

ITERATED PERFECT-SET FORCING

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0. Introduction

In [7] Sacks introduced the notion of forcing with perfect closed sets of real numbers as conditions. This notion has since found wide application, particularly in descriptive set theory. Forcing with perfect sets has become known as *Sacks forcing*, and the associated generic real numbers are called *Sacks reals*.

Many people discovered independently that it is possible to enlarge the continuum by adding an arbitrary number κ of Sacks reals. Conditions are functions with countable support belonging to the product of κ copies of P , the partial ordering of perfect sets of reals. This notion has also proved useful, but it is still unknown whether it shares many of the combinatorial properties of forcing with P alone. This notion of forcing is said to add κ Sacks reals *simultaneously*, or *side-by-side*.

Here we will discuss another way of enlarging the continuum with Sacks reals, namely adding Sacks reals *iteratively*, so that each real is generic over the model obtained by adjoining the ones before it. Details are in Sections 1 to 3.

Our principal results are as follows:

First we show (Theorem 4.5) that if one starts with a model of $\text{ZFC} + 2^{\aleph_0} = \aleph_1$ and adjoins $\aleph_2$ Sacks reals iteratively, then in the resulting model $2^{\aleph_0} = \aleph_2$, every selective ultrafilter is $\aleph_1$ -generated, and every selective ultrafilter in the ground model generates a selective ultrafilter in the extension. Hence in the extension there are only $2^{\aleph_0}$ selective ultrafilters. It is also true (Theorem 4.3) that if $f: \omega \rightarrow 2$ and f is in the extension, then some infinite subset of f lies in the ground model.

Analogous results are unknown for simultaneous Sacks forcing (unless $\kappa < \omega!$).

We also show (Theorem 5.3) that if one forces with P as defined in the model mentioned above, then cardinals are collapsed. Hence it is relatively consistent that perfect-set forcing collapses cardinals. It is not known whether it is relatively consistent that $2^{\aleph_0} > \aleph_1$ and perfect-set forcing preserves cardinals.

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Finally we show (Theorem 6.4) that if κ is weakly compact in the ground model and κ Sacks reals are added iteratively, then in the resulting model κ becomes $\aleph_2$ and there are no ω_2 -Aronszajn trees. The relative consistency of the latter assertion was first proved by Silver in a model of Mitchell [6]. This model is quite different.

We assume that the reader is familiar with the theory of forcing and generic sets (see [4], for example) but not the theory of iterated forcing. In a partial ordering $(P, \leq)$ we write p extends q as $p \geq q$. Thus, for example, a set $D \subseteq P$ is *dense* in P iff $\forall p \in P \exists q \in D q \geq p$.

We consider forcing as taking place over V , the universe of all sets. If P is a partial ordering and G is P -generic, then $V[G]$ denotes the generic extension. Forcing with respect to P is denoted by $\Vdash_P$. We abbreviate $(\forall p) p \Vdash_P \varphi$ to $\Vdash_P \varphi$. For typographical convenience we generally use $\Vdash_\alpha$ instead of $\Vdash_{P_\alpha}$.

If Q is a term such that $\Vdash_P Q$ is a partial ordering, then the *iterated forcing* first by P , then by Q , can be accomplished by a single partial ordering $P \hat{\otimes} Q$. Elements of $P \hat{\otimes} Q$ are pairs $(p, \dot{q})$ such that $p \in P$ and $\Vdash_P \dot{q} \in Q$. We let $(p_1, \dot{q}_1) \geq (p_2, \dot{q}_2)$ iff $p_1 \geq p_2$ and $p_1 \Vdash_P \dot{q}_1 \geq \dot{q}_2$. We identify pairs $(p_1, \dot{q}_1)$ and $(p_2, \dot{q}_2)$ such that $(p_1, \dot{q}_1) \geq (p_2, \dot{q}_2) \geq (p_1, \dot{q}_1)$. More generally, if $\leq$ is a pre-ordering being used for forcing, then we consider $\leq$ as a partial ordering by identifying p and q when $p \geq q \geq p$.

