References:

Hartley Rogers, Theory of recursive functions
Robert Soare, Recursively enumerable sets and degrees
Barry Cooper, Computability theory

UR-Basic Programming Language

Variables are any string of letters or numerals, A-Za-z0-9.
Statements are of the form

Let $X = X + 1$
Let $X = X \dot{-} 1$
If $X \leq Y$ then goto k

where X and Y are any variables and k is a nonnegative integer, i.e. $k \in \omega$.

A UR-Basic program is a sequence S_0, S_1, S_2, ..., S_n of statements. Variables only take on nonnegative integer values. The symbol $\dot{-}$ means subtraction unless the result is negative and then it yields zero. The program halts if we go to any line $k > n$.

A function $f : \omega \to \omega$ is UR-Basic computable iff there exists a program P, designated input variable X and output variable Y such that for any $n \in \omega$ if we put $X = n$ and all other variables zero and start with the first statement of P, then P eventually halts with $f(n)$ in variable Y. There is a similar definition for $f : \omega^m \to \omega$ to be UR-Basic computable.

Basic:	UR-Basic:
Go to k	If $X \leq X$ then goto k
Continue	Let Donothing=Donothing+1
Let Y=X	1 If $X \leq Y$ then go to 4
	2 Let Y=Y+1
	3 Go to 1
	4 If $Y \leq X$ then go to 7
	5 Let $Y = Y \dot{-} 1$
	6 Go to 4
	7 Continue
Constants	
0	this is a variable - we agree never to change it

1	let $1 = 1 + 1$
2	Let $2 = 2 + 1$
	Let $2 = 2 + 1$
If $X < Y$ then goto k	Let $tempX = X$
	Let $tempX = tempX + 1$
	if $tempX \leq Y$ then goto k
If $X = Y$ then goto k	1 If $X < Y$ then goto 4
	2 If $Y < X$ then goto 4
	3 Go to k
	4 continue
For $i = 1$ to n	1 If $n = 0$ then goto 7
S_1	2 Let $i = 1$
...	3 S_1
S_k	...
Next i	4 S_k
	5 Let $i = i + 1$
	6 If $n < i$ then goto 3
	7 continue

Theorem 1 *The functions $Z = X + Y$, $Z = XY$ and $Z = X^Y$ are UR-Basic computable. The functions $X \dot{-} Y$ is UR-Basic computable. The pair of functions remainder and quotient are UR-Basic computable i.e., input n, m then output q, r with $n = qm + r$ and $0 \leq r < m$.*

Proof
$Z = X + Y$:
    Let $Z = X$
    For $i = 1$ to Y
        Let $Z = Z + 1$
    Next i

$Z = XY$:

Let $Z = 0$
For $i = 1$ to Y
    Let $Z = Z + X$
Next i

$Z = X^Y$:
Let $Z = 1$
For $i = 1$ to Y
    Let $Z = ZX$
Next i

$Z = X \dot{-} Y$:
0 Let saveY=Y
1 If $X \leq Y$ then goto 7
2 Let $Z = X$
3 If $Y = 0$ then goto 8
4 Let $Y = Y \dot{-} 1$
5 Let $Z = Z - 1$
6 Go to 3
7 Let $Z = 0$
8 Continue
9 Let $Y = saveY$

$n = qm + r$:
1 Let $q = 0$
2 Let $r = n$
3 If $r < m$ then goto 7
4 Let $r = r \dot{-} q$
5 Let $q = q + 1$
6 go to 3
7 continue

QED

Hmwk 1. (Fri 9-3) Prove that the greatest common divisor function $d = gcd(n, m)$ is UR-Basic computable. Or if you prefer the function $f(n) =$ the n^{th} prime. Or you can prove that your favorite function is UR-Basic computable.

Primitive recursive functions

The class of primitive recursive functions is the smallest set of functions $f : \omega^m \rightarrow \omega$ of arbitrary arity m which contain

1. the constant zero function, $Z : \omega \rightarrow \omega$, $Z(n) = 0$ all n,
2. the successor function, $S : \omega \rightarrow \omega$ with $S(n) = n + 1$ all n (which we usually write $n + 1$), and
3. the projections $\pi^n_m(x_1, \ldots, x_n) = x_m$ for $1 \leq m \leq n < \omega$

and is closed under

- composition: h is primitive recursive, if

  $$h(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

  where f is n-ary and each g_i is m-ary are primitive recursive, and
- primitive recursion: h is primitive recursive, if

  $$\begin{array}{rcl} h(0, x_1, \ldots, x_m) & = & g(x_1, \ldots, x_m) \\ h(y + 1, x_1, \ldots, x_m) & = & f(y, x_1, \ldots, x_m, h(y, x_1, \ldots, x_m)) \end{array}$$

  where g is m-ary and f is $(m + 2)$-ary primitive recursive.

Note that by using the projections and compositions we may swap variables around and introduce dummy variables, e.g.

$$h(x, y, z) = f(g(x, y), z, k(z, x)) = f(g_1(x, y, z), g_2(x, y, z), g_3(x, y, z))$$

where

$$\begin{array}{rcl} g_1(x, y, z) & = & g(\pi^3_1(x, y, z), \pi^3_2(x, y, z)) \\ g_2(x, y, z) & = & \pi^3_3(x, y, z) \\ g_3(x, y, z) & = & k(\pi^3_3(x, y, z), \pi^3_2(x, y, z)) \end{array}$$

A predicate $P \subseteq \omega^n$ is primitive recursive iff its characteristic function $\chi_P(\vec{x})$ is where

$$\chi_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ 0 & \text{if } \neg P(\vec{x}) \end{cases}$$

Constant functions of any arity are primitive recursive. E.g.

$$f(x, y, z) = S(S(Z(\pi^3, 1(x, y, z)))) = 2.$$

Define $z = x + y$:
$x + 0 = x$
$x + (y + 1) = (x + y) + 1$

Define $z = xy$:
$x0 = 0$
$x(y + 1) = xy + x$

Define $z = x^y$:
$x^0 = 1$
$x^{y+1} = x^y x$

Define $z = x^{(y)} = x^{x^{\cdot^{\cdot^{x}}}}$:
$x^{(0)} = x$
$x^{(y+1)} = x^{x^{(y)}}$

Define $z = x!$:
$0! = 1$
$(x + 1)! = (x + 1)x!$

Define $z = x \dot{-} 1$:
$0 \dot{-} 1 = 0$
$(x + 1) \dot{-} 1 = x$

Define $z = y \dot{-} x$:
$y \dot{-} 0 = y$
$y \dot{-} (x + 1) = (y \dot{-} x) \dot{-} 1$

Define

$$sign(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

by $sign(x) = 1 \dot{-} (1 \dot{-} x)$.

Proposition 2 *The predicates $x \leq y$, $x = y$, $x < y$ are primitive recursive. If P and Q are primitive recursive predicates, then so is $P \vee Q$ and $\neg P$. If $P(\vec{x}, y)$ is a primitive recursive predicate and $f(\vec{x})$ a primitive recursive function, then $Q(\vec{x}) \equiv P(\vec{x}, f(\vec{x}))$ is a primitive recursive predicate.*

Proof
$\chi_{\leq}(x, y) = 1 \dot{-} (x \dot{-} y)$
$\chi_{P \vee Q} = sign(\chi_P + \chi_Q)$
$\chi_{\neg P} = 1 \dot{-} \chi_P$
$x = y$ iff $x \leq y$ and $y \leq x$
$x < y$ iff $\neg y \leq x$
$\chi_Q(\vec{x}) = \chi_P(\vec{x}, f(\vec{x}))$
QED

Proposition 3 *If $P(\vec{x}, y)$ is a primitive recursive predicate and $f(\vec{x})$ a primitive recursive function, then*

$$\exists y \leq f(\vec{x})\ P(\vec{x}, y) \text{ and } \forall y \leq f(\vec{x})\ P(\vec{x}, y)$$

are both primitive recursive predicates.

Proof
Let

$$Q(\vec{x}, z) \equiv \exists y \leq z \ \ P(\vec{x}, y)$$

Then χ_Q has the recursive definition:
$\chi_Q(\vec{x}, 0) = \chi_P(\vec{x}, 0)$
$\chi_Q(\vec{x}, z + 1) = sign(\chi_Q(\vec{x}, z) + \chi_P(\vec{x}, z + 1))$

Note that

$$Q(\vec{x}, h(\vec{x}) \equiv \exists y \leq h(\vec{x}))\ P(\vec{x}, y)$$

and

$$\forall y \leq h(\vec{x})\ P(\vec{x}, y) \equiv \neg \exists y \leq h(\vec{x})\ \neg P(\vec{x}, y)$$

QED
For example,
x divides y iff $\exists z \leq y\ y = xz$.
x is a Prime iff $x > 1$ and $\forall y \leq x$ if y divides x, then $y = 1$ or $y = x$.
are primitive recursive predicates.

Bounded search: define $f(\vec{x}, z) = \mu y \leq z \; P(\vec{x}, y)$ where f is the least $y \leq z$ which satisfies $P(\vec{x}, y)$ and $f = 0$ if no $y \leq z$ can be found.

Proposition 4 *Suppose Q is a primitive recursive predicate and h a primitive recursive function. Then*

$$g(\vec{x}) = \mu y \leq h(\vec{x}) \; P(\vec{x}, y)$$

is primitive recursive.

