Spring 2009

Homework problems are due in class one week from the day assigned (which is in parentheses).

Theorem *(Ehrenfeucht-Fräisse 1960 [8]). If M and N are $\mathcal{L}$-structures and $M \equiv_n N$, then M and N model the same $\mathcal{L}$-sentences of quantifier depth $\leq n$.*

Problem 1. (1-21 W)

For structures M and N in the language of pure equality, prove that $M \equiv_n N$ iff $||M|| = ||N||$ or ($||M|| \geq n$ and $||N|| \geq n$).

Problem 2. (1-21 W)

Let M be an equivalence relation with exactly one equivalence class of size n for each positive integer n and no infinite classes. Let N be the same, except in addition it has one infinite equivalence class. Use Ehrenfeucht games to prove that $M \equiv N$.

Theorem *(Ehrenfeucht-Fräisse 1960). If $\mathcal{L}$ is a finite language which contains only predicate symbols and constant symbols, then for every $n \in \omega$ there exist a finite set F_n of $\mathcal{L}$-sentences each with quantifier depth $\leq n$ such that for any two $\mathcal{L}$-structures M and N, if ($M \models \theta$ iff $N \models \theta$) for every θ in F_n, then $M \equiv_n N$.*

Problem 3. (1-26 M)

Let L_n be a linear order of size n and $L_\infty = \omega + \omega^*$ where ω^* is the order type of the negative integers.

(a) Prove that for every $n < \omega$ there is an $N < \omega$ such that $L_k \equiv_n L_\infty$ for all $k > N$.

(b) Use the part (a) to prove that the linear orders ω and $\omega+\omega^*+\omega = \omega+\mathbb{Z}$ are elementarily equivalent.

(c) Use part (b) to prove that (ω, S) and $(\omega + \mathbb{Z}, S)$ are elementarily equivalent where S is the successor operation, $S(x) = x + 1$.

Theorem *(Cantor 1880) If M and N are countable $\mathcal{L}$-structures, then $M \simeq N$ iff $M \equiv_\infty N$.*

Problem 4. (1-26 M)

Let T be a $\mathcal{L}$-theory such that T has no finite models and $\mathcal{L}$ is countable. Prove that the following are equivalent:

1. T is ω-categorical
2. $M \equiv_\infty N$ for every pair of models M and N of T.

Hint: You may use without proof a consequence of the Ryll-Nardzewski Theorem, namely if M is a model of T and $a_1, \ldots, a_n$ a tuple from $|M|$, then

$$Th(M, a_1, \ldots, a_n)$$

is ω-categorical. You may also use the Downward-Lowenheim-Skolem-Tarski Theorem.

Theorem *(Carol Karp 1965). Given $\mathcal{L}$-structures M and N the following are equivalent:*

1. *M and N satisfy the same $\mathcal{L}_{\infty,\omega}$ sentences*
2. *$M \equiv_\infty N$*

Problem 5. (1-28 W)
Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures.
Prove that $\mathcal{K}$ is EC iff both $\mathcal{K}$ and $\overline{\mathcal{K}}$ are EC_Δ.

Theorem *(Łos-Tarski 1955) A first-order theory T is $\forall$-axiomatizable iff the models of T are closed under taking substructures.*

Corollary *The class of models of a sentence θ is closed under taking substructures iff θ is logically equivalent to a $\forall$-sentence.*

Corollary *The class of models of a sentence θ is closed under taking superstructures iff θ is logically equivalent to a $\exists$-sentence.*

Problem 6. (2-02 M)
Show that if a first-order theory T is preserved by taking superstructures, then it can be axiomatized by existential sentences, i.e. $\exists$-sentences.
Hint: Suppose $M \models (\exists - sent) \cap T$. Prove that $Th_\forall(M) \cup T$ is consistent.

Theorem *(Elementary Chain Lemma Tarski-Vaught 1957) If*

$$M_0 \preceq M_1 \preceq M_2 \preceq M_3 \preceq \cdots$$

is a chain of elementary substructures and

$$N = \bigcup_{n<\omega} M_n$$

then $M_k \preceq N$ *for all* $k < \omega$.

