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M776 Model Theory
April 18,1988

Theorem 1 *(Morley) Suppose T is a theory in a countable language L with built in Skolem functions and $\langle \Sigma_n \ : \ n \in \omega \rangle$ is a countable family of types such that for every $\alpha < \omega_1$ T has a model $\mathcal{A}$ omitting each Σ_n and $|A| = \beth_\alpha$. Then T has a model with an infinite set of indiscernibles omitting each Σ_n.*

Hence by the expanding technique of indiscernibles T has arbitrarily large models omitting each Σ_n and so the Hanf number of omitting types is $\beth_{\omega_1}$.

Lemma 2 *Suppose N is a model of ZFC such that ω^N is standard but ω_1^N contains an infinite descending sequence. Suppose that T and the Σ_n are in N and N models:*

> *"T is a theory in a countable language L with built in Skolem functions and $< \Sigma_n \ : \ n \in \omega >$ is a countable family of types such that for every $\alpha < \omega_1$ T has a model $\mathcal{A}$ omitting each Σ_n and $|A| = \beth_\alpha$."*

Then T has a model with an infinite set of indiscernibles omitting each Σ_n.

proof: For simplicity assume we just have a single 1-type, $\Sigma(x)$. Let α_n for $n \in \omega$ be such that $N \models$ "$\omega_1 > \alpha_n > \alpha_{n+1} + \omega$". Hence for every $n \in \omega$ and $k \in \omega$

$$N \models \beth_{\alpha_n} \to (\beth_{\alpha_{n+1}})^k_\omega$$

Let $N \models$ "$\mathcal{A}$ models T, omits $\Sigma(x)$, $|A| = \beth_{\alpha_0}$". Build a sequence X_n of elements of N such that $N \models$ "$X_{n+1} \subset X_n \subset A$; $|X_n| = \beth_{\alpha_n}$" and also for every L-formula $\theta(\vec{x})$ there exists $n \in \omega$ such that

$$N \models \forall \vec{a} \in [X_n]^{<\omega} \mathcal{A} \models \theta(\vec{a})$$

or

$$N \models \forall \vec{a} \in [X_n]^{<\omega} \mathcal{A} \models \neg\theta(\vec{a})$$

and for every L-term $\tau(\vec{x})$ there exists $n \in \omega$ and $\theta_\tau(x) \in \Sigma$ such that

$$N \models \forall \vec{a} \in [X_n]^{\omega} \mathcal{A} \models \neg\theta_\tau(\tau(\vec{a}))$$

Let $I = \{c_n \ : \ n \in \omega\}$ and consider the theory:

$$T \cup \{\theta(\vec{a}) \iff \theta(\vec{b}) \ : \ \vec{a}, \vec{b} \in I^{<\omega}\} \cup \{\neg\theta_\tau(\tau(\vec{a})) \ : \ \vec{a} \in I^{<\omega}\}$$

Each finite subset of this is consistent and it gives an E-M type which does the job.

Lemma 3 *Suppose M is a countable model of ZFC. Then there exists a model N such that $M \preceq N$ and*

$$\{n \in M \ : \ M \models n \in \omega\} = \{n \in N \ : \ N \models n \in \omega\}$$

but ω_1^N contains an infinite descending sequence.

proof: Let

$$T = D_{elem}(M) \cup \{c_0 < \omega_1^M, c_{n+1} < c_n \ : \ n \in \omega\} \cup \{c_n > \alpha \ : \ \alpha < \omega_1^M, n \in \omega\}$$

and let

$$\Sigma(x) = \{x < \omega^M\} \cup \{x \neq n \ : \ n < \omega^M\}$$

By the omitting types theorem it is enough to show that Σ is not isolated over T.

Claim For any sentence $\theta(\vec{c})$

$$T \cup \{\theta(c_0, c_1, \ldots, c_n)\} \text{ is consistent}$$

iff

M models:

$$\forall\alpha < \omega_1 \exists x_0, x_1, \ldots, x_n[(\alpha < x_n < \ldots < x_1 < x_0 < \omega_1) \wedge (\theta(x_0, x_1, \ldots, x_n))]$$

The proof is left to reader. Abbreviate the last statement:

$$\exists^{cof\omega_1}\vec{x}\ \theta(\vec{x})$$

Suppose that $\psi(x, \vec{c})$ isolates the type $\Sigma(x)$. Hence $T \vdash \psi(x, \vec{c}) \to x < \omega$. By the claim

$$M \models \exists^{cof\omega_1}\vec{y}\ \exists x < \omega\ \psi(x, \vec{y})$$

Since ω_1 is uncountable $\exists n \in M$ such that

$$M \models n < \omega \wedge \exists^{cof\omega_1} \vec{y}\, \psi(n, \vec{y})$$

But then $T \cup \{\psi(n, \vec{c})\}$ is consistent, contradiction.

To prove Morley's theorem, first note that both lemmas are true for any sufficiently nice fragment of ZFC. For example let ZFC* be the theory of $(V_\kappa, \in)$ where κ is any regular cardinal greater than $\beth_{\omega_1}$. So given any T a theory in a countable language L with built in Skolem functions and $< \Sigma_n \ : \ n \in \omega >$ a countable family of types such that for every $\alpha < \omega_1 \ \ T$ has a model $\mathcal{A}$ omitting each Σ_n and $|A| = \beth_\alpha$, use the Lowenheim-Skolem theorem to find a countable model $M \preceq V_\kappa$ which contains T and $< \Sigma_n \ : \ n \in \omega >$. Apply Lemma 3 to get a model N and then apply Lemma 2 to get the conclusion of the theorem for T and $< \Sigma_n \ : \ n \in \omega >$.