}$ into countable sets there exists a Hamel base H such that $|p \cap H| \leq 1$ for every $p \in \mathcal{P}$.*

Theorem 18 *(Erdos, Komjath [13]) CH is equivalent to there exists S_k such that*

$$\mathbb{R}^2 = \bigcup_{k<\omega} S_k$$

and no S_k contains the vertices of a right triangle.

Question 19 *(Erdos, Komjath [13]) Is this result true for $\mathbb{R}^n$ for every n?*

Theorem 20 *(Sierpinski, see Simms [41] for references and history) The following are equivalent:*

1. *CH*
2. *$\mathbb{R}^2 = H \cup V$ where every horizontal line meets H in a countable set and every vertical line meets V in a countable set.*
3. *$\mathbb{R}^3 = A_1 \cup A_2 \cup A_3$ where every line parallel to axis x_i meets A_i in a finite set.*
4. *$\mathbb{R}^3 = B_1 \cup B_2 \cup B_3$ where every plane perpendicular to the x_i axis meets B_i in a countable set.*

Theorem 21 *(Kuratowski, see Simms [41]) The following are equivalent:*

1. *$|\mathbb{R}| \leq \omega_2$*
2. *$\mathbb{R}^3 = A_1 \cup A_2 \cup A_3$ where every line parallel to axis x_i meets A_i in a countable set.*
3. *$\mathbb{R}^4 = B_1 \cup B_2 \cup B_3 \cup B_4$ where every every line parallel to axis x_i meets B_i in a finite set.*

And the obvious generalizations to $|\mathbb{R}| \leq \omega_n$.

Exercise 22 *Suppose that* $\kappa \leq \aleph_m$ *is equivalent to:*
There exists A_p *such that*

$$\kappa^n = \bigcup_{s \in [n]^k} A_s$$

and $|A_s(p)| < \aleph_l$ *for each* $s \in [n]^k$ *and* $p \in \kappa^s$ *where*

$$A_s(p) = \{(\alpha_i : i < n) \in \kappa^n \quad : \quad \forall i \in s \;\; \alpha_i = p(i)\}$$

Find $m = m(n, k, l)$ *as a function of* n, k, l.

Hint: $m = l + k - 1$ for $k < n$ according to David Milovich.

Question 23 *(Ristow, see Simms [41] 5.2) Characterize the sequences* σ_i, κ_i *for* $i < n$ *such that* $\omega \leq \sigma_i \leq \kappa_i$ *and there exists* E_i *such that*

$$\prod_{i<n} \kappa_i = \bigcup_{i<n} E_i$$

and $|l \cap E_i| < \sigma_i$ *for each* i *and line* l *parallel to the ith axis.*

Theorem 24 *(Komjath [30]) CH is equivalent to* $\mathbb{R}^2$ *is the union of three clouds. A cloud is set of points* C *such that for some point* p *every line thru* p *meets* C *in a finite set. The plane is the union of* $n+2$ *clouds if* $|\mathbb{R}| \leq \omega_n$.

Theorem 25 *(Bagemihl-Davies [1][8]) CH is equivalent to* $\mathbb{R}^2$ *is the union of three fogs. A fog* $F \subseteq \mathbb{R}^2$ *is a set with the property that for some slope* m *every line with slope* m *meets* F *in a finite set.*

Theorem 26 *(Schmerl [39]) CH implies that* $\mathbb{R}^2$ *is the union of three sprays. A spray* $S \subseteq \mathbb{R}^2$ *is a subset with the property that for some* $p \in \mathbb{R}^2$ *every circle centered at* p *meets at most finitely many elements of* S.

Question 27 *(Schmerl) If the plane is the union of three sprays, then does CH hold?*

Two sprays cannot cover the plane. Ditto for two fogs. If S^2 can be covered by three "sprays" whose center lies on a great circle, then CH is true. To see this project the sphere to the plane of the great circle and get 2-1 map to cover the disk with three fogs.

Exercise 28 *Define: A set $W \subseteq \mathbb{R}^2$ is a wringer iff there are two points p_1, p_2 such that $C \cap W$ is finite for every circle containing p_1 and p_2. (The circles travel between the two points which wrings them dry.) A set $P \subseteq \mathbb{R}$ is a pinched wringer iff there is a line l_0 and point $p_0 \in l_0$ such that $P \cap C$ is finite for every circle C with $p_0 \in C$ and center of C on l_0. A set $F \subseteq \mathbb{R}$ is a flower iff there is a point p_0 and radius r such that $F \cap C$ is finite for every circle C with $p_0 \in C$ and radius r.*

Assume CH holds then

1. *There exists three wringers W_1, W_2, W_3 determined by pairs of three points p_1, p_2, p_3 which cover every point in $\mathbb{R}^2$ not on the circle containing p_1, p_2, p_3.*
2. *Three pinched wringers cover the plane.*
3. *Given a triangle the interior of the triangle is covered by three flowers at the vertices.*

Do any of these imply CH?