Theorem *(Chang-Łos-Suszko 1959) A first-order theory* T *is axiomatizable by* $\forall\exists$*-sentences iff the models of* T *are closed under chains of substructures.*

Problem 7. (2-04 W)

(Directed Unions) Suppose $\mathcal{D}$ is a directed set of $\mathcal{L}$-structures and $M_\infty = \bigcup \mathcal{D}$. Prove:

(a) Every $\forall\exists$-sentence which is true in every $M \in \mathcal{D}$, is true in M_∞.

(b) If for every $M \subseteq N$ in $\mathcal{D}$ we have $M \preceq N$, then $M \preceq M_\infty$ for every $M \in \mathcal{D}$.

Problem 8. (2-04 W)

(Direct Limits). Let $\mathbb{P} = (\mathbb{P}, \leq)$ be a poset (partially ordered set), $(M_p : p \in \mathbb{P})$ a family of $\mathcal{L}$-structures, and $j_{pq} : M_p \to M_q$ be maps for each $p \leq q$ in $\mathbb{P}$. State the appropriate conditions on $\mathbb{P}$, these structures, and these maps, so as to naturally generalize problem above (a) and (b).

Problem 9. (2-06 F)

Show that $T = Th(\mathbb{Q}, \leq, S)$ where S is the successor function is finitely axiomatizable. Warning: it is not categorical in any power.

Theorem *(Lowenheim 1915) If* T *is a theory in countable language and has a model, then it has a countable model.*

Theorem *(Lowenheim-Skolem) If* T *is an* $\mathcal{L}$*-theory which has an infinite model, then* T *has models of all cardinality* $\kappa \geq |\mathcal{L}| + \omega$.

Theorem *(Upward-Downward Lowenheim-Skolem-Tarski 1950s) See*
`http://www.math.wisc.edu/~miller/old/m776-97/preq.pdf`

Theorem*(Łos-Vaught Test 1954) If* T *is an* $\mathcal{L}$*-theory which has no finite models and is* κ*-categorical for some* $\kappa \geq |\mathcal{L}| + \omega$*, then* T *is complete.*

Theorem *(McKinsey 1943) A first-order theory T is axiomatizable by universal Horn sentences iff the class of models of T is closed under substructure and products.*

Problem 10. (2-09 M)
Prove that the class of well-orderings is not PC_Δ but its complement is.

Theorem *(Keisler Sandwich 1960) An $\mathcal{L}$-theory T is $\exists\forall$-axiomatizable iff for any $\mathcal{L}$-structures $M_1 \subseteq M_2 \subseteq M_3$ with $M_1 \preceq M_3$, if $M_1 \models T$, then $M_2 \models T$.*

Problem 11. (2-11 W)
Let M_1 and M_2 be $\mathcal{L}$-structures. Prove that $M_1 \equiv M_2$ iff there are $\mathcal{L}$-structures N_1 and N_2 such that $M_1 \preceq N_1$, $M_2 \preceq N_2$, and $N_1 \simeq N_2$.

Theorem *(Lyndon 1959) A first-order theory T is axiomatizable by positive sentences iff the class of models of T is closed under homomorphic images.*
Key Lemma. *Suppose $B \models Th_{POS}(A)$ then*
(a) there exists $B' \succeq B$ and $f : A \to B'$ such that

$$(B, f(a))_{a\in|A|} \models Th_{POS}(A, a)_{a\in|A|}$$

(b) there exists $A' \succeq A$ and $g : B \to A'$ such that

$$(B, b)_{b\in|B|} \models Th_{POS}(A', g(b))_{b\in|B|}$$

Problem 12. (2-13 F)
(a) Prove that

$$B \models Th_{POS}(A) \text{ iff } A \models Th_{\neg POS}(B)$$

(b) Find A and B such that $B \models Th_{POS}(A)$ but $A \not\models Th_{POS}(B)$.

Problem 13. (2-13 F)
Prove Key Lemma part (b).

