Sacks forcing collapses $\mathfrak{c}$ to $\mathfrak{b}$

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Abstract. We shall prove that Sacks algebra is nowhere $(\mathfrak{b},\mathfrak{c},\mathfrak{c})$-distributive, which implies that Sacks forcing collapses $\mathfrak{c}$ to $\mathfrak{b}$.

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A. Rosłanowski and S. Shelah recently proved that Sacks forcing $\mathbb{S}$ collapses $\mathfrak{c}$ to $\mathfrak{b}^{+\epsilon}$ [RS]. The aim of the present note is to prove the theorem from the title. Since Rosłanowski and Shelah showed also the consistency of the inequality $\mathfrak{b}^{+\epsilon} > \mathfrak{b}$, our theorem improves that result and answers a question from their paper. To put the things to the right perspective, let us mention first that PFA implies that Sacks forcing does not collapse cardinals at all [A]. Next, it is consistent that MA+$\neg$CH holds (hence $\mathfrak{b} = \mathfrak{c} > \omega_1$) and $\mathfrak{c}$ is still collapsed to ω_1 [JMS, Theorem 2.1]. Hence the question, whether $\mathbb{S}$ collapses $\mathfrak{c}$ below $\mathfrak{b}$ is undecidable.

Let us start with some definitions. A *binary tree* is a subset of $\bigcup_{n\in\omega} {}^n2$ such that $\emptyset \in T$ and whenever $s \in T$ and $n \in \operatorname{dom} s$, then $s \restriction n \in T$. There is a natural partial order of elements of a tree given by $\subseteq$. For a (binary) tree T, a subset $V \subseteq T$ is called a *branch*, if V is a maximal linearly ordered subset of T.

A binary tree T is called *perfect*, if it satisfies the following: For every $s \in T$ there are $q, r \in T$, $q \neq r$ both extending s, i.e., $s \subseteq q$, $s \subseteq r$. Notice that in a perfect tree, all branches are infinite.

A Sacks forcing is a partially ordered set $\mathbb{S}$ of all perfect trees ordered by inclusion. Since every partially ordered set determines uniquely a complete Boolean algebra, we shall use the same symbol $\mathbb{S}$ to denote the complete Boolean algebra, whose dense subset is isomorphic to the set of all perfect trees.

Let us recall a three-parameter distributivity of Boolean algebras. Suppose that $\mathcal{B}$ is a Boolean algebra, κ, λ, μ are cardinal numbers. $\mathcal{B}$ is called to be (κ,λ,μ)-distributive, if for every collection $\{P_\alpha : \alpha \in \kappa\}$ of partitions of $\mathbf{1}_{\mathcal{B}}$ with $|P_\alpha| \leq \lambda$ for all $\alpha \in \kappa$ there is a partition of unity Q such that for every $q \in Q$ and for every $\alpha \in \kappa$, $|\{p \in P_\alpha : q \wedge p \neq \mathbf{0}_{\mathcal{B}}\}| < \mu$. A bit stronger property than just the negation of being (κ,λ,μ)-distributive, is the following. A Boolean algebra $\mathcal{B}$ is (κ,λ,μ)-nowhere distributive, if there is some collection $\{P_\alpha : \alpha \in \kappa\}$ of partitions of $\mathbf{1}_{\mathcal{B}}$ with $|P_\alpha| \leq \lambda$ for all $\alpha \in \kappa$ such that for every non-zero $q \in \mathcal{B}$ there is some $\alpha \in \kappa$ such that $|\{p \in P_\alpha : q \wedge p \neq \mathbf{0}_{\mathcal{B}}\}| \geq \mu$. It is well-known and easy to prove

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that if $\kappa < \mu$ and $\mathcal{B}$ is (κ, μ, μ)-nowhere distributive, then forcing with $\mathcal{B}$ changes the cofinality of μ to κ. If moreover the density of $\mathcal{B}$ does not exceed μ, then forcing with $\mathcal{B}$ collapses μ to κ.

Before stating the Theorem, let us note that the letter $\mathfrak{c}$ stands for the cardinal 2^ω and the cardinal number $\mathfrak{b}$ is defined by $\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \,\&\, \mathcal{F}$ has no upper bound in the order $<$ mod *fin*$\}$.

Theorem. *The Boolean algebra* $\mathbb{S}$ *is* $(\mathfrak{b}, \mathfrak{c}, \mathfrak{c})$*-nowhere distributive.*

To begin the proof of the theorem, we shall introduce some notation and observe several easy facts.