If $a \in V$, then we use a itself as the canonical term of the language of forcing which denotes a . Terms not necessarily denoting specific elements of V usually have dots over the top. For instance, it is quite possible to have $\Vdash_P \dot{a} \in V$ without having $\Vdash_P \dot{a} = a$ for any specific $a \in V$ (although necessarily $(\forall p \in P)(\exists q \geq p)(\exists a \in V) q \Vdash_P \dot{a} = a$). Following this convention, the Q of the preceding paragraph should have been $\dot{Q}$.

We shall assume that the set of terms of the language of forcing is *full* in the sense that if $p \Vdash_P \exists x \varphi(x)$, then there is a term $\dot{x}$ such that $p \Vdash_P \varphi(\dot{x})$. This requires the axiom of choice, which is assumed throughout the paper (see [4]).

Finally, if G is P -generic and $\dot{x}$ is a term of the language of forcing with $P \hat{\otimes} Q$ over V , then $\dot{x}$ canonically determines a term of the language of forcing with Q over $V[G]$. The latter term will also be denoted by $\dot{x}$.

1. The partial ordering

Let $\text{Sq} = \bigcup \{ \langle n : n \in \omega \rangle \}$, the set of finite sequences of zeroes and ones. A set $p \subseteq \text{Sq}$ is *perfect* iff

- (1) $\forall s \in p \forall n s \restriction n \in p$, and
- (2) $\forall s \in p \exists t, u \in p s \subseteq t, u$ and $t \not\subseteq u$ and $u \not\subseteq t$.

We express (2) by saying that p *forks* below each $s \in p$. Note that $p \subseteq \text{Sq}$ is perfect iff $\{f \in {}^\omega 2 : \forall n f \restriction n \in p\}$ is a perfect subset of ${}^\omega 2$ with the product topology.

Let $P = \{p \subseteq \text{Sq} : p \text{ is perfect}\}$. We order P by letting $p \geq q$ (read " p extends q ") iff $p \subseteq q$. Forcing with this partial ordering was introduced by Sacks in [7] and has found many applications.

We will be interested in iterated forcing with P . For ordinals $\alpha > 1$, we define P_α by induction as follows. Let $P_1 = P$. Given P_α , let $P_{\alpha+1} = P_\alpha \hat{\otimes} P$, the canonical partial ordering associated with the extension obtained by forcing first with P_α and then with P as defined in the extension via P_α . (Note that, 'the definition of P is not absolute.) If α is a limit ordinal, let P_α be the inverse limit of $\langle P_\beta : \beta < \alpha \rangle$ if $\text{cf } \alpha = \omega$, and the direct limit otherwise.

As is well-known, P_α may be realized as the set of all functions p such that $\text{domain}(p)$ is a countable subset of α and $\forall \beta \in \text{domain}(p) \ p \restriction_\beta p(\beta) \in P$, where $\restriction_\beta$ denotes forcing with respect to P_β . We let $p \geq q$ iff $\text{domain}(q) \subseteq \text{domain}(p)$ and $\forall \beta \in \text{domain}(q) \ p \restriction_\beta p(\beta) \geq q(\beta)$. We identify p and q if $p \leq q$ and $q \leq p$. We also identify P_1 and P .

The reader should be warned that we will frequently declare that $p \in P_\alpha$ when we have only checked that $\forall \beta \in \text{domain}(p) \ p \restriction_\beta p(\beta) \in P$. The reason for doing this is that if $p \restriction_\beta p(\beta) \in P$, then there is $q(\beta)$ such that $\restriction_\beta q(\beta) \in P$ and $p \restriction_\beta p(\beta) = q(\beta)$, and therefore p could be replaced by q .

If $p, q \in P$ and $m, n \in \omega$, let $(p, m) >_t (q, n)$ iff $p \geq q$, $m > n$, and $(\forall s \in q \cap {}^n 2)(\exists t, u \in p \cap {}^m 2) s \subseteq t, u$ and $t \neq u$. The following is a variant of Lemma 1.4 of [7], frequently called the Fusion Lemma.

Lemma 1.1. *Suppose $\langle (p_i, n_i) : i \in \omega \rangle$ is a sequence such that $p_i \in P$, $n_i \in \omega$ and for each i , $(p_{i+1}, n_{i+1}) >_t (p_i, n_i)$. Then $\bigcap \{p_i : i \in \omega\} \in P$ (and clearly $\bigcap \{p_i : i \in \omega\} \geq p_i$ for all i).*

The proof is left to the reader.