Theorem *(Craig's Interpolation Lemma 1957) Suppose θ_1 is an $\mathcal{L}_1$-sentence, θ_2 is an $\mathcal{L}_2$-sentence, and $\vdash \theta_1 \to \theta_2$. Then there exists ρ an $\mathcal{L}_1 \cap \mathcal{L}_2$-sentence such that $\vdash \theta_1 \to \rho$ and $\vdash \rho \to \theta_2$.*

Problem 14. (2-16 M)

(Prove) Suppose T_0 is a complete $\mathcal{L}_0$-theory, T_1 is a complete $\mathcal{L}_1$-theory, and $\mathcal{L} = \mathcal{L}_0 \cap \mathcal{L}_1$. Then:

$T_0 \cup T_1$ is consistent iff $(T_0 \cap (\mathcal{L} - sent)) \cup (T_1 \cap (\mathcal{L} - sent))$ is consistent.

Problem 15. (2-16 M)

(Millar) (Disprove) Suppose T_i (for $i = 1, 2, 3$) is a complete consistent $\mathcal{L}_i$-theory. Then:

$T_1 \cup T_2 \cup T_3$ is consistent iff $T_i \cup T_j$ is consistent for all i and j.

Problem 16. (2-18 W)

Suppose that M is an infinite $\mathcal{L}$-structure, $\leq$ is a binary relation symbol in $\mathcal{L}$, and $\leq^M$ is a linear order with no greatest element. Prove there exists $N \succeq M$ with $||N|| \leq \omega_1 + |\mathcal{L}| + ||M||$ and the cofinality of $\leq^N$ is ω_1.

Theorem *(Beth Definability) With respect to theories, implicitely definable implies explicitely definable.*

Theorem *(Addison 1960 [1]) Let $\mathcal{L}$ be a language containing at least one constant symbol. Suppose θ_0 is a universal $\mathcal{L}$-sentence and θ_1 an existential $\mathcal{L}$-sentence such that $\vdash \theta_0 \rightarrow \theta_1$. Then there exists a quantifier free $\mathcal{L}$-sentence ρ such that $\vdash \theta_0 \rightarrow \rho$ and $\vdash \rho \rightarrow \theta_1$.*

Problem 17. (2-20 F)

Suppose $\mathcal{L}$ is language containing at least one relation or operation symbol but no constant symbols. Show there exists θ_0 a universal $\mathcal{L}$-sentence and θ_1 an existential $\mathcal{L}$-sentence such that

1. $\theta_0 \rightarrow \theta_1$ is a logical validity,
2. θ_1 is not a logical validity, and
3. $\neg\theta_0$ is not a logical validity.

Show that there is no such pair of sentences in the language of pure equality.

Theorem *(Shoenfield 1960 in [2], [16] p. 97) Suppose θ_0 is a $\forall\exists$-$\mathcal{L}$-sentence and θ_1 is an $\exists\forall$-$\mathcal{L}$-sentence such that $\vdash \theta_0 \rightarrow \theta_1$. Then there exists an $\mathcal{L}$-sentence ρ which is a boolean combination of existential and universal sentences such that $\vdash \theta_0 \rightarrow \rho$ and $\vdash \rho \rightarrow \theta_1$. (Similar result holds for higher prenex classes.)*

Problem 18. (2-23 M)

Let R be a binary relation symbol. Note that

$$\exists x \forall y\ R(x,y) \rightarrow \forall y \exists x\ R(x,y)$$

Prove that there does not exist a sentence ρ which is a boolean combination of existential and universal sentences and interpolates between them.

Hint: Consider R-structures in which every finite R-structure embedds.

Theorem *(Rabin 1959 [15], [6] p. 136.) There is a complete theory T in a language of size continuum, which is categorical in power ω and has no model of size κ with $\omega < \kappa < |2^\omega|$.*

Problem 19. (2-25 W)

Give an example of a theory T with arbitrarily large finite models but no model of cardinality κ with $\omega \leq \kappa < |2^\omega|$.

Problem 20. (2-25 W)

Suppose that the continuum $|2^\omega|$ is larger than $\aleph_\omega$. Prove that for every $A \subseteq \omega$ there is a first order theory T_A such that for every $n < \omega$

T_A has a model of cardinality ω_n iff $n \in A$.

Open Question. Can we find T_A which is complete?