Theorem 29 *(Komjath [30]) The plane is the union of countably many starcircles. A starcircle is a set of points C such that for some point p every ray starting at p meets C in at most one point.*

Theorem 30 *(Schmerl [39]) The plane is not the union of finitely many starcircles. The plane is the union of four clouds iff $|\mathbb{R}| \leq \omega_2$. Similarly, the union of $n+2$ clouds iff $|\mathbb{R}| \leq \omega_n$.*

Analogous results holds for n fogs. A fog F can be thought of as the graph of a finitely multi-valued function which has been rotated. I.e., fix a line l orthogonal to the lines determining the fog, then every line orthogonal to l meets F in a finite set.

Theorem 31 *(Davies [10]) There is a function $f : \mathbb{R} \to \mathbb{R}$ such that the plane is the countable union of sets, each of which is isometric to the graph of f.*

Remark. We can construct f so that countably many rotations of its graph cover the plane. Erik Andrejko notes that in addition f can be made uniformly bounded, i.e., $|f(x)| \leq 1$ for all x.

Question 32 *(Davies, see Simms [41] 5.1) Is the plane the countable union of sets congruent to the graph of a function which is connected?*

Theorem 33 *(Mazurkiewicz 1933, see Simms [41] for reference) The plane is not the finite union of curve-graphs. A curve graph is a subset of the plane which is isometric to the graph of some function $f : \mathbb{R} \to \mathbb{R}$.*

This result follows easily from Gallai's Theorem.

Theorem 34 *(Bagemihl [2]) There exists S_k such that*

$$\mathbb{R}^\omega = \bigcup_{n<\omega} S_n$$

and $|S_n \cap l| \leq 1$ for every n and line l parallel to e_n.

Theorem 35 *(Gruenhage [18]) also in Komjath [28]) There exists $F(l) \subseteq l$ finite for each line l in the plane and finite $G(p) \subseteq \{l : p \in l\}$ for each point $p \in \mathbb{R}^2$ such that for every line l and point $p \in l$ either $p \in F(l)$ or $l \in G(p)$.*

Theorem 36 *(Galvin) If $\mathbb{R} \leq \omega_n$ then*

(a) There are F and G as in Gruenhage's Theorem such that $|F(l)| \leq n+1$ for every line l and

(b) There are F and G as in Gruenhage's Theorem such that $|G(p)| \leq n+1$ for every point p.

Question 37 *In the Galvin result*

(a) Is it consistent to have $|F(p)| \leq 1$ all p?

(b) Does $|F(p)| \leq 2$ all p imply CH?

(c) Similarly for $|G(l)|$ in (a) and (b)?

(d) Is it consistent to have $|F(p)| + |G(l)| < n$ for all p, l?

I don't have a copy of the Gruenhage-Galvin unpublished manuscript so perhaps these are known. Bergman-Hrushovski [3] also has a proof of Galvin's Theorem as well as some questions.

Theorem 38 *(Davies [9]) Suppose $\mathcal{L}_k$ for $k < \omega$ is a partition of the lines of $\mathbb{R}^2$. Then there exists S_k such that $\mathbb{R}^2 = \bigcup_k S_k$ and for every k and $S_k \cap l$ is finite for every $l \in \mathcal{L}_k$. In addition, if $|\mathbb{R}| \leq \omega_n$ then we can find S_k so that $|S_k \cap l| \leq n+1$ for each $k < \omega$ and $l \in \mathcal{L}_k$.*

Remark. Same proof for any $\mathbb{R}^m$ replacing $\mathbb{R}^2$. This result implies that the plane is the countable union of starcircles.

Example 39 *(Davies) There exists $\mathcal{L}_k$ for $k < \omega$ a partition of the lines of $\mathbb{R}^2$ such that there does not exists S_k such that $|S_k \cap l| \leq 1$ for each $k < \omega$ and $l \in \mathcal{L}_k$.*

Theorem 40 *(Erdos-Jackson-Mauldin [15]) Assume MA. Suppose $\mathcal{L}_k$ for $k < \omega$ is a partition of the lines of $\mathbb{R}^2$. Then there exists S_k such that $\mathbb{R}^2 = \bigcup_k S_k$ and for every k and $|S_k \cap l| \leq 3$ for every $l \in \mathcal{L}_k$.*

So the "3-point" property does not imply any bound on the size of the continuum. Later Schmerl showed that the "2-point" property is just true in ZFC.