If $n < m$ are integers, we shall denote by $[n, m)$ the set of all integers i satisfying $n \le i < m$. Two infinite sets are called *almost disjoint*, if their intersection is finite.

If $T \in \mathbb{S}$, define a mapping $f_T \in {}^\omega\omega$ by induction as follows. $f_T(0) = 0$. If $f_T(n)$ is known, then $f_T(n+1)$ is the minimal $k \in \omega$ such that for every $s \in T$ with $\operatorname{dom} s = f_T(n)$ there are at least two distinct $r, q \in T$ satisfying $\operatorname{dom} r = \operatorname{dom} q = k$, $s \subseteq r$, $s \subseteq q$.

If T is a binary tree and if $A \subseteq \omega$, we shall denote by $T[A]$ the subtree of T defined by induction on nodes. $\emptyset \in T[A]$; if $s \in T[A]$ and $\operatorname{dom} s = n$, then we distinguish two cases: If $n \in A$, then $r \in T[A]$ for all $r \in T$ with $\operatorname{dom} r = n + 1$ and $r \supseteq s$. If $n \notin A$ and $s^\frown 0 \in T$, then $s^\frown 0 \in T[A]$ but $s^\frown 1 \notin T[A]$; if $s^\frown 0 \notin T$, then $s^\frown 0 \notin T[A]$ and $s^\frown 1 \in T[A]$ only if $s^\frown 1 \in T$. So $s \in T[A]$ branches in $T[A]$ only if $\operatorname{dom} s \in A$ and s branches in T.

The symbols f_T and $T[A]$ will have the meaning just described till the end of the proof. Let us notice without proofs a few observations concerning the notions just introduced.

Fact 1. Let $T \in \mathbb{S}$ and suppose that $A \in [\omega]^\omega$ satisfies $A \supseteq [f_T(n), f_T(n+1))$ for infinitely many $n \in \omega$. Then $T[A] \in \mathbb{S}$.

Fact 2. Let T_0, T_1 be binary trees, A_0, A_1 subsets of ω. Then $T_0[A_0] \cap T_1[A_1] = (T_0 \cap T_1)[A_0 \cap A_1]$.

An immediate consequence of Fact 2 is the next Fact 3. The trivial Fact 4 is mentioned for the sake of completeness.

Fact 3. If $A, B \subseteq \omega$ are almost disjoint, then for arbitrary binary trees T_0, T_1, $T_0[A] \cap T_1[B] \notin \mathbb{S}$.

Fact 4. Let $\{R_n : n \in \omega\}$ be a pairwise disjoint family of finite sets. If $A, B \in [\omega]^\omega$ are almost disjoint, then so are the sets $\bigcup_{n \in A} R_n$ and $\bigcup_{n \in B} R_n$.

Let $\mathcal{R} = \{R_n : n \in \omega\}$ be a partition of ω. We shall denote by $\mathcal{J}^+(\mathcal{R})$ the set of all subsets of ω, which are large if measured by $\mathcal{R}$, precisely, $\mathcal{J}^+(\mathcal{R}) = \{X \subseteq \omega : \limsup_{n\to\infty} |X \cap R_n| = \infty\}$. Two facts are necessary to be mentioned:

Fact 5. Let $X \in [\omega]^\omega$ be arbitrary, let $\mathcal{F} \subseteq {}^\omega\omega$ be a family without an upper bound consisting of strictly increasing functions. Then there is an $f \in \mathcal{F}$ such that $X \in \mathcal{J}^+(\mathcal{R})$ for $\mathcal{R} = \{[f(n), f(n+1)) : n \in \omega\}$.

Indeed, one may write $X = \{x_0 < x_1 < \cdots < x_n < \dots\}$ and put $g(n) = x_{n^2}$. By the assumption, the mapping g does not dominate the family $\mathcal{F}$, so there is some $f \in \mathcal{F}$ with $f(n) \geq g(n)$ for infinitely many integers n. We may assume that $f(0) = 0$. If $K \in \omega$ is arbitrary, find $n > K$ with $g(n) \leq f(n)$. The number of intervals $[f(j), f(j+1))$ covering the interval $[0, f(n))$ is n, but $[0, f(n))$ contains at least n^2 points of X. So $|X \cap [f(j), f(j+1))| \geq n > K$ for some $j < n$. As all sets $[f(n), f(n+1))$ are finite, $\limsup_{n\to\infty} |X \cap [f(n), f(n+1))| = \infty$.