Theorem *(Łos 1955) For any $f_1, \ldots, f_n \in \prod_i A_i$ and formula θ*
$\prod_i A_i/\mathcal{U} \models \theta([f_1], \ldots, [f_n])$ *iff* $\{i \in I : A_i \models \theta(f_1(i), \ldots, f_n(i))\} \in \mathcal{U}$.

Problem 21. (2-27 F)

Prove Łos's Theorem for ultraproducts $\prod_{n\in\omega} A_n/\mathcal{U}$ and the language $\mathcal{L}(Q_{c^+})$ where $Q_{c^+}x$ is the quantifier which means "There are more than continuum many x such that".

Theorem *(Keisler unpublished see [7] p.472) If T is a first-order theory with a model of size $\kappa \geq \omega$, then for every $\lambda \geq \kappa^\omega$ T has a model of size λ.*

Theorem *(Keisler 1959) (CH) If $A \equiv B$ are countable and $\mathcal{U}$ is a nonprincipal ultrafilter on ω, then $A^\omega/\mathcal{U} \simeq B^\omega/\mathcal{U}$.*

Theorem *(Morley, Vaught 1962) If A and B are κ-saturated models of size κ, then $A \simeq B$.*

Problem 22. (3-04 W)

Suppose A is an $\mathcal{L}$ structure and $\leq$ is a binary relation symbol in $\mathcal{L}$ such that A reducted to $\leq$ is a linear order of uncountable cofinality. Prove:

(a) There exists a proper elementary extension $B \succeq A$ such that $|A|$ is cofinal in $|B|$, i.e., no new elements come at the end.

(b) There exists elementary extensions $B \succeq A$ of arbitrarily large cardinality such that $|A|$ is cofinal in $|B|$.

Theorem *(Hausdorf 1936 see [12]) There are $2^{\mathfrak{c}}$ many ultrafilters on ω.*

Problem 23. (3-06 F)

Prove there exists $f_\alpha : \omega \to \omega$ for $\alpha < \mathfrak{c} = |2^\omega|$ such that for any $F \in [\mathfrak{c}]^{<\omega}$ and $s : F \to \omega$ there exists $n < \omega$ such that $f_\alpha(n) = s(\alpha)$ for all $\alpha \in F$.

Problem 24. (3-06 F)

Prove that for any infinite cardinal κ there are 2^{2^κ} ultrafilters on κ.

Theorem *(Morley-Vaught 1962) If $\kappa \geq |\mathcal{L}| + \omega$ and A an $\mathcal{L}$-structure of cardinality 2^κ, then there exists $B \succeq A$ of cardinality 2^κ which is κ^+-saturated.*

Theorem *(Vaught) Let T be theory in a countable language. Then the following are equivalent:*

1. *T has a countable ω-saturated model*
2. *T has a countable weakly-saturated model*
3. *$S_n(T)$ is countable for all n*

Theorem *(Vaught) A structure A is ω-saturated iff it is weakly saturated and ω-homogenous.*

Theorem *(Vaught) If $\mathcal{L}$ is countable and A is a countable $\mathcal{L}$-structure, then there exists a countable ω-homogeneous $B \succeq A$.*

Problem 25. (3-11 W)

Suppose T is a consistent $\mathcal{L}$-theory with only infinite models. Suppose $\leq$ is a binary relation symbol in $\mathcal{L}$ such that $T \vdash$"$\leq$ is a linear order". Prove that every ω_1-saturated model has cardinality at least continuum.

Theorem *(Vaught's Two Cardinal) If a theory T in a countable language with a distinguished predicate U admits (κ, κ^+), then it admits (ω, ω_1).*

Theorem *(Henkin-Orey 1954) If T is a consistent theory in a countable language and $(\Sigma_n : n < \omega)$ are nowhere dense partial types, then T has a model omitting all Σ_n.*

Problem 26. (3-13 F)
Suppose that T is an $\mathcal{L}$-theory and $S_n(T)$ is countable. Prove there exists a countable $\mathcal{L}_0 \subseteq \mathcal{L}$ such that every $\mathcal{L}$-formula $\theta(x_1, \ldots, x_n) = \theta(\overline{x})$ there exists an $\mathcal{L}_0$-formula $\theta_0(\overline{x})$ such that $T \vdash \forall \overline{x} \ \ (\theta(\overline{x}) \ \leftrightarrow \ \theta_0(\overline{x}))$.