We will need a version of Lemma 1.1 for the partial orderings P_α . If $p, q \in P_\alpha$, $m, n \in \omega$, and F is a finite subset of $\text{domain}(q)$, let $(p, m) >_t (q, n)$ iff $p \geq q$, $m > n$, and $(\forall \beta \in F) \ p \restriction_\beta p(\beta), m > (q(\beta), n)$. Note that if $k \geq m$ and $(p, m) >_t (q, n)$, then $(p, k) >_t (q, n)$.

Lemma 1.2 (Fusion Lemma). *Suppose $\langle (p_i, n_i, F_i) : i \in \omega \rangle$ is a sequence such that $p_i \in P_\alpha$, $n_i \in \omega$, $F_i \subseteq F_{i+1}$, $\bigcup \{F_i : i \in \omega\} = \bigcup \{\text{domain}(p_i) : i \in \omega\}$, and for each i , $(p_{i+1}, n_{i+1}) >_{F_i} (p_i, n_i)$. Define p so that $\text{domain}(p) = \bigcup \{\text{domain}(p_i) : i \in \omega\}$ and $\forall \beta \in \text{domain}(p) \ p(\beta) = \bigcap \{p_i(\beta) : i \in \omega, \beta \in \text{domain}(p_i)\}$. Then $p \in P_\alpha$ and $p \geq p_i$ for each i .*

Proof. To show $p \in P_\alpha$ we will show that $\forall \beta \in \text{domain}(p) \ p \restriction_\beta p(\beta) \in P$. This is an example of the abuse of terminology referred to earlier. But the proof is a trivial induction on β , using Lemma 1.1 at each stage.

A sequence $\langle (p_i, n_i, F_i) : i \in \omega \rangle$ satisfying the hypothesis of Lemma 1.2 will be called a *fusion sequence*, and p will be referred to as the *fusion* of the sequence.

Now if $\beta < \alpha$, let $P_{\beta\alpha} = \{p \in P_\alpha : \text{domain}(p) \subseteq \{\gamma : \beta \leq \gamma < \alpha\}\}$. If $p \in P_\alpha$, then $p^\beta = p - (p \restriction_\beta) \in P_{\beta\alpha}$, and the mapping which carries p into $(p \restriction_\beta, p^\beta)$ is an isomorphism of P_α with a dense subset of $P_\beta \hat{\otimes} P_{\beta\alpha}$, where the ordering on $P_{\beta\alpha}$ in $V[G_\beta]$ is

given by letting $f \Vdash g$ iff $(\exists p \in G_\beta) p \cup f \Vdash p \cup g$. To see that the image is dense, suppose $(p, \dot{f}) \in P_\beta \otimes P_{\beta\alpha}$. Then there must be $p' \Vdash p$ and $f \in P_{\beta\alpha}$ such that $p' \Vdash_\beta \dot{f} = f$. Now let $q \in P_\alpha$ be such that $q \restriction \beta = p'$ and $q^\beta = f$. Then clearly $(q \restriction \beta, q^\beta) \Vdash (p, \dot{f})$. Thus forcing with P_α is the same as forcing with $P_\beta \otimes P_{\beta\alpha}$.

2. Preservation of ω_1

In this section we will show that forcing with P_α does not collapse ω_1 .

Some explanation of our forcing terminology may be in order first. We always consider forcing as taking place over V , the universe of set theory, and we denote by $V[G_\alpha]$ the extension of V via forcing with P_α (of course G_α denotes the canonical generic subset of P_α). The reader uncomfortable with this approach may substitute for V any countable transitive model M of a sufficiently large fragment of ZFC.

If $p \in P$ and $s \in {}^n 2$ for some $n \in \omega$, then let $p_s = \{t \in p : s \subseteq t \text{ or } t \subseteq s\}$. Of course $p_s \in P$ iff $s \in p$.