Fact 6. Let $\mathcal{R} = \{R_n : n \in \omega\}$ be a partition of ω. Then there is a family $\mathcal{A} \subseteq [\omega]^\omega$ such that:

(i) $\mathcal{A}$ is almost disjoint;
(ii) every $A \in \mathcal{A}$ is a transversal of $\mathcal{R}$, i.e., $|A \cap R_n| \leq 1$ for each $n \in \omega$;
(iii) for every $X \in \mathcal{J}^+(\mathcal{R})$, the set $\{A \in \mathcal{A} : A \subseteq X\}$ is of size $\mathfrak{c}$.

Fact 6 is a special case of more general Theorem 4.6 from [BS]. This fact is rather nontrivial; we shall not indicate a proof here.

For the proof of the Theorem, fix a family $\mathcal{F} \subseteq {}^\omega\omega$ such that $\mathcal{F}$ has no upper bound, all mappings in $\mathcal{F}$ are strictly increasing, all $f \in \mathcal{F}$ satisfy $f(0) = 0$ and $|\mathcal{F}| = \mathfrak{b}$.

We shall assign to every $T \in \mathbb{S}$ two mappings from $\mathcal{F}$ and a subset of ω: By Fact 5, there is a mapping $h_T \in \mathcal{F}$ such that $\operatorname{rng} f_T \in \mathcal{J}^+(\mathcal{R})$, where $\mathcal{R} = \{[h_T(n), h_T(n+1)) : n \in \omega\}$. Since $\operatorname{rng} f_T \in \mathcal{J}^+(\mathcal{R})$, we conclude that the set X_T defined by $X_T = \{n \in \omega : |[h_T(n), h_T(n+1)) \cap \operatorname{rng} f_T| \geq 2\}$ is infinite. Applying once more Fact 5, we can find the second mapping $g_T \in \mathcal{F}$ such that $X_T \in \mathcal{J}^+(\mathcal{Q})$, where $\mathcal{Q}$ stands now for the partition $\{[g_T(n), g_T(n+1)) : n \in \omega\}$.

In order to prove the Theorem, we need to find the family of partitions witnessing the $(\mathfrak{b}, \mathfrak{c}, \mathfrak{c})$-nowhere distributivity of $\mathbb{S}$. We shall use as an index set the square $\mathcal{F} \times \mathcal{F}$ and, instead of a partition of unity, we shall find only a subset of the desired partition, having the required properties. (It should be clear that this suffices.) For $(h, g) \in \mathcal{F} \times \mathcal{F}$, denote by $\mathbb{S}(h, g)$ the set of all perfect trees $T \in \mathbb{S}$ satisfying $h_T = h, g_T = g$. Consider a partition $\mathcal{R}(g) = \{[g(n), g(n+1)) : n \in \omega\}$. Using Fact 6, there is an almost disjoint family $\mathcal{A}$ satisfying (i), (ii) and (iii). Since $|\mathbb{S}(h, g)| \leq \mathfrak{c}$, one may choose for each $T \in \mathbb{S}(h, g)$ a subset $\mathcal{A}(T) \subseteq \mathcal{A}$ such that for each $A \in \mathcal{A}(T)$, $A \subseteq X_T$, $|\mathcal{A}(T)| = \mathfrak{c}$ and $\mathcal{A}(T) \cap \mathcal{A}(T') = \emptyset$ for $T \neq T'$, $T, T' \in \mathbb{S}(h, g)$.

For $A \in \mathcal{A}$, let $B_A = \bigcup_{n \in A} [h(n), h(n+1))$. The desired disjoint family $P_{(h,g)}$ will be now the set of all $T[B_A]$ for $T \in \mathbb{S}(h, g)$ and $A \in \mathcal{A}(T)$.

By Fact 6 (i), by Fact 4 and by Fact 3, $P_{(h,g)}$ is pairwise disjoint. By Fact 1, all members from $P_{(h,g)}$ are perfect trees. Finally, every tree $T \in \mathbb{S}(h, g)$ contains all $T[B_A]$ for $A \in \mathcal{A}(T)$, so by Fact 6 (iii), T meets $\mathfrak{c}$ many members from $P_{(h,g)}$.

To conclude the proof notice that, by Fact 5, for every perfect tree T there is a pair $(h, g) \in \mathcal{F} \times \mathcal{F}$ with $T \in \mathbb{S}(h, g)$. □

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