Proof
Let

$$Q(\vec{x}, y) \equiv P(\vec{x}, y) \wedge \forall u < y \; \neg P(\vec{x}, u).$$

Then if we define

$$f(\vec{x}, z) = \mu y \leq z \; P(\vec{x}, y)$$

then

$$f(\vec{x}, z) = \sum_{y=0}^{z} y \cdot \chi_Q(\vec{x}, y)$$

which has the following primitive recursive definition:
$f(\vec{x}, 0) = \chi_Q(\vec{x}, 0)$
$f(\vec{x}, z+1) = f(\vec{x}, z) + \chi_Q(\vec{x}, z+1)$
Hence

$$g(\vec{x}) = f(\vec{x}, h(\vec{x})) = \mu y \leq h(\vec{x}) \;\; P(\vec{x}, y).$$

QED

Proposition 5 *If $f : \omega \rightarrow \omega$ is primitive recursive, the graph(f) is a primitive recursive predicate. If graph(f) is a primitive recursive predicate and there is a primitive recursive function g which bounds f, then f is primitive recursive.*

Proof
Graph(f) has characteristic function $\chi_=(\vec{x}, f(\vec{x}))$. If f is bounded by g then

$$f(\vec{x}) = \mu y \leq g(\vec{x}) \;\; (\vec{x}, y) \text{ is in the graph of } f.$$

QED
Examples:

z=max(x,y) iff ($x = z$ and $x \geq y$) or ($y = z$ and $y \geq x$)
has primitive recursive graph and is bounded by $x + y$, so it is a primitive recursive function.

Division,Quotient: input $n, m > 0$ output q, r with $n = qm+r$ and $r < m$. $q =$ quotient(n, m) and $r =$ remainder(n, m) both have primitive recursive graphs bounded by $n + m$ so they are primitive recursive.

Hmwk 2. (Wed 9-8) Let $r(n) = n^{th}$ digit of $\sqrt{2} = 1.4142136\ldots$, so $r(0) = 1$, $r(1) = 4$, and so on. Prove that r is primitive recursive. If you prefer you may use $e = 2.7182818\ldots$ instead of $\sqrt{2}$. Does every naturally occurring constant in analysis have this property?

Coding pairs and sequences.

Coding pairs. $\langle x, y\rangle = 2^x(2y + 1) - 1$ is a bijection between ω^2 and ω. Both unpairing functions are primitive recursive since if $x = \langle x_0, x_1\rangle$, then $x_0, x_1 \leq x$.
Triples can be coded by $\langle x, y, z\rangle = \langle x, \langle y, z\rangle\rangle$ and similarly $n \geq 4$-tuples.
To code arbitrary length finite sequences we can use primes:
Define: nextprime$(x) = \mu y \leq x! + 1$ $y > x$ and y is prime

Note that if there is no prime between x and $x!$ then $x! + 1$ is prime. Actually there is always a prime between x and $2x$.
Define: $p_0 = 2$ and p_n is the n^{th} odd prime, primitively recursively by:

$p_0 = 2$
p_{n+1} =nextprime(p_n).

Sequences are coded by $c : \omega \times \omega \to \omega$ where

$c(y, i) = \mu k \leq y$ p_i^{k+1} does not divide y

We often use y_i to denote $c(y, i)$.

Theorem 6 *Every primitive recursive function is UR-Basic computable.*

Proof
The empty program with input x and output y, computes the constant zero function. Similarly for the projections. The successor function is computed by the one-line program "Let x=x+1", with input and output variable x.

For closure under composition: $z = f(g_1(\vec{x}), \ldots, g_n(\vec{x}))$ use the basic program:

Let $z_1 = g_1(\vec{x})$
Let $z_2 = g_2(\vec{x})$

. . .
Let $z_n = g_n(\vec{x})$
Let $y = f(z_1, \ldots, z_n)$
where appropriate substitution of UR-Basic code has been done.

The basic code for a primitive recursion looks like
Let $z = g(\vec{x})$
For $i = 1$ to y
Let $z = h(i, z, \vec{x})$
next i
QED

Theorem 7 *(Kleene) There exists a primitive recursive predicate $Q(e, x, y)$ and primitive recursive g such that for every partial UR-Basic computable $f : \omega \to \omega$ there exists an e with*

$$f(x) = g(\mu y \;\; Q(e, x, y)).$$

Proof
We can assume that the UR-Basic program only uses the variable v_i for $i < \omega$ and that the input variable is v_0 and output variable v_1.
1. $S = \langle 0, i\rangle \in \omega$ codes the statement "Let $v_i = v_i + 1$".
2. $S = \langle 1, i\rangle \in \omega$ codes the statement "Let $v_i = v_i \dot{-} 1$".
3. $S = \langle n, i, j, k\rangle$ for $n \geq 2$ codes the statement "If $v_i \leq v_j$ then goto k".
For $e \in \omega$ let $e = \langle n, S\rangle$ and let $S_0, S_1, \ldots, S_{n-1}$ be the program statements with S_i coded by $c(S, i)$.

Next we define three primitive recursive predicates:

In the tuple (e, x, y), e codes the program, x is the input value and y is pair $\langle k, V\rangle$ coding the line k in the program which is being executed and V coding the values of the variables.

$$Init(e, x, y) \equiv$$
$$\exists V < y \;\; y = \langle 0, V\rangle \text{ and } c(V, 0) = x \text{ and } \forall i < e \; (i > 0 \to c(V, i) = 0)$$

Since this is the start we want to start with Statement 0, i.e., $y = (0, V)$ and $v_0 = x$ and $v_i = 0$ for all i with $0 < i < e$. Note that we can bound this by e since e cannot refer to any variables with index higher than e.

$$Halt(e, y) \equiv \exists n, S < e \; \exists k, V < y \;\; y = \langle k, V\rangle \text{ and } e = \langle n, S\rangle \text{ and } k \geq n$$

All this says is we halt when we try to execute a line of the program beyond its length.

$$Onestep(e, y, y')$$

This just says we take one step in executing the program. So it will be: $\exists k, V, k', V' < y + y'$ and $\exists n, S < e$ such that all of the following are true:

1. $y = \langle k, V\rangle$, $y' = \langle k', V'\rangle$, and $e = \langle n, S\rangle$
2. $k < n$ (we don't take a step if program has halted)
3. If $c(S, k)$ codes "Let $v_i = v_i + 1$" then
   $c(V, i) = c(V', i) + 1$,
   $c(V, j) = c(V', j)$ for all $j < e$ with $j \neq i$, and
   $k' = k + 1$,
4. If $c(S, k)$ codes "Let $v_i = v_i \dot{-} 1$" then
   $c(V, i) = c(V', i) \dot{-} 1$,
   $c(V, j) = c(V', j)$ for all $j < e$ with $j \neq i$, and
   $k' = k + 1$.
5. If $c(S, k)$ codes "If $v_i \leq v_j$ then goto l" then
   $V = V'$ and
   $k' = l$ if $c(V, i) \leq c(V, j)$ or $k' = k + 1$ if $c(V, i) > c(V, j)$.

$$Q(e, x, y)$$

This says that y codes a computation using program e and input x.

$$Q(e, x, y) \equiv$$

$$\exists N, Y < y \;\; y = \langle N, Y\rangle \text{ and } Init(e, x, c(Y, 0)) \text{ and } Halt(e, c(Y, N - 1)) \text{ and}$$

$$\forall i < N \;\; Onestep(e, c(Y, i), c(Y, i + 1))$$

The function g simply extracts the value of v_1 the output variable from the computation sequence y. Since $g(y) \leq y$ it is enough to see that its graph is primitive recursive:

$$g(y) = v \text{ iff}$$

$$\exists N, Y, V, k < y \;\; \langle N, Y\rangle = y \text{ and } c(Y, N-1) = \langle k, V\rangle \text{ and } c(V, 1) = v$$

QED

Hmwk 3. (Fri 9-11) Prove that there exists a (total) $f : \omega \to \omega$ whose graph is a primitive recursive predicate but f is not a primitive function.

Hmwk 4. (Mon 9-13) Prove there exists a primitive recursive bijection $f : \omega \to \omega$ such that f^{-1} is not primitive recursive.

Corollary 8 *The family of (partial) UR-Basic computable functions is the same as the family of (partial) recursive functions.*

Church-Turing Thesis:

Every intuitively computable function is recursive.

Proposition 9 *There exists a recursive function $f : \omega \to \omega$ which is not primitive recursive.*

Proof
Make an effective list $f_n : \omega^{k_n} \to \omega$ of all the primitive recursive functions. Define $f(n) = f_n(n) + 1$ if f_n is a 1-ary function, otherwise put $f(n) = 0$. Since the listing can be effectively done by the Church-Turing Thesis the function f is recursive.
QED

Proposition 10 *There exists a universal partial recursive function*

$$\psi : \omega \to \omega$$

i.e. if we define $\psi_e(x) = \psi(\langle e, x\rangle)$ then $\{\psi_e \; : \; e \in \omega\}$ is a uniformly computable listing of all partial recursive functions.

Proof
$\psi(\langle e, x\rangle) = g(\mu y\, Q(e, x, y))$.
QED

Proposition 11 *(Padding Lemma) There exists a 1-1 recursive function p such that $\psi_e = \psi_{p(e,n)}$ for every e, n.*
(S-n-m Theorem). There exists a recursive function $S : \omega^2 \to \omega$ such that $\psi_{e_0}(e_1, x) = \psi_{S(e_0,e_1)}(x)$ for all e_0, e_1, x.