Lemma 2.1. *Suppose $p \in P$, $n \in \omega$, and $p \Vdash \dot{a} \in V$. Then there are $q \in P$, $m \in \omega$, and finite $x \in V$ such that $(q, m) \Vdash (p, n)$ and $q \Vdash \dot{a} \in x$. In fact, q can be chosen so that for every $s \in {}^n 2 \cap p$ there is $a_s \in V$ such that $q_s \Vdash \dot{a} = a_s$ (so $x = \{a_s : s \in {}^n 2 \cap p\}$).*

Proof. For each $s \in {}^n 2 \cap p$, find $q_s \Vdash p_s$ and $a_s \in V$ such that $q_s \Vdash \dot{a} = a_s$. Let $q = \bigcup \{q_s : s \in {}^n 2 \cap p\}$. Then $a_s = q_s$ for each $s \in {}^n 2 \cap p$. Now choose m large enough so that $(q, m) \Vdash (p, n)$.

In order to generalize Lemma 2.1 to arbitrary P_α (see Lemma 2.3(ii) below) we need an analogue of p_s .

Suppose $p \in P_\alpha$, F is a finite subset of $\text{domain}(p)$, and $\sigma : F \rightarrow {}^n 2$. Then $p \restriction \sigma$ is the function with the same domain as p such that

$$p \restriction \sigma(\beta) = \begin{cases} p(\beta) & \text{if } \beta \notin F, \\ p(\beta)_{\sigma(\beta)} & \text{if } \beta \in F. \end{cases}$$

As with p_s it need not be the case that $p \restriction \sigma \in P_\alpha$. Let us say that σ is *consistent* with p iff $p \restriction \sigma \in P_\alpha$, i.e., iff $\forall \beta \in F (p \restriction \sigma \restriction \beta \Vdash_\beta \sigma(\beta) \in p(\beta))$.

Let us say that p is (F, n) -*determined* iff for any $\sigma : F \rightarrow {}^n 2$, either σ is consistent with p or else there is $\beta \in F$ such that $\sigma \restriction (F \cap \beta)$ is consistent with p and $(p \restriction \sigma) \restriction \beta \Vdash \sigma(\beta) \notin p(\beta)$.

The following lemma collects some easy properties of these notions.

Lemma 2.2. *Suppose $p \in P_\alpha$, F is a finite subset of $\text{domain}(p)$, $n \in \omega$, and $\sigma : F \rightarrow {}^n 2$. Then:*

- (i) *If $\max F < \beta < \alpha$, then $(p \restriction \sigma) \restriction \beta = (p \restriction \beta) \restriction \sigma$.*
- (ii) *If $\alpha = 1$, then p is $(\{0\}, n)$ -determined for every $n \in \omega$.*

- (iii) If $k \geq n$, $G \Vdash F$, $(q, m) \succ_{\varepsilon} (p, k)$ and p is (F, n) -determined, then so is q .
- (iv) If $\max F < \beta < \alpha$, then p is (F, n) -determined iff $p \restriction \beta$ is (F, n) -determined.
- (v) There is $q \in P_\alpha$ such that $q \geq p$ and for some $\sigma: F \rightarrow {}^2$, $\dot{q} = q \restriction \sigma$.
- (vi) If p is (F, n) -determined and $q \geq p$, then there is $\sigma: F \rightarrow {}^2$ such that σ is consistent with p , and q and $p \restriction \sigma$ are compatible.

Proof. The only non-trivial parts are (v) and (vi), and (v) easily implies (vi) so we only prove (v). Suppose the elements of F are $\beta_1, \dots, \beta_k$ in increasing order. By induction on $i \leq k$ we find q_i and $\sigma_i: \{\beta_1, \dots, \beta_i\} \rightarrow {}^2$ so that $p \leq q_0 \leq \dots \leq q_k$ and $q_i \restriction \sigma_i = q_i$. Let $q_0 = p$ and $\sigma_0 = \emptyset$. Given q_i and σ_i , find $q \geq q_i \restriction \beta_{i+1}$ and $s \in {}^2$ so that $q \restriction \beta_{i+1} \leq q_i \restriction \beta_{i+1}$. Then let $\sigma_{i+1} = \sigma_i \cup \{(\beta_{i+1}, s)\}$, and let q_{i+1} be defined by $q_{i+1} \restriction \beta_{i+1} = q_i \restriction \beta_{i+1}$ and $q_{i+1}(\gamma) = q_i(\gamma)$ for $\gamma \succ \beta_{i+1}$.