Theorem 41 *(Schmerl [36, 38]) Suppose $\mathcal{L}_k$ for $k < \omega$ is a partition of the lines of $\mathbb{R}^2$. Then there exists S_k such that $\mathbb{R}^2 = \bigcup_k S_k$ and for every k and $|S_k \cap l| \leq 2$ for every $l \in \mathcal{L}_k$.*

Theorem 42 *(Erdos-Jackson-Mauldin [14]) Assume $|\mathbb{R}| \leq \omega_{n-2}$. Then for every partition*

$$\bigcup_{k<n} \mathcal{L}_k = lines(\mathbb{R}^2)$$

there exists S_k such that $\mathbb{R}^2 = \bigcup_{k<n} S_k$ and $|l \cap S_k| < \omega$ for every $k < n$ and $l \in \mathcal{L}_k$.

This results is also true for any $\mathbb{R}^m$ in place of the plane. It also implies that the plane is the union of n-clouds and n-fogs.

Theorem 43 *(Sierpinski see Komjath [28]) There exists $A, S \subseteq \mathbb{R}^2$ nontrivial so that for every B an isometric copy of A we have $|A \cap S| = 1$.*

This answered a question of Steinhaus who also asked if A could be taken to be $\mathbb{Z}^2$.

Theorem 44 *(Komjath [26, 29])*

(a) There exists $S \subseteq \mathbb{R}^2$ such that for every isometric copy $L \subseteq \mathbb{R}^2$ of $\mathbb{Z}$ we have that $|S \cap L| = 1$.

(b) There exists $S \subseteq \mathbb{R}^2$ such that for every isometric copy $L \subseteq \mathbb{R}^2$ of $\mathbb{Q}^2$ we have that $|S \cap L| = 1$.

(c) There exists $S \subseteq \mathbb{R}^2$ such that for every isometric copy $L \subseteq \mathbb{R}^2$ of $\mathbb{Q}$ we have that $|S \cap L| = 1$.

Schmerl [37] proves some generalizations of these for $\mathbb{Q}^n \subseteq \mathbb{R}^n$, and for isometric copies of the $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{R}^n$.

Proposition 45 *(Schmerl [37]) For $4 \leq m < n$ there does not exist $S \subseteq \mathbb{R}^n$ such that for every isometric copy $L \subseteq \mathbb{R}^n$ of $\mathbb{Z}^m$ we have that $|S \cap L| = 1$.*

Similarly, for $\mathbb{Q}^m$ and $\mathbb{R}^n$.

Proposition 46 *(Jackson-Mauldin) Suppose $A \subseteq \mathbb{R}^n$ and S is Steinhaus for A, i.e., S meet every isometric copy of A in exactly one point. Then*

$$\{p + S : p \in A\}$$

is a partition of $\mathbb{R}^n$.

Hence if there S in the plane meeting every isometric copy of $\mathbb{Q}$ in exactly one point, then the plane can be partitioned into countably many such sets. Similarly, if a Steinhaus set for a countable A has the Property of Baire (is Lebesgue measurable) then it is nonmeager (positive measure). If there is a Steinhaus set for A then for each positive $n < |A|$ there is a set S meeting every isometric copy of A in exactly n points.

Theorem 47 *(Jackson-Mauldin [20, 21, 22]) There exists $S \subseteq \mathbb{R}^2$ such that for every isometric copy $L \subseteq \mathbb{R}^2$ of $\mathbb{Z}^2$ we have that $|S \cap L| = 1$.*

Lemma 48 *(Lemma B) Suppose a_1, a_2, a_3 are distinct points in the plane. Suppose $r_1, r_2, r_3 > 0$ and C_i is the circle centered at a_i with radius r_i. Fix $\lambda_{i,j} > 0$. Suppose there are infinitely many collinear triples*

$$(p_1, p_2, p_3) \in C_1 \times C_2 \times C_3$$

with $d(p_i, p_j) = \lambda_{i,j}$. Then a_1, a_2, a_3 are collinear, $r_1 = r_2 = r_3$, and $\lambda_{i,j} = d(a_i, a_j)$ for all $i, j \in 1, 2, 3$.