Proof
To pad the program $S_0, S_1, \ldots, S_m$ coded by e just add the statement

$$S_{m+1} = \mathrm{LetDonothing}\langle \mathrm{e}, \mathrm{n}\rangle = \mathrm{Donothing}\langle \mathrm{e}, \mathrm{n}\rangle + 1$$

and let $p(e, n)$ code this new program.

Given $\mathcal{P}$ the program coded by e_0 and input e_1 make-up a new program coded by $S(e_0, e_1)$ which puts e_1 into $\mathcal{P}$'s first input variable and then pops into program $\mathcal{P}$.
QED

This proposition can be used as follows: Suppose we have described a partial recursive function $\theta(e, x)$. Then there exists a one-to-one recursive function $f : \omega \to \omega$ such that

$$\forall e, x \quad \psi_{f(e)}(x) = \theta(e, x)$$

When use it this way we should call it the 1-1-S-1-1 Theorem.

Definition 12 *$A \subseteq \omega$ is recursively enumerable iff either A is empty or A is the range of a recursive function, i.e., $A = \{a_0, a_1, a_2, \ldots\}$ where the function $n \mapsto a_n$ is recursive. This is abbreviated r.e.*

$A \subseteq \omega$ is recursive iff its characteristic function χ_A is recursive.

$A \subseteq \omega$ is Σ^0_1 iff there exists a recursive predicate $R \subseteq \omega^2$ such that $A = \{n : \exists m\, R(n, m)\}$.

Proposition 13 *For $A \subseteq \omega$ the following are equivalent:*

(1) A is recursively enumerable.

(2) A is the domain of a partial recursive function.

(3) A is Σ^0_1.

(4) A is finite or A has a one-to-one recursive enumeration.

Proof
(1) $\rightarrow$ (2): Given a recursive enumerable listing a_n describe a partial recursive function f by input x and look for x on the list. Halt if you find it, otherwise continue looking forever.

(2) $\rightarrow$ (1): Define $\psi_{e,s}(x)\downarrow= y$ to mean that $e, x, y < x$ and the e^{th} UR-Basic program with input x converges and outputs y in fewer than s steps. The predicate

$$P(e, x, y, s) \equiv \psi_{e,s}(x)\downarrow= y$$

is primitive recursive. If A is the domain of ψ_e, then either A is empty or let $x_0 \in A$ be arbitrary and define a recursive enumeration of A by $a_n = x$ if $n = \langle x, y, s\rangle$ and $\psi_{e,s}(x)\downarrow= y$ otherwise $a_n = x_0$.

(1) $\rightarrow$ (3): Let $f : \omega \rightarrow \omega$ be recursive and have range A. Let R be the graph of f, then $y \in A$ iff $\exists x\; R(x, y)$.

(3) $\rightarrow$ (2): Suppose $x \in A$ iff $\exists y\; R(x, y)$. Then $f(x) = \mu y\; R(x, y)$ is partial recursive with domain A.

(1) $\rightarrow$ (4): Given $\{a_n : n < \omega\}$ a *recursive* enumeration of A, define a recursive enumeration $\{b_n : n < \omega\}$ by $b_{n+1} = a_m$ where m is the least such that $a_m \notin \{b_i : i \leq n\}$.
QED

Definition 14 *For $A \subseteq \omega$, $\overline{A} = \omega \setminus A$ the complement of A.*
$A \subseteq \omega$ is Π^0_1 iff $\overline{A}$ is Σ^0_1.
$\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$.

Proposition 15 *For $A \subseteq \omega$ the following are equivalent:*
(1) A is recursive.
(2) A and $\overline{A}$ are both r.e.
(3) A is Δ^0_1.
(4) A is finite or A has a strictly increasing recursive enumeration.

Proof
(1) $\rightarrow$ (2): Since recursive implies r.e.

(2) $\rightarrow$ (1): Input x. Effectively list A and $\overline{A}$ simultaneously until x shows up.

(2) iff (3): Trivial

(1) $\rightarrow$ (4): Take a_n to be the n^{th} element of A.

(4) $\rightarrow$ (1): Let $\{a_n : n < \omega\}$ be a strictly increasing *recursive* enumeration of A.

Input x. Find n such that $a_n > x$. Then $x \in A$ iff $x \in \{a_i : i < n\}$.
QED

Hmwk 5. (Wed 9-15) Prove that every nonempty recursively enumerable set A is the range of a primitive recursive function. Extra Credit: prove that not every infinite recursively enumerable set is the range of a one-to-one primitive recursive function.

Proposition 16 *Every infinite r.e. set contains an infinite recursive set.*

Proof
Given $\{a_n : n < \omega\}$ a *recursive* enumeration of A, define a strictly increasing recursive enumeration $\{b_n : n < \omega\}$ by $b_{n+1} = a_m$ where m is the least such that $a_m > b_n$.
QED

Proposition 17 *If A and B are r.e. sets, then $A \cap B$ is r.e. and $A \cup B$ is r.e. If A and B are recursive sets, then $A \cap B$, $A \cup B$, and $\overline{A}$ are all recursive sets.*

Proof
Domain of $f + g$ is the intersection of domain f and domain g. Enumerate $A \cup B$ by $x_{2n} = a_n$ and $x_{2n+1} = b_n$.
QED

Hmwk 6. (Fri 9-17) Suppose that $V \subseteq \omega$ is r.e. For each n define $V_n = \{x : \langle n, x\rangle \in V\}$. Prove that $\cup_n V_n$ is r.e.

Example 18 *There exists an r.e. set K which is not recursive.*

Proof

$$K = \{e : \psi_e(e) \downarrow\}$$

If $\overline{K}$ is the domain of ψ_e, then $e \in K$ iff $e \notin K$.
QED

Example 19 *There exists disjoint r.e. sets K_0 and K_1 which are recursively inseparable, i.e., there is not exists a recursive set $R \subseteq \omega$ with $K_0 \subseteq R$ and $K_1 \subseteq \overline{R}$.*

Proof

$$K_0 = \{e \ : \ \psi_e(e) \downarrow= 0\} \text{ and } K_1 = \{e \ : \ \psi_e(e) \downarrow= 1\}$$

QED

Definition 20 *For any $\Gamma \subseteq P(\omega)$ define $\widetilde{\Gamma}$ to be the set of all $\overline{A}$ for $A \in \Gamma$ and define $\Delta = \Gamma \cap \widetilde{\Gamma}$. $Sep(\Gamma)$ is the property that for every $A, B \in \Gamma$ disjoint there exists $C \in \Delta$ with $A \subseteq C$ and $B \subseteq \overline{C}$. $Red(\Gamma)$ (the reduction principle) is the property that for every $A, B \in \Gamma$ there exists disjoint $A' \subseteq A$ and $B' \subseteq B$ with $A', B' \in \Gamma$ and $A \cup B = A' \cup B'$.*

Proposition 21 *$Red(\Gamma)$ implies $Sep(\widetilde{\Gamma})$.*

Proof
Apply reduction to the complements.
QED

Proposition 22 *$Red(\Sigma^0_1)$ and hence $Sep(\Pi^0_1)$.*

Proof
$A = \{x \ : \ \exists u \ R(u, x)\}$ and $B = \{x \ : \ \exists v \ S(v, x)\}$. Put

$$x \in A' \leftrightarrow \exists u \ R(u, x) \text{ and } \forall v \leq u \neg S(v, x)$$

$$x \in B' \leftrightarrow \exists v \ S(v, x) \text{ and } \forall u < v \neg R(u, x)$$

QED

In example 19 it follows that K_0 and K_1 cannot be separated by disjoint Π^0_1 sets B_0 and B_1 because such a B_0 and B_1 could be recursively separated.

Hmwk 7. (Mon 9-20) Prove $Sep(\Gamma)$ for $\Gamma = \{A \cup B \ : \ A \in \Sigma^0_1, \ B \in \Pi^0_1\}$.

Definition 23 *$A \leq_m B$ iff there exists a recursive function f such that*

$$\forall x \in \omega \ \ x \in A \leftrightarrow f(x) \in B.$$

If the f can be taken one-to-one, then we write $A \leq_1 B$.

Note that $A \leq_m B$ and B is recursive, then A is recursive.

Definition 24 *$W = \{\langle e, x\rangle \ : \ \psi(\langle e, x\rangle) \downarrow\}$. Then $\{W_e \ : \ e \in \omega\}$ where $W_e = \{x \ : \ \langle e, x\rangle \in W\}$ is a uniform listing of the r.e. sets.*

Example 25 *Empty* $= \{e : W_e = \emptyset\}$ *is not recursive.*

Proof
Define

$$\theta(e,x) = \begin{cases} \downarrow = 0 & \text{if } e \in K \\ \uparrow & \text{otherwise} \end{cases}$$

By the S-n-m theorem there exists f recursive such that

$$\forall e, x \quad \psi_{f(e)}(x) = \theta(e,x)$$

But then $e \in K$ iff $W_{f(e)} \neq \emptyset$ iff $f(e) \notin E$ so $K \leq_m \overline{E}$ and therefor E not recursive.
QED

Proposition 26 *(Rice) If A is a nontrivial index set, then A is not recursive.*

Proof
Like proof for Empty.
QED

Theorem 27 *(Myhill) $A \leq_1 B$ and $B \leq_1 A$ iff there exists a recursive bijection $\pi : \omega \to \omega$ with $\pi(A) = B$.*

Proof
The Schroeder-Bernstein Theorem says: if there exists a 1-1 $f : A \to B$ and 1-1 $g : B \to A$, then there exists a bijection $h : A \to B$. One way to prove this is to assume A and B are disjoint and define a bipartite graph on the vertices $A \cup B$. Put $a \in A$ connected to b iff either $f(a) = b$ or $g(b) = a$. As f and g are 1-1 the order of every vertex is either 1 or 2. The connected components of this graph come in 4 types, see figure 1. Note that in Type 1 the point $a \in A$ is not in the range of g and in Type 2 the point $b \in B$ is not in the range of f. Type 4 components are infinite in both 'directions' while Type 3 is the only finite component.