Lemma 2.3. For each $\alpha \geq 1$, the following hold:

- (i) If $p \in P_\alpha$, $n \in \omega$, $p \restriction_\alpha \dot{a} \in V$, and F is a finite subset of domain (p) , then there exists (q, m) such that $(q, m) \succ_F (p, n)$, q is (F, n) -determined, and $\forall \sigma: F \rightarrow {}^2$, if σ is consistent with q , then there exists a_σ such that $q \restriction \sigma \restriction_\alpha \dot{a} \approx a_\sigma$. Hence if $x = \{a_\sigma: \sigma \text{ is consistent with } q\}$, then $q \restriction_\alpha \dot{a} \in x$.
- (ii) If $p \in P_\alpha$, $n \in \omega$, $p \restriction_\alpha \dot{f}: \omega \rightarrow V$, and F is a finite subset of domain (p) , then there exists (q, m) and a sequence $\langle x_i: i \in \omega \rangle$ of finite sets such that $(q, m) \succ_1 (p, n)$ and $q \restriction_\alpha \forall i f(i) \in x_i$.
- (iii) If $p \in P_\alpha$, $n \in \omega$, $p \restriction_\alpha \dot{a}$ is a countable subset of V , and F is a finite subset of domain (p) , then there exists (q, m) and a countable set A such that $(q, m) \succ_1 (p, n)$ and $q \restriction_\alpha \dot{a} \subseteq A$.
- (iv) If $p \in P_\alpha$, $n \in \omega$, $\alpha < \gamma$, $p \restriction_\alpha \dot{f} \in P_{\alpha, \gamma}$, and F is a finite subset of domain (p) , then there exist (q, m) and $f \in P_{\alpha, \gamma}$ such that $(q, m) \succ_F (p, n)$ and $q \restriction_\alpha \dot{f} = f$.

Proof. The proof is by induction on α .

- (i) For $\alpha = 1$ this is Lemma 2.1 (and Lemma 2.2(ii)). Assume $\alpha > 1$.

Suppose $\alpha = \beta + 1$. Without loss of generality we may assume $\beta \in F$. By Lemma 2.1 applied in $V[G_\beta]$, we see that there are terms $\dot{q}$, $\dot{m}$, and $\dot{a}_s$ for each $s \in {}^2$ such that $p \restriction \beta \restriction_\beta \dot{\varphi}$, where φ is the following assertion:

$$''(\dot{q}, \dot{m}) \succ (p(\beta), n), \dot{a}_s \in V, \text{ and } \forall s \in p(\beta) \cap {}^2 \dot{q}_s \restriction \dot{a} = \dot{a}_s''.$$

Now, by inductive hypothesis applied to P_β , we can find $(q', m') \succ_{F \restriction \beta} (p \restriction \beta, n)$ such that q' is $(F \restriction \beta, n)$ -determined and for each $\sigma: F \restriction \beta \rightarrow {}^2$, if σ is consistent with q' , then there are x_σ , m_σ , and $\{a_{\sigma \restriction \gamma}: \gamma \in s, \gamma \in V\}$ such that

$$q' \restriction \sigma \restriction \dot{q} \cap {}^2 \approx x_\sigma, \quad \dot{m} = m_\sigma, \quad \text{and} \quad \forall s \in {}^2 \dot{a}_s \approx a_{\sigma \restriction \gamma}.$$

Here $\sigma \restriction \gamma$ denotes the function $\tau: F \rightarrow {}^2$ such that $\tau \restriction \beta = \sigma$ and $\tau(\beta) = \gamma$.

Now define $q \in P_\alpha$ by $q \restriction \beta = q'$ and $q(\beta) = \dot{q}$, and let $m \in \omega$ be large enough so that $m \geq m'$ and $m \geq m_\sigma$ for all σ consistent with q' . But then $(q, m) \succ_1 (p, n)$, q is (F, n) -determined, and for any $\sigma: F \rightarrow {}^2$, if σ is consistent with q , then $q \restriction \sigma \restriction_\alpha \dot{a} \approx a_\sigma$. This completes the proof if α is a successor ordinal.