Theorem *(Henkin-Orey 1954) If T is an ω-complete consistent theory, then T has an ω-model. If T is complete and has an ω-model, then T is ω-complete.*

Problem 27. (3-23 M)
Prove or Disprove. Suppose T is an $\mathcal{L}$-theory where $\mathcal{L}$ is countable and Σ_n for $n < \omega$ are partial types. Suppose for every $N < \omega$ that T has a model omitting $(\Sigma_n : n < N)$. Then T has a model omitting $(\Sigma_n : n < \omega)$.

Theorem *(Keisler [18]) Suppose A is a countable $\mathcal{L}$-structure, $\mathcal{L}$ countable, and $\leq$ is a binary relation symbol in $\mathcal{L}$ with the properties:*

1. *$\leq^A$ is a linear order without a greatest element and*
2. *for any $\theta(x, y)$ with parameters from A and $a \in A$ if*

$$A \models \forall x < a \ \exists y \ \theta(x, y),$$

then there is a $b \in A$ such that

$$A \models \forall x < a \ \exists y < b \ \theta(x, y).$$

Then A has a proper elementary end extension.

Problem 28. (3-25 W)
Prove the converse to this theorem. If A has a proper elementary end extension, then (1) and (2) must hold.

Theorem *(Keisler) Two cardinal theorem for sentences of $\mathcal{L}_{\omega_1,\omega}$.*

Theorem *(MacDowell-Specker 1961) Every model of Peano Arithmetic, has a proper elementary end extension. (proof for countable models only)*

Theorem *(Vaught) The set of logical validities for $\mathcal{L}(Q)$ is computably enumerable.*

Theorem *(Fuhrken) $\mathcal{L}(Q)$ is countably compact.*

Problem 29. (3-27 F)

(a) Prove that for $\mathcal{L}$ countable that for any uncountable A $\mathcal{L}$-structure, there is B of cardinality ω_1 with

$$B \preceq_{\mathcal{L}(Q)} A.$$

Here Qx means "there are uncountably many x such that".

(b) Prove that for any countable family $\mathcal{F}$ of $\mathcal{L}_{\omega_1,\omega}$-formulas, each with only finitely many free variables, for any $\mathcal{L}$-structure A there is a countable B with

$$B \preceq_{\mathcal{F}} A.$$

Theorem *(Ryll-Nardzewski 1959) Suppose T is a countable,complete, consistent $\mathcal{L}$-theory without finite models. Then the following are equivalent:*

1. *T is ω-categorical*
2. *$S_n(T)$ is finite for all $n < \omega$*
3. *every $p \in S_n(T)$ is principal for all $n < \omega$.*

Problem 30. (3-30 M)

Suppose T_2 is a countable, complete, consistent $\mathcal{L}_2$-theory without finite models, $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and $T_1 = T_2 \cap (\mathcal{L}_1\text{-sentences})$.

(a) (Prove) T_2 ω-categorical implies T_1 ω-categorical.

(b) (Disprove) T_1 ω-categorical implies T_2 ω-categorical.

(c) (Prove) T_1 ω-categorical implies T_2 ω-categorical, if $\mathcal{L}_2 = \mathcal{L}_1 \cup \{c\}$.

Theorem *Suppose T is a countable, complete, consistent $\mathcal{L}$-theory without finite models. Any two prime models of T are isomorphic. If A is the prime model of T, then A elementarily embedds in every model of T. Conversely, if A embedds into model of T, then A is the prime model of T.*

Theorem *(Vaught 1961) Suppose T is a countable,complete, consistent $\mathcal{L}$-theory without finite models. Then T has a prime model iff the principal types in $S_n(T)$ are dense for for all $n < \omega$.*

Theorem *(Vaught Never Two) Suppose T is a countable, complete, consistent $\mathcal{L}$-theory without finite models. Then $I(\omega, T) \neq 2$.*

Example *(Ehrenfeucht) For each n with $3 \leq n < \omega$ there is a countable, complete theory T with $I(\omega, T) = n$.*

Problem 31. (4-01 W)
Suppose T is a countable, complete, consistent $\mathcal{L}$-theory without finite models. Suppose that every countable model of T is ω-homogeneous. Prove that $I(\omega, T) = 1$ or $I(\omega, T) \geq \omega$.