To get h simply define $h = f$ on any component of type 1,3, or 4 and $h = g^{-1}$ on components of type 2.

The proof of Myhill's theorem is similar except we may never know exactly which type of component we looking at.

Suppose f and g are 1-1 recursive functions reducing A to B and B to A.

Effectively construct a sequence π_s of bijections with

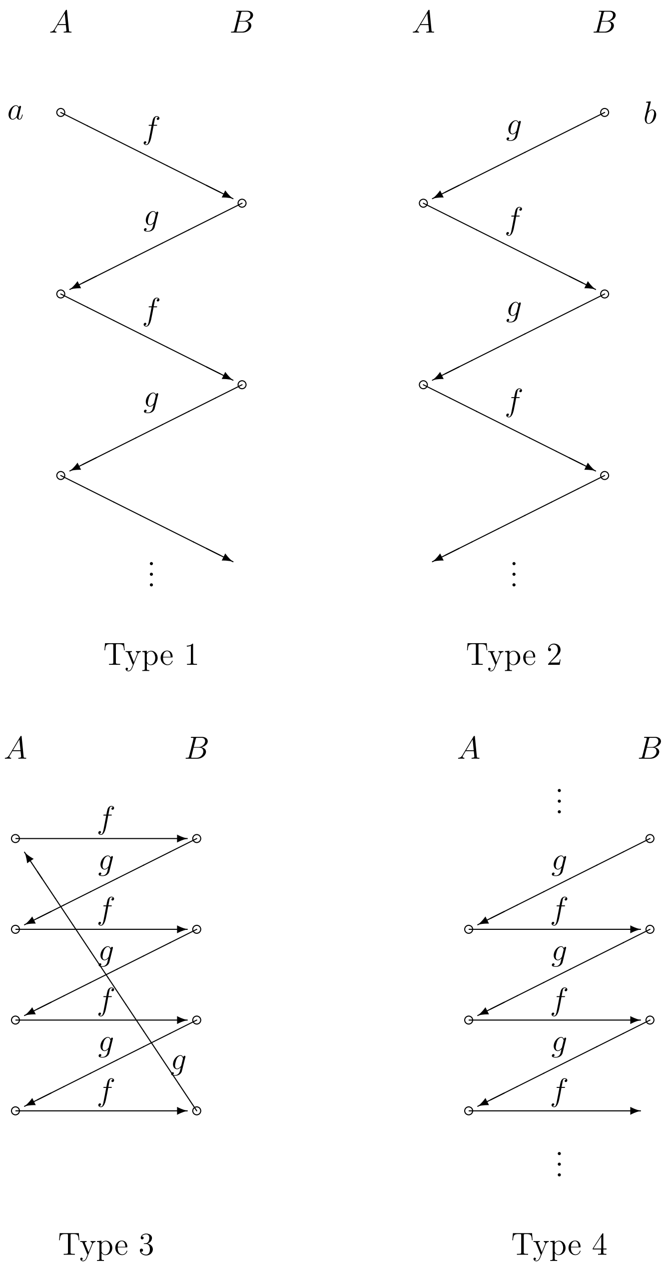

Figure 1: Schroeder-Bernstein connected components

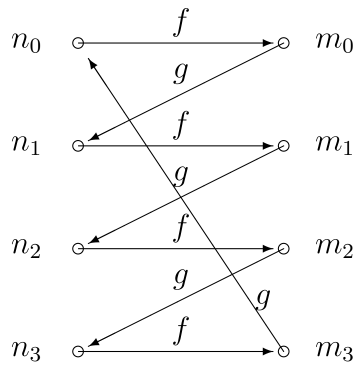

Figure 2: Myhill back and forth

1. $\pi_s : D_s \to E_s$ is a bijection.
2. D_s and E_s are finite subsets of ω.
3. $\pi_s \subseteq \pi_{s+1}$.
4. $n \in D_{2n}$ and $n \in E_{2n+1}$.
5. if $\pi_s(n) = m$, then either $m = fgfg \cdots fn$ or $n = gfgf \cdots gm$.

In the condition 5 we have dropped the parentheses to make it more readable.

If we then take $\pi = \cup_s \pi_s$ then π is a recursive bijection since we effectively constructed the sequence. It takes A to B, because suppose $\pi(n) = m$. Then if $m = fgfg \cdots fn$

$$n \in A \text{ iff } fn \in B \text{ iff } gfn \in A \text{ iff } fgfn \in B \text{ iff } \cdots \text{ iff } m = fgfg \cdots fn \in B$$

similarly if $n = gfgf \cdots gm$

$$m \in B \text{ iff } gm \in A \text{ iff } fgm \in B \text{ iff } gfgm \in A \text{ iff } \cdots \text{ iff } n = gfgf \cdots gm \in A$$

either way $n \in A$ iff $m \in B$.

At stage s=0 we take π_0 to be the empty function.

At stage s+1 suppose we are given $\pi_s : D_s \to E_s$. If $s = 2n$ we try to extend π_s to include $n \in D_{s+1}$. If its already there we let $\pi_{s+1} = \pi_s$. Otherwise consider the following sequences:

Let $n = n_0$, $fn_0 = m_0$ and in general $f(n_k) = m_k$ and $g(m_k) = n_{k+1}$, see figure 2.

Case 1. For some k we have that $m_k \notin E_s$.
In this case we put $\pi_{s+1} = \pi_s \cup \{\langle n_0, m_k\rangle$.

Case 2. Not case 1.

In this case the connected component of the graph (see Figure 1) must be of Type 3, i.e., a finite closed loop. Suppose $g(m_k) = n_0$. But by condition 5 if all the m_k are in E_s, then they must map via π_s^{-1} to the set $\{n_0, n_1, \ldots, n_k\}$ (although not in any particular order). But this is a contradiction, since $n = n_0 \notin D_s$. Hence Case 2 cannot happen.

The construction at stage s+1 where $s = 2n + 1$ is entirely analogous except we make sure $n \in E_{s+1}$.
QED

Theorem 28 *(Rogers) Suppose $\rho : \omega \to \omega$ is partial recursive and we define $\rho_e(x) = \rho(e, x)$. Suppose*

1. *ρ is universal, i.e., $\{\rho_e \ : \ e \in \omega\}$ includes all partial recursive functions.*
2. *ρ satisfies padding, i.e., there exists one-to-one recursive $p : \omega \times \omega \to \omega$ such that*
$$\forall e, n \ \ \rho_e = \rho_{p(e,n)}$$
3. *ρ satisfies S-1-1, i.e., there exists a recursive $S : \omega \times \omega \to \omega$ such that*
$$\forall e_1, e_2, x \ \ \rho_{e_1}(\langle e_2, x\rangle) = \rho_{S(e_1,e_2)}(x)$$

Then there exists a recursive bijection $\pi : \omega \to \omega$ such that

$$\forall e \ \ \psi_e = \rho_{\pi(e)}$$

Proof
Let $\psi = \rho_{e_0}$. Using padding and S-1-1 for ρ we can find a 1-1 recursive function $f(e) = p(S(e_0, e))$ such that

$$\forall e \ \ \psi_e = \rho_{S(e_0,e)} = \rho_{f(e)}$$

similarly there is a 1-1 recursive function g such that

$$\forall e \;\; \rho_e = \psi_{g(e)}.$$

By the proof of Theorem 27 there is a recursive bijection $\pi : \omega \to \omega$ with the property that whenever $\pi(n) = m$ then either $m = fgfg \cdots fn$ or $n = gfgf \cdots gm$. But

$$\psi_n = \rho_{fn} = \psi_{gfn} = \ldots = \rho_{fgfg \cdots fn} = \rho_m$$

and

$$\rho_m = \psi_{gm} = \rho_{fgm} = \ldots = \psi_{gfgf \cdots gm} = \psi_n$$

so in either case $\psi_n = \rho_{\pi(n)}$.
QED

Hmwk 8. (Wed 9-22) Find an example of a partial recursive ρ which is universal but fails to satisfy padding. Find an example which is universal, satisfies padding but fails to satisfy S-1-1. (S-1-1 implies padding see Soare p.25-26.)