Assume α is a limit ordinal. Choose β so that $\max(F) < \beta < \alpha$. We use the symbol $\Vdash^*$ to denote forcing over $V[G_\beta]$ with respect to $P_{\beta\alpha}$. Using the isomorphism between P_α and the dense subset of $P_\beta \hat{\otimes} P_{\beta\alpha}$ constructed earlier, we see that

$$p \mid \beta \Vdash_\beta \neg(\exists f \in P_{\beta\alpha})(\exists b \in V) f \Vdash p^\beta \text{ and } f \Vdash^* \dot{a} = b''.$$

It follows that there are terms $\dot{f}$ and $\dot{b}$ of the language of forcing with respect to P_β such that

$$p \mid \beta \Vdash_\beta \neg \dot{f} \Vdash p^\beta, \dot{b} \in V, \text{ and } \dot{f} \Vdash^* \dot{a} = \dot{b}''.$$

Applying the inductive hypothesis to P_β we find $(q_1, m) >_F (p \mid \beta, n)$ such that q_1 is (F, n) -determined and for every $\sigma : F \rightarrow {}^n 2$, if σ is consistent with q_1 , then there is $b_\sigma \in V$ such that $q_1 \mid \sigma \Vdash_\beta \dot{b} = b_\sigma$.

Now, applying part (iv) of the lemma to P_β , we find $f \in P_{\beta\alpha}$ and $(q_2, k) >_F (q_1, m)$ such that $q_2 \Vdash_\beta \dot{f} = f$. But then $(q_2, k) >_1 (p \mid \beta, n)$, so if we let $q = q_2 \cup f$, then $(q, k) >_F (p, n)$, q is (F, n) -determined (by Lemma 2.2(iii)) and for any $\sigma : F \rightarrow {}^n 2$, if σ is consistent with q , then $q \mid \sigma \Vdash \dot{a} = b_\sigma$.

The final sentence of (i) follows from Lemma 2.2(vi).

(ii) Using part (i) repeatedly, it is easy to construct a fusion sequence $\langle (p_i, n_i, F_i) : i \in \omega \rangle$, and a sequence $\langle x_n : n \in \omega \rangle$ so that $p_0 = p$, $n_0 = n$, $F_0 = F$ and $\forall i p_{i+1} \Vdash_\alpha \dot{f}(i) \in x_i$. If q is the fusion of the sequence, then by Lemma 1.2, $q \Vdash_\alpha \forall i \dot{f}(i) \in x_i$.

(iii) is a trivial consequence of (ii).

(iv) By part (iii) there is $(q, m) >_F (p, n)$ and a countable set A such that $q \Vdash_\alpha \text{domain}(\dot{f}) \subseteq A$. Now define $f \in P_{\alpha\gamma}$ so that $\forall \beta \in A$, $f(\beta)$ is a term denoting the same object denoted by $\dot{f}(\beta)$ if $\beta \in \text{domain}(\dot{f})$ (in $V[G_\alpha]$), and otherwise $\dot{f}(\beta) = 0$. Then clearly $q \Vdash_\alpha \dot{f} \Vdash f \Vdash \dot{f}$ so $q \Vdash_\alpha \dot{f} = f$ by our convention for identifying equivalent objects.

Theorem 2.4. *Forcing with P_α does not collapse ω_1 .*

Proof. By Lemma 2.3(ii) or (iii).

Remark. Lemma 2.3 is all we will need for the rest of the paper, but it should be remarked that for many applications of perfect-set forcing a somewhat stronger version of part (ii) is required, namely: if $p \Vdash_\alpha \dot{f} : \omega \rightarrow V$, and if $g \in {}^\omega \omega$ is eventually arbitrarily large (i.e., $\forall m \exists n \forall k \geq n \ g(k) > m$) and $\forall n \ g(n) > 0$, then $\exists q \geq p \exists \langle z_n : n \in \omega \rangle q \Vdash_\alpha \forall n \dot{f} \mid n \in z_n$, and $\forall n \mid z_n \leq g(n)$. The proof is rather involved, since one must ensure that each condition “forks very slowly”, but the reader familiar with perfect-set forcing should have little difficulty reconstructing it.

Lemma 2.3 also allows us to conclude that forcing with the P_α is iterated forcing in the full sense of the word.