Example *(Kunen unpublished) There is a pseudo elementary class with exactly ω_1 countable models up to isomorphism. (Homogeneous linear orders).*

Theorem *Ramsey's Theorem, finite version of Ramsey's Theorem.*

Theorem *(Ehrenfeucht-Mostowski) Suppose T is a first-order theory with an infinite model and $(I, \leq)$ is a linear-order. Then T has a model A with $I \subseteq |A|$ order-indiscernibles.*

Problem 32. (4-06 M)
Let $\mathcal{L} = \{R\}$ where R is a binary relation symbol. Prove there are finitely many infinite $\mathcal{L}$-structures M_i for $i < N$ such that for every universal $\mathcal{L}$ theory T with an infinite model some $M_i \models T$. Extra credit: prove the same for R 3-ary and find the smallest N.

Theorem *(Ehrenfeucht-Mostowski) Suppose T is a countable first-order theory with an infinite model. Then for any $\kappa \geq \omega$ T has a model of size κ which realizes only countably many types and has 2^κ automorphisms.*

Problem 33. (4-13 M)
Suppose T is a countable first-order theory with an infinite model. Prove that for every $\kappa > \omega$ that T has a model A of size κ such that for every countable $X \subseteq |A|$, the structure $(A, c)_{c \in X}$ realizes only countably many types.

Theorem *(Erdos-Rado)* $\beth_n^+(\kappa) \to (\kappa^+)^{n+1}_\kappa$.

Example *(Sierpinski)* $2^\omega \not\to (3)^2_\omega \quad 2^\omega \not\to (\omega_1)^2_2$.

Problem 34. (4-15 W)

Suppose T is a countable first-order theory with an infinite model. Prove that there exists countable models of T, $A(X)$ for $X \subseteq \omega$, such that for any $X, Y \subseteq \omega$

$$X \subseteq Y \text{ iff } A(X) \preceq A(Y).$$

Theorem *(Vaught's two cardinals far apart) Suppose T is a countable theory with distinguished predicate U. Suppose for every n there is a $\kappa \geq \omega$ such that T has a model of type $(\beth_n(\kappa), \kappa)$. Then for every $\gamma \geq \kappa \geq \omega$, T has a model of type (γ, κ).*

Theorem *(Morley) The Hanf Number of $\mathcal{L}_{\omega_1,\omega}$ is $\beth_{\omega_1}$.*

Theorem *(Silver, Erdos, Rowbottom) Let κ_0 be the least κ such that*

$$\kappa \to (\omega)^{<\omega}_2$$

Assume κ_0 exists, then

1. *κ_0 is strongly inaccessible.*
2. *The Hanf Number of $\mathcal{L}_{\omega_1,\omega_1}$ is at least κ_0.*
3. *There are unboundedly many weakly compact cardinals less than κ_0, however κ_0 is not weakly compact.*

Problem 35. (4-22 W)

Let κ_1 be least such that $\kappa_1 \to (\omega+1)^{<\omega}_2$. Prove $\kappa_1 > \kappa_0$. Extra credit: prove it is strongly inaccessible.

Theorem *(Morley) If a countable first-order theory is categorical in some uncountable power, then it is categorical in all uncountable powers.*

Problem 36. (4-27 M)

Let $T_n = Th(\omega^\omega, P_s)_{s \in \omega^{\leq n}}$ where P_s is the unary predicate

$$P_s = \{x \in \omega^\omega \;:\; s \subseteq x\}.$$

Prove that

(a) $\text{rank}_{T_n}(x = x) = n + 1$.

(b) Give an example of a T such that $\text{rank}_T(x = x) = \omega$.

Theorem *(Shelah) Suppose T is a countable theory.*

(a) If there exists $\kappa \geq \omega$ such that T is κ-stable, then T is κ-stable for every κ such that $\kappa^\omega = \kappa$.

(b) T is unstable iff T has the order property.

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