Theorem 29 *(Kleene - Recursion Theorem) For any recursive function f there exists an e with $\psi_e = \psi_{f(e)}$.*

Proof
Define a partial recursive function θ by

$$\theta(u, x) = \psi_{\psi_u(u)}(x) = \psi(\langle \psi(\langle u, u \rangle), x \rangle)$$

By padding-S-1-1 we can find a (one-to-one) recursive function $d : \omega \to \omega$ such that

$$\forall u \;\; \psi_{d(u)}(x) = \theta(u, x)$$

Let v be an index for $f \circ d$, i.e.,

$$\forall x \;\; \psi_v(x) = f(d(x))$$

Put $e = d(v)$ then

$$\psi_e(x) = \psi_{d(v)}(x) = \theta(v, x) = \psi_{\psi_v(v)}(x) = \psi_{f \circ d(v)}(x) = \psi_{f(e)}(x)$$

QED

From the proof we can get an infinite recursive set of fixed points e, since we can take any v' such that $\psi_{v'} = f \circ d$ and set $e' = d(v')$. Also note that our fixed point e is obtained effectively from an index for f, so given a recursive $f : \omega \times \omega \to \omega$ if we let $f_n : \omega \to \omega$ be defined by $f_n(x) = f(n, x)$ then we get a fixed points e_n

$$\psi_{e_n} = \psi_{f_n(e_n)}$$

and the function $n \mapsto e_n$ is recursive. This is called the recursion theorem with parameters:

$$\forall n \quad \psi_{e(n)} = \psi_{f(n,e(n))}.$$

Example 30 *There are infinitely many e such that $\psi_e(0) = e$. There are infinitely many e such that $W_e = \{e\}$.*

Proof
Define $\theta(e, x) = e$ for all e. By the S-n-m Theorem there exists a recursive f such that

$$\forall e, x \;\; \psi_{f(e)} = \theta(e, x)$$

By the Recursion Theorem there are infinitely many fixed points for f, i.e.,

$$\psi_e = \psi_f(e)$$

and for each of these ψ_e is the constant function e.

Define a partial recursive function θ by

$$\theta(e, x) = \begin{cases} \downarrow = 0 & \text{if } e = x \\ \uparrow & \text{otherwise} \end{cases}$$

By S-n-m theorem there is a recursive function g with $\psi_g(e)(x) = \theta(x)$. By the definition of θ we see that for every e:

$$W_{g(e)} = \{e\}$$

By the Recursion Theorem there are infinitely many fixed points for g and for any of them

$$W_e = W_{g(e)} = \{e\}.$$

Hmwk 9. (Fri 9-24) Prove:

(a) for every f, g recursive functions, there exists e_1 and e_2 such that $\psi_{f(e_1)} = \psi_{e_2}$ and $\psi_{g(e_2)} = \psi_{e_1}$

(b) $\exists e_1 \neq e_2 \quad W_{e_1} = \{e_2\},\ W_{e_2} = \{e_1\}$

(c) $\exists e_1 > e_2 > e_3 \quad W_{e_1} = \{e_2\},\ W_{e_2} = \{e_3\},\ W_{e_3} = \{e_1\}$

Example 31 *(Smullyan) For any recursive functions $f(x,y)$ and $g(x,y)$ there exists $a, b \in \omega$ such that*

$$\psi_{f(a,b)} = \psi_a \text{ and } \psi_{g(a,b)} = \psi_b$$

Proof
By the recursion theorem

$$\forall x \ \exists y \ \ \psi_{g(x,y)} = \psi_y$$

but since the fixed point y is obtained effectively from x and an index for g there exists a recursive function h such that

$$\forall x \ \ \psi_{g(x,h(x))} = \psi_{h(x)}$$

Apply the fixed point theorem to $f(x, h(x))$ there exists $a \in \omega$ such that

$$\psi_{f(a,h(a))} = \psi_a$$

Letting $b = h(a)$ does the job.
QED

Hmwk 10. (Mon 9-27) Prove
(a) $\exists e_1 < e_2 < e_3 \quad W_{e_1} = \{e_2\},\ W_{e_2} = \{e_3\},\ W_{e_3} = \{e_1\}$
(b) $\exists e_1 \neq e_2 \quad W_{e_1} = \{e_1, e_2\} = W_{e_2}$
(c) $\exists e_1 < e_2 < e_3 \quad W_{e_1} = \{e_2, e_3\},\ W_{e_2} = \{e_1, e_3\},\ W_{e_3} = \{e_1, e_2\}$

Definition 32 *A r.e. set A is m-complete iff $B \leq_m A$ for every r.e. B. Similarly 1-complete. Define C is creative iff C is r.e. and there exists a recursive function $q \in \omega^\omega$ such that for every e*

$$W_e \cap C = \emptyset \rightarrow q(e) \notin C \cup W_e.$$

Theorem 33 *(Myhill) For $C \subseteq \omega$ r.e. the following are equivalent:*

1. *C is creative*
2. *$C \equiv_1 K$*
3. *C is 1-complete*
4. *C is m-complete*

Proof
(2) $\to$ (3): It is enough to see that K is 1-complete, since then for any B r.e. we would have $B \leq_1 K \leq_1 A$. Define a partial recursive function ρ as follows:

$$\rho(e,x) = \begin{cases} \downarrow = 0 & \text{if } e \in B \\ \uparrow & \text{otherwise} \end{cases}$$

ρ is partial recursive because we enumerate B looking to see if e ever turns up, if not the computation never halts. Using the 1-1-S-1-1 Theorem there exists a 1-1 recursive function f such that

$$\forall e, x \;\; \psi_{f(e)}(x) = \rho(e,x) = \begin{cases} \downarrow = 0 & \text{if } e \in B \\ \uparrow & \text{otherwise} \end{cases}$$

Then $e \in B$ iff $\psi_{f(e)}(f(e)) \downarrow$ iff $f(e) \in K$.

(3) $\to$ (4): Trivial

(4) $\to$ (1): The creativity of K is witnessed by the identity function, i.e.,

$$W_e \cap K = \emptyset \to e \notin W_e \cup K.$$

Suppose $K \leq_m A$ is witnessed by the function f. Then there exists a recursive function g such that

$$\text{for all } e \quad W_{q(e)} = f^{-1}(W_e)$$

(Use S-1-1 to get $\psi_{q(e)} = \psi_e \circ f$.) Then

$$W_e \cap A = \emptyset \to$$

$$f^{-1}(W_e) \cap K = \emptyset \to$$

$$W_{q(e)} \cap K = \emptyset \to$$

$$q(e) \notin f^{-1}(W_e) \cup K \to$$

$$f(q(e)) \notin W_e \cup A$$

so $f \circ q$ witnesses the creativity of A.

(1) $\to$ (2):

Claim The creativity function for A can be taken to be 1-1.
Proof
Given any creativity function d for A. Construct a recursive function f such that

$$\forall x \quad W_{f(x)} = W_x \cup \{d(x)\}.$$

To do this use

$$\forall x, y \;\; \psi_{f(x)}(y) = \rho(x,y) = \begin{cases} \downarrow = 0 & \text{if } y \in W_x \text{ or } y = d(x) \\ \uparrow & \text{otherwise} \end{cases}$$

Now we get our 1-1 creativity function $\hat{d}$ recursively as follows: Input e put $e = e_0$ and effectively generate the sequence e_{s+1} where $W_{e_{s+1}} = W_{e_s} \cup \{d(e_s)\}$, i.e. put $e_{s+1} = f(e_s)$. and look for e_s such that. Simultaneously enumerate A and W_e looking for something in their intersection.

Search for the least s such that either

1. $e_s > \hat{d}(e-1)$ or
2. $A_s \cap W_{e,s} \neq \emptyset$

If the first happens put $\hat{d}(e) = e_s$. If the second happens put $\hat{d}(e) = \hat{d}(e-1)+1$.

This proves the Claim.

QED

Now we show that $K \leq_1 A$. Define a partial recursive function θ as follows:

$$\psi_{f(n,x}(y) = \theta(n,x,y) = \begin{cases} \downarrow = 0 & \text{if } n \in K \text{ and } y = \hat{d}(x) \\ \uparrow & \text{otherwise} \end{cases}$$

It follows that

$$W_{f(n,x)} = \begin{cases} \{\hat{d}(x)\} & \text{if } n \in K \\ \emptyset & \text{otherwise} \end{cases}$$

By the uniform proof of the recursion theorem and by padding we get a 1-1 recursive sequence $n \mapsto e_n$ of fixed points so that

$$\forall n \;\; W_{f(n,e_n)} = W_{e_n} = \begin{cases} \{\hat{d}(e_n)\} & \text{if } n \in K \\ \emptyset & \text{otherwise} \end{cases}$$

But then $n \in K$ iff $\hat{d}(e_n) \in A$. So $K \leq_1 A$.

QED

Most naturally occurring nonrecursive r.e. sets are m-complete.

Hmwk 11. (Wed 9-29) Prove or disprove: there exists a recursive function $d : \omega \to \omega$ such that for every e

$$W_e \cap K \text{ finite } \to d(e) \notin W_e \cup K$$

Definition 34 *A is simple iff A is r.e. , $\overline{A}$ is infinite, and $\overline{A}$ does not contain an infinite r.e. set.*

Theorem 35 *(Post) There exists a simple set.*

Proof
Define a recursive sequence $A_s \subseteq s$ of increasing finite sets as follows. $A_0 = \emptyset$. At stage $s+1$ find the least $e < s$ (if any) such that $W_{e,s} \cap A_s = \emptyset$ and $\exists x > 2e\ x \in W_{e,s}$. Put $A_{s+1} = A_s \cup \{x\}$ for the least e and x for which this is true. If this happens we say that e has acted at stage $s+1$. If there no such e, then put $A_{s+1} = A_s$.

The set $A = \cup_s A_s$ is simple. Note that each e can act at most once. Hence if W_e is infinite and $W_e \cap A = \emptyset$, eventually there will come a stage s where $\exists x > 2e\ x \in W_{e,s}$ and all smaller e's which will ever act have already acted at a previous stage. But then e will act, which is a contradiction.