Theorem 2.5. *Let $\dot{P}_\beta$ denote the result of defining P_β in $V[G_\alpha]$. Then for any $\alpha, \beta \geq 1$, $\Vdash_\alpha P_{\alpha, \alpha+\beta}$ is isomorphic to $\dot{P}_\beta$.*

Proof. The proof is by induction on β . It is easy to see that $\Vdash_\alpha P_{\omega, \alpha, \beta} \subseteq \{\tilde{f} : \exists f \in \dot{P}_\beta \forall \gamma \in \text{domain}(f) \tilde{f}(\alpha + \gamma) = f(\gamma)\}$. The reverse inclusion follows from Lemma 2.3(iv).

3. The $\aleph_2$ -chain condition

In this section we show that $2^{\aleph_0} = \aleph_1$ implies that for each $\alpha \leq \omega_2$, P_α has the $\aleph_2$ -chain condition, and hence preserves cardinals. This result is sharp, for it is shown in Section 5 that P_{ω_1+1} collapses ω_2 onto ω_1 (if $2^{\aleph_0} = \aleph_1$). We also remark that if $2^{\aleph_0} = \aleph_1$, then $\Vdash_{\omega_2} 2^{\aleph_0} = \aleph_2$.

Lemma 3.1. *For each $\alpha < \omega_2$, there is a dense set $W_\alpha \subseteq P_\alpha$ such that $|W_\alpha| = 2^{\aleph_0}$.*

Proof. Let $W_\alpha = \{p \in P_\alpha : \forall n \in \omega \quad \forall \beta \in \text{domain}(p) \quad \exists m \geq n \exists \text{ finite } F \subseteq \text{domain}(p) \text{ } p \text{ is } (F, m)\text{-determined and } \beta \in F\}$.

First we show W_α is dense in P_α . Let $p \in P_\alpha$ be arbitrary. Using Lemma 2.3(i) repeatedly, it is easy to find a fusion sequence $\langle (p_i, n_i, F_i) : i \in \omega \rangle$ such that $p_0 = p$, F_0 and n_0 are arbitrary, and for each i , p_{i+1} is (F_i, n_i) -determined. Let q be the fusion of the sequence. We claim $q \in W_\alpha$. Fix $n \in \omega$ and $\beta \in \text{domain}(q)$. Choose i large enough so that $n_i \geq n$ and $\beta \in F_i$. Then since p_{i+1} is (F_i, n_i) -determined it follows that q is (F_i, n_i) -determined (by Lemma 2.2(iii)).

Now we show by induction on $\alpha < \omega_2$ that $|W_\alpha| = 2^{\aleph_0}$. Clearly $W_1 = P_1$ (by Lemma 2.2(ii)) so $|W_1| = 2^{\aleph_0}$. Note that if $p \in W_\alpha$ and $\beta < \alpha$, then $p \restriction \beta \in W_\beta$. Hence $|W_\alpha| = 2^{\aleph_0}$ if α is a limit ordinal. Suppose $\alpha = \beta + 1$. If $p \in W_\alpha$, then $\forall n \exists m_n \geq n \exists \text{ finite } F_n \subseteq \text{domain}(p \restriction \beta) \forall \sigma : F_n \rightarrow {}^{m_n}2$ if σ is consistent with $p \restriction \beta$, then $\exists x_n \in V(p \restriction \beta) \restriction \sigma \restriction_{\beta} p(\beta) \cap {}^{m_n}2 = x_n$. But then clearly p is completely determined by $p \restriction \beta$, the sequence $\langle (m_n, F_n) : n \in \omega \rangle$, and the correspondence that sends σ to x_n . Hence $|W_\alpha| = 2^{\aleph_0}$.

Theorem 3.2. *Assume $2^{\aleph_0} = \aleph_1$. Then for every $\alpha \leq \omega_2$, P_α has the $\aleph_2$ -chain condition, and therefore preserves cardinals.*

Proof. For $\alpha < \omega_2$ this follows from Lemma 3.1, since P_α has a dense subset W_α of cardinality $\aleph_1$. For $\alpha = \omega_2$ this follows from an easy Δ -system argument.