Lemma B is somewhere in between the original Komjath [26] version which used 5 circles and the stronger version in Jackson-Mauldin [21] which applies to noncollinear triples. The Maple proof for our version is slightly simpler than the one given in Jackson-Mauldin [21] since it avoids Grobner basis.

```
eq1:= (x+u*f-a)^2+ (y+u*g)^2  = r^2;
eq2:= (x+v*f-b)^2+(y+v*g-c)^2 = s^2;
eq1:= collect(expand(eq1),u):
eq2:= collect(expand(eq2),v):
eq1:= simplify(eq1,{x^2+y^2=1,f^2+g^2=1}):
eq2:= simplify(eq2,{x^2+y^2=1,f^2+g^2=1}):
eq1:=sort(collect(eq1,[x,y],'distributed'));
eq2:=sort(collect(eq2,[x,y],'distributed'));
sol:=solve({eq1,eq2},{x,y}): sol[1]; sol[2];
assign(sol):
z  := x^2+y^2-1:
bot:=denom(z):
bot:=factor(bot);
top:=numer(z):
top:=simplify(top,{g^2=1-f^2}):
top:=sort(top,[f,g],'plex');
coeffs(expand(top),[f,g],'terms'): terms;
p:=lcoeff(top,[f,g],'t03'); t03;
q:=lcoeff(top,[g,f],'t12'); t12;
p:=factor(p);
q:=factor(q);
```

Theorem 49 *(Jackson-Mauldin [22]) If $S \subseteq \mathbb{R}^2$ has the property of Baire, then there exists an isometry α of the plane such that $|\alpha(\mathbb{Z}^2) \cap S| \neq 1$.*

This is open for Lebesgue measure, however Croft [7] (and independently Beck) has shown that there is no Lebesgue measurable Steinhaus set which is measure zero outside a bounded set. It seems to be open if there can be a bounded Steinhaus set.

Jackson and Mauldin also prove that there are Steinhaus sets for the lattice $n\mathbb{Z} \times m\mathbb{Z}$ provided every prime which divides n or m is 1 mod 4. It seems to be open for $\mathbb{Z} \times 5\mathbb{Z}$.

References

[1] Bagemihl, Frederick; A proposition of elementary plane geometry that implies the continuum hypothesis. Z. Math. Logik Grundlagen Math. 7 1961 77–79.

[2] Bagemihl, Frederick; A decomposition of an infinite-dimensional space. Z. Math. Logik Grundlag. Math. 31 (1985), no. 5, 479–480.

[3] Bergman, George M.; Hrushovski, Ehud; Identities of cofinal sublattices. Order 2 (1985), no. 2, 173–191.

[4] Bešlagić, Amer; Partitions of vector spaces. Proc. Amer. Math. Soc. 110 (1990), no. 2, 491–493.

[5] Ceder, Jack; Finite subsets and countable decompositions of Euclidean spaces. Rev. Roumaine Math. Pures Appl. 14 1969 1247–1251.

[6] Carlson, Timothy J.; Some unifying principles in Ramsey theory. Discrete Math. 68 (1988), no. 2-3, 117–169.

[7] Croft, H. T.; Three lattice-point problems of Steinhaus. Quart. J. Math. Oxford Ser. (2) 33 (1982), no. 129, 71–83.

[8] Davies, Roy O.; Equivalence to the continuum hypothesis of a certain proposition of elementary plane geometry. Z. Math. Logik Grundlagen Math. 8 1962 109–111.

[9] Davies, Roy O.; On a denumerable partition problem of Erdös. Proc. Cambridge Philos. Soc. 59 1963 501–502.

[10] Davies, Roy O.; Covering the plane with denumerably many curves. J. London Math. Soc. 38 1963 433–438.

[11] Davies, Roy O.; Partitioning the plane into denumberably many sets without repeated distances. Proc. Cambridge Philos. Soc. 72 (1972), 179–183.

[12] Erdös, P.; Kakutani, S.; On non-denumerable graphs. Bull. Amer. Math. Soc. 49, (1943). 457–461.

[13] Erdös, P.; Komjáth, P. Countable decompositions of R^2 and R^3. Discrete Comput. Geom. 5 (1990), no. 4, 325–331.

[14] Erdös, P.; Jackson, Steve; Mauldin, R. Daniel; On partitions of lines and space. Fund. Math. 145 (1994), no. 2, 101–119.

[15] Erdös, P.; Jackson, Steve; Mauldin, R. Daniel; On infinite partitions of lines and space. Fund. Math. 152 (1997), no. 1, 75–95.

[16] Gowers, W. T.; A new proof of Szemeredi's theorem. Geom. Funct. Anal. 11 (2001), no. 3, 465–588.