Also we see that $\overline{A}$ is infinite because for all e $|A \cap 2e| \leq e$ since the only $e^p r$ which can put an x into A with $x \leq 2e$ are those e' with $e' < e$.
QED

Definition 36 *$A \leq_T B$ or A is Turing reducible to B. Add to the UR-Basic programming language statements of the form:*

$$\textit{Let } y = \chi_B(x)$$

for any variables x, y. This programming language is called Oracle UR-Basic. Then $A \leq_T B$ iff there is an Oracle UR-Basic program with Oracle for B which computes the characteristic function χ_A of A.

Hmwk 12. (Fri 10-1) Suppose A is a simple set and $A = \{a_n\ :\ n \in \omega\}$ is a 1-1 recursive enumeration of A. Prove there exists infinitely many n such that $W_{a_n} = \{a_m\ :\ m > n\}$. (Hint: it is easier to show there exists $e \in A$ such that $W_e = \{e\}$.)

Proposition 37 *(Dekker Deficiency Set) For every r.e. set A which is not recursive there exists a simple set B with $B \equiv_T A$.*

Proof

Let $\{a_n \ : \ n \in \omega\}$ be a 1-1 recursive enumeration of A. Define

$$B = \{n \ : \ \exists m > n \ \ a_m < a_n\}$$

It is easy to see that B is r.e.

$\overline{B}$ is infinite: Otherwise there would be an N such that $a_{n+1} > a_n$ for all $n > N$ and then A would be recursive.

$A \leq_T B$: Input x. Find $n \in \overline{B}$ such that $a_n > x$. Then $x \in A$ iff $x \in \{a_i : i < n\}$.

$\overline{B}$ does not contain an infinite recursive set: Suppose $R \subseteq \overline{B}$ is an infinite recursive set. But then the argument we just gave for $A \leq_T B$ shows that $A \leq_T R$ which would make A recursive.

$B \leq_T A$: Input n. Using an Oracle for A check if

$$\{a_i \ : \ a_i < a_n \text{ and } i < n\} = A \cap \{x \ : \ x < a_n\}$$

if they are equal, then $n \notin B$, otherwise $n \in B$.
QED

Hmwk 13. (Mon 10-4) Define $B \subseteq \omega$ is intro-reducible iff $B \leq_T C$ for every infinite $C \subseteq B$. Prove that for every A there exists $B \equiv_T A$ intro-reducible.

Definition 38 *For $A \subseteq \omega$ define the Turing degree of A to be*

$$a = deg(A) = \{B \ : \ B \equiv_T A\}.$$

Let $\mathcal{D} = \{deg(A) \ : \ A \subseteq \omega\}$ be the Turing Degrees. $(\mathcal{D}, \leq)$ is the partial order where $a \leq b$ iff $A \leq_T B$.

Definition 39 *For $\sigma \in 2^{<\omega}$ and $e, x, y, s \in \omega$ we write*

$$\{e\}_s^\sigma(x) \downarrow = y$$

to mean that the e^{th} oracle machine with input x and using σ to answer Oracle questions, converges in less than s steps and outputs y. We also require that $e, x, y < s$ and that in this computation the oracle is not asked about $n \notin dom(\sigma)$ or $n \geq s$.

Proposition 40 *The predicate $O(\sigma, e, x, y, s)$ defined by*

$$O(\sigma, e, x, y, s) \text{ iff } \{e\}_s^\sigma(x) \downarrow = y$$

is primitive recursive.

Definition 41 *For $A \subseteq \omega$ the jump of A is defined by*

$$A' = \{e : \exists s \ \ e_s^{A \restriction s}(x) \downarrow\}$$

Proposition 42 *(1) $A \leq_T B$ implies $A' \leq_1 B'$.*
(2) $A <_T A'$

Proof
(1) Define

$$\theta(e,x) = \begin{cases} \downarrow = 0 & \text{if } e^A(e) \downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Then θ is partial recursive in A and since $A \leq_T B$ we have that θ is partial recursive in B. By the 1-1-S-1-1 Theorem relativized to B there exists a 1-1 recursive function f such that

$$\forall e, x \ \ \{f(e)\}^B(x) = \theta(e,x).$$

But then $e \in A'$ iff $\{e\}^A(e) \downarrow$ iff $\{f(e)\}^B(f(e)) \downarrow$ iff $f(e) \in B'$.

(2) To see $A \leq_1 A'$ construct a 1-1 recursive function f so that $f(n)^A(?)$ has the same computation on any input and it converges iff $n \in A$. Then $n \in A$ iff $f(n) \in A'$. To see that $A' \not\leq_T A$, suppose that it is. Define $f = 1 - \chi A'$. Then since $f \leq_T A' \leq_T A$ there is an e_0 with $\{e_0\}^A = f$. But then $e_0 \in A'$ iff $e_0 \notin A'$.
QED

Corollary 43 *If $A \equiv_T B$, then $A' \equiv_T B'$. Hence, letting $a' \in \mathcal{D}$ be the Turing degree of A' is well-defined and $a < a'$ for every $a \in \mathcal{D}$.*

Similarly, a'' is the jump of the jump of a, and $a^{(n)}$ is n jumps of a.

Definition 44 *$a|b$ iff not $a \leq b$ and not $b \leq a$. I.e. the degrees a and b are Turing incomparable.*

Proposition 45 *(Kleene-Post) There exists $a, b \in \mathcal{D}$ with $a|b$.*

Proof

Construct sequences $(\sigma_s \in 2^{<\omega} : s \in \omega)$, $(\tau_s \in 2^{<\omega} : s \in \omega)$ with the property that $\sigma_s \subseteq \sigma_{s+1}$ and $\tau_s \subseteq \tau_{s+1}$ for each s. For $s = 0$ take τ_s and σ_s to be the empty sequence.

At stage $s+1$ we are given τ_s and σ_s and we do as follows:

Case $s = 2e$:
Let $n = |\tau_s|$.
Case a. There exists $\sigma \supseteq \sigma_s$ such that $\{e\}^\sigma(n)\downarrow$. In this case put $\sigma_{s+1} = \sigma$ and put $\tau_{s+1} = \tau_s i$ where $i = 0, 1$ whichever is different from $\{e\}^\sigma(n)$.
Case b. No such σ. Put $\sigma_{s+1} = \sigma_s$ and $\tau_{s+1} = \tau_s 0$.

Case $s = 2e+1$:
Let $n = |\sigma_s|$ and proceed similarly to $s = 2e$ with the roles of σ_s and τ_s reversed.
This ends the construction. We put $A = \cup_{s\in\omega}\sigma_s$ and $B = \cup_{s\in\omega}\tau_s$.
QED
It is easy to see that the entire construction is recursive in o' and hence there are incomparable Turing degrees beneath o'.

Proposition 46 *(Kleene-Post) For every $a \in \mathcal{D}\setminus\{o\}$ there exists $b \in \mathcal{D}$ with $a|b$.*

Let $deg(A) = a$. Construct $(\tau_s \in 2^{<\omega} : s \in \omega)$ as follows. $\tau_0 = \langle\rangle$.
At stage $s+1$ we are given τ_s.

Case $s = 2e$. Let $n = |\tau_s|$. Take $i = 0$ or $i = 1$ so that $i \neq \{e\}^A(n)$. Put $\tau_{s+1} = \tau_s i$.

Case $s = 2e+1$.
Case a. There exists $n < \omega$, ρ_1, ρ_2 with $\tau_s \subseteq \rho_i$ and

$$\{e\}^{\rho_1}(n)\downarrow \neq \{e\}^{\rho_2}(n)\downarrow$$

In this case we put $\tau_{s+1} = \rho_1$ or $\tau_{s+1} = \rho_2$ which ever that case is that

$$\{e\}^{\tau_{s+1}}(n) \neq A(n).$$

Case b. There is no such n and ρ_i. Put $\tau_{s+1} = \tau_s 0$.

This ends the construction. Now we check that $B = \cup_s\tau_s$ is Turing incomparable to A. The cases $2e$ easily show that it is not the case that $B \leq_G = TA$. Suppose $A \leq_T B$ and choose e so that $\{e\}^B = A$ and consider stage $s+1$ where $s = 2e+1$. In case (a) we get that $\{e\}^B(n) \neq A(n)$ so that it is impossible. Now we show that case (b) cannot happen. Define

$$f(n) = i \text{ iff } \exists\tau \supseteq \tau_s \{e\}^\tau(n) \downarrow= i$$

Note that f is well-defined because we are in case (b) and f is total because we are assume that $\{e\}^B$ is the characteristic function of A. Hence f which is recursive is the characteristic function of A, which contradicts the assumption that A is not recursive.
QED

Hmwk 14. (Wed 10-6) Prove that for every countable $\mathcal{A} \subseteq \mathcal{D} \setminus \{0\}$ there exists $b \in \mathcal{D}$ such that $a|b$ for all $a \in \mathcal{A}$.

Definition 47 $A \oplus B = \{2n \ : \ n \in A\} \cup \{2n+1 \ : \ n \in B\}$.

Proposition 48 *$A_0 \leq_T A_1$ and $B_0 \leq_T B_1$ implies $A_0 \oplus B_0 \leq_T A_1 \oplus B_1$.*
$A \leq_T C$ and $B \leq_T C$ iff $A \oplus B \leq_T C$.