Theorem 3.3. (a) *If $f \in {}^\omega 2 \cap V[G_\alpha]$ and cf $\alpha \geq \omega_1$, then $\exists \beta < \alpha \ f \in V[G_\beta]$.*

(b) *Assume $2^{\aleph_0} = \aleph_1$. Then $\Vdash_{\omega_2} 2^{\aleph_0} = \aleph_2$.*

Proof. Suppose $\Vdash_\alpha \dot{f} : \omega \rightarrow 2$. As in the proof of Lemma 3.1, there is $q \geq p$ and a sequence $\langle (F_n, m_n) : n \in \omega \rangle$ such that $\forall n \forall \sigma : F_n \rightarrow {}^{m_n}2$ if σ is consistent with q , then for some $x_n, q \restriction \sigma \restriction_{\beta} \dot{f} \restriction n = x_n$. Then $\dot{f}$ is completely determined by q , $\langle (F_n, m_n) : n \in \omega \rangle$, and the mapping carrying each σ to x_n , provided that $q \in G_\alpha$.

If $q \in G_\alpha$ and cf $\alpha \geq \omega_1$, then $\exists \beta < \alpha \ q \in G_\beta$. This proves (a).

To prove (b) simply note that q may be chosen to be in W_α for some $\alpha < \omega_2$.

4. Selective ideals and ultrafilters

If I is an ideal of subsets of ω , then let $I' = \{A \in \omega : A \notin I\}$. An ideal I is *selective* iff $(\forall A \in I')(\forall f: A \rightarrow \omega)$ f is one-to-one or constant on a set in I' . An ultrafilter on ω is *selective* (or *Ramsey*) iff its dual ideal is selective. (Note: Some authors use the term "weakly selective" instead of "selective"; our terminology is that of Grigorieff [3]. Throughout this section we consider only *non-trivial* ideals, i.e., ideals containing all finite subsets of ω .)

The object of this section is to show that under forcing with respect to the partial orderings P_α , selective ideals remain, in some sense, selective ideals. A precise statement is in Theorem 4.2.

Examples of selective ideals are:

- (a) the dual ideal to a selective ultrafilter,
- (b) the ideal of finite subsets of ω , and
- (c) the ideal generated by a maximal (infinite) collection of almost-disjoint subsets of ω .

It should be remarked that selective ultrafilters can be proved to exist only under special set-theoretic assumptions, such as the continuum hypothesis or Martin's Axiom (see [1] or [4]). For a model with no selective ultrafilters, see Kunen [5].

It will be convenient to observe that I is a selective ideal iff

- (1) $\forall A \in I, \forall B \subseteq A, B \in I$,
- (2) $\forall A \in I, \forall$ finite $X \subseteq \omega, A \cup X \in I$, and
- (3) $\forall A \in I', \forall f: A \rightarrow \omega$, if $\forall n \{a \in A : f(a) \leq n\} \in I$, then $\exists B \in I'$ f is one-to-one on B .

The proof is trivial except for the fact that any set I satisfying (1)–(3) is closed under finite unions. If $A, B \in I$ but $A \cup B \notin I$, then choose $f: A \cup B \rightarrow \omega$ such that $f^{-1}\{0\} = A$ and f is one-to-one on $B - A$, and apply (3) for a contradiction.

Another characterization of selective ideals, to be found in Grigorieff [3], is the following:

A set $T \subseteq \bigcup \{\omega : n \in \omega\}$ is a *tree* iff $\forall s \in T \forall n \in \omega \{s \restriction n \in T\}$. For each $s \in \omega$ and each $m \in \omega$, let $s \smallfrown (m)$ denote the concatenation of s and m . If T is a tree and $s \in T$, let $T_s = \{m : s \smallfrown (m) \in T\}$. If I is an ideal and T is a tree, then T is a *strong I-tree* iff every finite intersection of sets in $\{T_s : s \in T\}$ lies in I' . A function $f \in \omega$ is a *'branch* through a tree T iff $\forall n f \restriction n \in T$.

Then I is a selective ideal iff every strong I -tree has a branch f with range $(f) \in I'$.

If X is a collection of subsets of ω , then the *upward closure* of X is $\{A \subseteq \omega : \exists B \in X B \subseteq A\}$.

Lemma 4.1. Suppose I is a selective ideal. Then

$\Vdash$ "the upward closure of I' is the complement of a selective ideal",
where " $\Vdash$ " refers to forcing with P .