[17] Graham, Ronald L.; Rothschild, Bruce L.; Spencer, Joel H.; **Ramsey theory**. Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1980. ix+174 pp. ISBN: 0-471-05997-8

[18] Gruenhage, Gary; Covering properties on $X^2 \setminus \Delta$, W-sets, and compact subsets of Σ-products. Topology Appl. 17 (1984), no. 3, 287–304.

[19] Hales, A. W.; Jewett, R. I.; Regularity and positional games. Trans. Amer. Math. Soc. 106 1963 222–229.

[20] Jackson, Steve; Mauldin, R. Daniel; Sets meeting isometric copies of the lattice Z^2 in exactly one point. Proc. Natl. Acad. Sci. USA 99 (2002), no. 25, 15883–15887.

[21] Jackson, Steve; Mauldin, R. Daniel; On a lattice problem of H. Steinhaus. J. Amer. Math. Soc. 15 (2002), no. 4, 817–856.

[22] Jackson, Steve; Mauldin, R. Daniel; Survey of the Steinhaus tiling problem. Bull. Symbolic Logic 9 (2003), no. 3, 335–361.

[23] Erdös, P.; Kakutani, S.; On non-denumerable graphs. Bull. Amer. Math. Soc. 49, (1943). 457–461.

[24] Komjáth, Péter; Tetrahedron free decomposition of R^3. Bull. London Math. Soc. 23 (1991), no. 2, 116–120.

[25] Komjáth, Péter; The master coloring. C. R. Math. Rep. Acad. Sci. Canada 14 (1992), no. 5, 181–182.

[26] Komjáth, Péter; A lattice-point problem of Steinhaus. Quart. J. Math. Oxford Ser. (2) 43 (1992), no. 170, 235–241.

[27] Komjáth, Péter; A decomposition theorem for R^n. Proc. Amer. Math. Soc. 120 (1994), no. 3, 921–927.

[28] Komjath, Peter; Set-theoretic constructions in Euclidean spaces. New trends in discrete and computational geometry, 303–325, Algorithms Combin., 10, Springer, Berlin, 1993.

[29] Komjáth, P.; A coloring result for the plane. J. Appl. Anal. 5 (1999), no. 1, 113–117.

[30] Komjáth, Péter; Three clouds may cover the plane. Dedicated to Petr Vopěnka. Ann. Pure Appl. Logic 109 (2001), no. 1-2, 71–75.

[31] Kunen, Kenneth; Partitioning Euclidean space. Math. Proc. Cambridge Philos. Soc. 102 (1987), no. 3, 379–383.

[32] Landman, Bruce M.; Robertson, Aaron; Ramsey theory on the integers. Student Mathematical Library, 24. American Mathematical Society, Providence, RI, 2004. xvi+317 pp. ISBN 0-8218-3199-2

[33] Marker, David; **Model theory. An introduction**. Graduate Texts in Mathematics, 217. Springer-Verlag, New York, 2002. viii+342 pp. ISBN: 0-387-98760-6

[34] Narasimhan, R.; **Analysis on real and complex manifolds**. North-Holland Mathematical Library, 35. North-Holland Publishing Co., Amsterdam, 1985. xiv+246 pp. ISBN: 0-444-87776-2

[35] Schmerl, James H.; Countable partitions of Euclidean space. Math. Proc. Cambridge Philos. Soc. 120 (1996), no. 1, 7–12.

[36] Schmerl, James H.; Countable partitions of the sets of points and lines. Fund. Math. 160 (1999), no. 2, 183–196.

[37] Schmerl, James H.; Coloring $\mathbb{R}^n$. Trans. Amer. Math. Soc. 354 (2002), no. 3, 967–974.

[38] Schmerl, James H.; Partitioning large vector spaces. J. Symbolic Logic 68 (2003), no. 4, 1171–1180.

[39] Schmerl, James H.; How many clouds cover the plane? Fund. Math. 177 (2003), no. 3, 209–211.

[40] Shoenfield, J. R.; Quantifier elimination in fields. Non-classical logics, model theory and computability (Proc. Third Latin-Amer. Sympos. Math. Logic. Campinas, 1976), pp. 243–252. Stud. Logic Found. Math., Vol. 89, North-Holland, Amsterdam, 1977.

[41] Simms, John C.; Sierpiński's theorem. Simon Stevin 65 (1991), no. 1-2, 69–163.

[42] Svetic, R. E.; The Erdos similarity problem: a survey. Real Anal. Exchange 26 (2000/01), no. 2, 525–539.