Definition 49 *$a \vee b = deg(A \oplus B)$ is the join or least upper bound of a and b.*

Meets, $a \wedge b$, in the Turing degrees may or may not exist.

Proposition 50 *(Kleene-Post) There exists $a, b \in \mathcal{D} \setminus \{o\}$ with $a \wedge b = 0$ i.e., for all c if $c \leq a$ and $c \leq b$ then $c = o$.*

Proof
As before construct sequences $(\sigma_s \in 2^{<\omega} : s \in \omega)$, $(\tau_s \in 2^{<\omega} : s \in \omega)$ with the property that $\sigma_s \subseteq \sigma_{s+1}$ and $\tau_s \subseteq \tau_{s+1}$ for each s. For $s = 0$ take τ_s and σ_s to be the empty sequence.

At stage $s+1$ we are given τ_s and σ_s and we do as follows:

Case $s = 3e$. Let $n = |\sigma_s|$. Let $i = 0$ or $i = 1$ so that $\psi_e(n) \neq i$. Put $\sigma_{s+1} = \sigma_s i$.

Case $s = 3e+1$. Similar to $3e$ but for τ_{s+1}.

Case $s = 3\langle e_1, e_2\rangle + 2$.

Case a. There exists $n < \omega$, $\sigma \supseteq \sigma_s$, and $\tau \supseteq \tau_s$ such that

$$\{e_1\}^\sigma(n) \downarrow \neq \{e_2\}^\tau(n) \downarrow$$

put $\sigma_{s+1} = \sigma$ and $\tau_{s+1} = \tau$.

Case b. Not case a. Put $\tau_{s+1} = \tau_s$ and $\sigma_{s+1} = \sigma_s$.

This ends the construction. We put $A = \cup_s \sigma_s$ and $B = \cup_s \tau_s$. The stages $3e, 3e+1$ guarantee that neither A nor B is recursive. Now suppose that $C \leq_T A$ and $C \leq_T B$. This will be witnessed by a pair e_1 and e_2. At stage $s = 3\langle e_1, e_2\rangle + 2$ it must have been that Case a. failed since we assume that

$$\{e_1\}^A = \{e_2\}^B = C.$$

But then we may define a total recursive function f by

$$f(n) = i \text{ iff } \exists \sigma \supseteq \sigma_s \;\; \{e_1\}^\sigma(n) \downarrow = i$$

and f must be the characteristic function of C and hence C is recursive.
QED

Proposition 51 *(Kleene-Post) For every $c \in \mathcal{D}$ there exists $a, b \in \mathcal{D}$ with $a \wedge b = c$ and $a|b$, i.e., $a > c$, $b > c$, and for all d if $d \leq a$ and $d \leq b$ then $d \leq c$.*

Proof
This is a relativization of the above argument. Construct A_0 and B_0 so that for every e

$$\{e\}^C \neq A_0 \oplus C \text{ and } \{e\}^C \neq B_0 \oplus C$$

and

$$\{e_1\}^{A_0 \oplus C} = \{e_2\}^{B_0 \oplus C} = D \rightarrow D \leq_T C$$

Then take $A = A_0 \oplus C$ and $B = B_0 \oplus C$.
QED

Hmwk 15. (Fri 10-8) Prove that for every $c \in \mathcal{D}$ there exists $a, b \in \mathcal{D}$ with $a|b$, $a \wedge b = 0$, and $a \vee b \geq c$. Hint: one way to code C into $A \oplus B$ is to use boot-strapping. Define

$$x_{2n} = \mu x > x_{2n-1} \; A(x) = 1$$

$$x_{2n+1} = \mu x > x_{2n} \; B(x) = 1$$

$$n \in C \text{ iff } x_n \text{ is even}$$

Proposition 52 *(Spector) Given* $(a_n : n < \omega)$ *in* $\mathcal{D}$ *with* $a_n < a_{n+1}$ *for all* n *there exists* $b, c \in \mathcal{D}$ *with*

(1) $a_n \leq b$ *and* $a_n \leq c$ *for all* n *and*

(2) for all $d \in \mathcal{D}$ *if* $d \leq b$ *and* $d \leq c$ *then there exists* n *with* $d \leq a_n$.

Proof

Let $deg(A_n) = a_n$ and set $A = \{\langle n, x\rangle : n < \omega, x \in A_n\}$. The key to this construction is to make B and C have the property that for each n

$$B_n =^* A_n =^* C_n$$

where $B_n = \{x : \langle n, x\rangle \in B\}$ and $C_n = \{x : \langle n, x\rangle \in C\}$.

As before construct sequences $(\sigma_s \in 2^{<\omega} : s \in \omega)$, $(\tau_s \in 2^{<\omega} : s \in \omega)$ with the property that $\sigma_s \subseteq \sigma_{s+1}$ and $\tau_s \subseteq \tau_{s+1}$ for each s. For $s = 0$ take τ_s and σ_s to be the empty sequence.

At stage $s + 1$ we will extend σ_s and τ_s so as to agree with A_i for $i < s$ on new elements of their domain. Define

$$f_s = \sigma_s \cup \{\langle\langle i, x\rangle, j\rangle : \langle i, x\rangle \notin dom(\sigma_s),\ \ i < s,\ \text{and}\ A_i(x) = j\}$$

$$g_s = \tau_s \cup \{\langle\langle i, x\rangle, j\rangle : \langle i, x\rangle \notin dom(\tau_s),\ \ i < s,\ \text{and}\ A_i(x) = j\}$$

Note that f_s is a partial function extending σ_s which agrees with the characteristic function of each A_i for $i < s$ except possible on the (finite) domain of σ_s. Similarly g_s.

Let $s = \langle e_1, e_2\rangle$.

Case a. There exists $n < \omega$, $\sigma \supseteq \sigma_s$ and $\tau \supseteq \tau_s$ such that $f_s \cup \sigma$ is a function (i.e., they are compatible) and $g_s \cup \tau$ is a function and

$$\{e_1\}^\sigma(n) \downarrow \neq \{e_2\}^\tau(n) \downarrow$$

Put $\sigma_{s+1} = \sigma$ and $\tau_{s+1} = \tau$.

Case b. Not Case a. Put $\sigma_{s+1} = \sigma_s$ and $\tau_{s+1} = \tau_s$.

This completes the construction, so put $B = \cup_s \sigma_s$ and $C = \cup_s \tau_s$.

Claim. For all n we have that $A_n \leq_T B$ and $A_n \leq_T C$. To see this note that in the construction that for all $s > n$ that $f_s(\langle n, m\rangle) = f_{n+1}(\langle n, m\rangle)$. Furthermore, except for the finitely many element of the domain of σ_{n+1} we have that $A_n(m) = f_{n+1}(\langle n, m\rangle)$. It follows that $A_n =^* B_n$ and so $A_n \leq_T B_n \leq_T B$. Similarly for C.

Claim. Suppose that $D \leq_T B$ and $D \leq_T C$. Then $D \leq_T A_n$ for some $n < \omega$. To see this suppose that

$$\{e_1\}^B = \{e_2\}^C = D$$

and $s = \langle e_1, e_2\rangle$. Since the characteristic functions of B and C extend σ_{s+1} and τ_{s+1} respectively it is evident that Case (a) could not have occurred. So we assume Case (b). Note that in this case it is impossible that there exists n, ρ_1, ρ_2 with $\sigma_s \subseteq \rho_1$ and $\sigma_s \subseteq \rho_2$, and each of ρ_1 and ρ_2 compatible with f_s such that

$$\{e_1\}^{\rho_1}(n)\downarrow \neq \{e_1\}^{\rho_2}(n)\downarrow.$$

This is because $\{e_2\}^C(n)\downarrow$ and so then we would be in Case (a).

It follows easily as before that $D = \{e_1\}^B \leq_T f_s$. But

$$f_s \leq_T A_0 \oplus A_1 \oplus \cdot \oplus A_{s-1} \leq_t A_{s-1}$$

so $D \leq_T A_{s-1}$.
QED

Proposition 53 *(Friedberg Jump Inversion) For every $a \in \mathcal{D}$ if $a \geq o'$ then there exists $b \in \mathcal{D}$ with $b' = a$.*

Proof
We construct sequence $(\tau_s : s \in \omega)$ recursive in $A \oplus 0' \equiv_T A$ as follows.

At stage $s+1$ we are given $\tau_s \in 2^{<\omega}$

(a) We put $\tau = \tau_s i$ where $i = A(s)$.

(b) Let $e = s$. We ask $0'$ if there exists $\sigma \supseteq \tau$ such that

$$\{e\}^{\sigma}_{|\sigma|}(e)\downarrow$$

If there is such a σ then we effectively find one and put $\tau_{s+1} = \sigma$.

More precisely, before the construction begins find a recursive function $f(e,\tau)$ such that

1. for any e, τ
$$\psi_{f(e,\tau)}(0)\downarrow \text{ iff } \exists\sigma \supseteq \tau \ \ \{e\}^{\sigma}_{|\sigma|}(e)\downarrow$$

2. when $\psi_{f(e,\tau)}(0)$ converges it outputs such a σ and

3. the algorithm $\psi_{f(e,\tau)}(?)$ ignores its input.

We put $\tau_{s+1} = \tau$ if $f(e,\tau) \notin 0'$, otherwise we put $\tau_{s+1} = \sigma =^{def} \psi_{f(e,\tau)}(0)$.
This ends the construction. We let $B = \cup_{s\in\omega}\tau_s$.

Claim.

1. $(\tau_s : s \in \omega) \leq_T A \oplus 0' \leq_T A$

2. $A \leq_T (\tau_s : s \in \omega)$

3. $(\tau_s : s \in \omega) \leq_T B \oplus 0'$

4. $B' \leq_T (\tau_s : s \in \omega)$

Proof

(1) The construction only requires oracles for $0'$ and A. Also $A \geq_T 0'$.

(2) We encoded the characteristic function of A at step (a). Hence

$$s \in A \text{ iff } \tau_{s+1}(|\tau_s|) = 1.$$

(3) Recursively construct the sequence $(\tau_s : s \in \omega)$ using oracles for $0'$ and B. Given τ_s we use that $\tau_{s+1} \subseteq B$ to figure out the first digit, i.e., τ of step (a). To do step (b) we only used $0'$ and the recursive function f.

(4) By our construction given any e let $s = e$, then we have that

$$e \in B' \text{ iff } \{e\}^B(e)\downarrow \text{ iff } \{e\}^{\tau_{s+1}}_{|\tau_{s+1}|}(e)\downarrow$$

This proves the Claim. But note that the Claim implies

$$B' \leq_T (\tau_s : s \in \omega) \leq_T A \leq_T (\tau_s : s \in \omega) \leq_T B \oplus 0' \leq_T B'$$

QED

Hmwk 16. (Mon 10-11) Prove that $\forall a \in \mathcal{D} \;\; a \geq o' \rightarrow \exists b, c \in \mathcal{D} \;\; b|c$ and $b' = a = c'$.

Theorem 54 *(Clifford Spector) There exists a minimal Turing degree, i.e., $\exists a \in \mathcal{D}$ with $o < a$ but no $b \in \mathcal{D}$ with $o < b < a$.*

Proof
For any $\sigma \in 2^n$, i.e., a finite sequence of zeros and ones, we can code σ by the number

$$x = 2^n + \sum \{2^i \ : \ i < n \text{ and } \sigma(i) = 1\}.$$

The extra 2^n is there to distinguish sequences ending in zeros from each other. We suppress this coding and just talk about recursive subsets of $2^{<\omega}$.

Definition 55 *$T \subseteq 2^{<\omega}$ is a perfect tree iff*

1. *T is nonempty,*
2. *$\sigma \subseteq \tau \in T$ implies $\sigma \in T$, and*
3. *$\forall \sigma \in T \ \ \exists \tau_0, \tau_1 \in T$ with $\sigma \subseteq \tau_0$, $\sigma \subseteq \tau_1$, and $\tau 0$ and $\tau 1$ are incomparable.*

Definition 56 *For $T \subseteq 2^{<\omega}$ a tree we define:*

1. *$\sigma \in T$ splits iff $\sigma 0, \sigma 1 \in T$*
2. *$\sigma = stem(T)$ iff σ splits but no shorter node of T splits*
3. *$[T] = \{x \in 2^\omega \ : \ \forall n \ \ x \upharpoonright n \in T\}$*
4. *for $\sigma \in T$ let*
$$T(\sigma) = \{\tau \in T \ : \ \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$$

To prove the Theorem construct a sequence $(T_s \ : \ s \in \omega)$ of recursive perfect trees as follows.

At stage $s = 0$ take $T_0 = 2^{<\omega}$.

At stage $s+1$ where $s = 2e$ let $\sigma = stem(T_s)$ and $n = |\sigma|$. If $\psi_e(n) \downarrow = 0$ then put $T_{s+1} = T_s(\sigma 1)$ otherwise put $T_{s+1} = T_s(\sigma 0)$.

At stage $s+1$ where $s = 2e+1$ we obtain $T_{s+1} \subseteq T_s$ a perfect recursive subtree as follows. We first ask the question:

> Does there exist $\sigma \in T_s$ such that for all $\sigma_1, \sigma_2 \in T(\sigma)$ and $n, m_1, m_2 < \omega$ if $\{e\}^{\sigma_1}(n) \downarrow = m_1$ and $\{e\}^{\sigma_2}(n) \downarrow = m_2$, then $m_1 = m_2$?

Case (a) If the answer is yes, we take $T_{s+1} = T_s(\sigma)$ for any such σ.

Case (b) If the answer is no, we construct recursive sequences
$(\sigma_\rho \in T : \rho \in 2^{<\omega})$ and $(n_\rho \in \omega : \rho \in 2^{<\omega})$
such that

1. $\{e\}^{\sigma_{\rho 0}}(n_\rho) \downarrow \neq \{e\}^{\sigma_{\rho 1}}(n_\rho) \downarrow$ and
2. $\sigma_\rho \subseteq \sigma_{\rho 0}$ and $\sigma_\rho \subseteq \sigma_{\rho 1}$.

Note that (1) implies that $\sigma_{\rho 0}$ is incomparable to $\sigma_{\rho 1}$. We put

$$T_{s+1} = \{\sigma \ : \ \exists \rho \in 2^{<\omega} \ \ \sigma \subseteq \sigma_\rho\}$$

then T_{s+1} is a recursive perfect subtree of T_s.

This ends the construction of the sequence of trees. Note that $T_{s+1} \subseteq T_s$. Take A to be the subset of ω whose characteristic function is the unique element of $\cap_{s\in\omega}[T_s]$. It is easy to see that stage $2e+1$ guarantees that A is not recursive, so it is enough to see stage $2e+2$ guarantees that if $B = \{e\}^A$ then either B is recursive or $A \leq_T B$.

Case (a) for all $\sigma_1, \sigma_2 \in T_{s+1}$ and $n, m_1, m_2 < \omega$ if $\{e\}^{\sigma_1}(n) \downarrow = m_1$ and $\{e\}^{\sigma_2}(n) \downarrow = m_2$, then $m_1 = m_2$. In this case B is recursive, since $A \in [T_{s+1}]$ and $B = \{e\}^A$ means that all we have to do to compute $B(n)$ is to search the recursive tree T_{s+1} for any σ for which $\{e\}^\sigma(n) \downarrow$ and then $B(n) = \{e\}^\sigma(n)$.

Case (b) In this case we show that $A \leq_T B$. We know $A \in [T_{s+1}]$. Suppose we know that $\sigma_\rho \subseteq A$. To decide whether $\sigma_{\rho 0} \subseteq A$ or $\sigma_{\rho 1} \subseteq A$, we compute both of

$$\{e\}^{\sigma_{\rho 0}}(n_\rho) \text{ and } \{e\}^{\sigma_{\rho 1}}(n_\rho).$$

Since these two computations are guaranteed to converge and to different values at most one of them can agree with $B(n_\rho)$. One of them must agree and so using an oracle for B we can determine the unique $i = 0, 1$ so that $\sigma_{\rho i} \subseteq A$.
QED

Hmwk 17. (Fri 10-15) Prove that there are uncountably many minimal degrees.

Theorem 57 *(Sacks) Minimal upper bounds exists. Given any sequence of degrees $(a_n \in \mathcal{D} \ : \ n < \omega)$ such that $a_n < a_{n+1}$ for all n there exists $b \in \mathcal{D}$ with $a_n < b$ all n but there is no $c \in \mathcal{D}$ with $a_n < c < b$ for all n.*

Proof
Here we use the notion of a recursively-pointed tree.

Definition 58 *$T \subseteq 2^{<\omega}$ is recursively-pointed iff T is a perfect tree and $T \leq_T A$ for every $A \in [T]$.*

The new ingredient required in this construction is

Claim. Suppose $T \subseteq 2^{<\omega}$ is recursively-pointed tree and $T \leq_T B$. Then there exists $T^* \subseteq T$ a recursively-pointed tree such that $T^* \equiv_T B$.
Proof
There exists a natural bijection $f : 2^{<\omega} \to Split(T)$ where $Split(T)$ are the splitting nodes of T. Note that f and T are Turing equivalent. Given $B \in 2^\omega$ let

$$T_B = \{\sigma \in 2^{<\omega} \ : \ \sigma(2n) = B(n) \text{ whenever } 2n < |\sigma|\}.$$

Now take T^* to be the tree generated by $f(T_B)$.
QED

Construct $(T_s : s \in \omega)$ a sequence of recursively-pointed trees as follows.

Suppose $T_s \equiv_T A_s$ and $e = s$. Relativizing Spector's proof above to T_s we can obtain $T^\circ \subseteq T_s$ with $T^\circ \leq_T T_s$ a perfect subtree so that for every $B \in [T^\circ]$: if $C = \{e\}^B$ then either $B \leq_T (C \oplus T^\circ)$ or $C \leq_T T^\circ$.

Note that T° is recursively-pointed and $T^\circ \leq_T A_s$. Hence by applying the Claim above we can obtain $T_{s+1} \subseteq T^\circ$ such that T_{s+1} is recursively-pointed and $T_{s+1} \equiv_T A_{s+1}$.

This ends the construction. We let B be the unique element of $\cap_{s\in\omega}[T_s]$.

First note that $A_s \leq_T B$ for each s, because $B \in [T_s]$, T_s is recursively-pointed and so $A_s \equiv_T T_s \leq_T B$.

Suppose that $A_s \leq_T C \leq_T B$ for every $s \in \omega$. Then at some stage $s = e$ we have that $C = \{e\}^B$. Hence by construction either $C \leq_T T^\circ \leq_T A_s$ or $B \leq_T (C \oplus T^\circ)$. The first is impossible since $A_s <_T A_{s+1} \leq_T C$ and so it must be that $B \leq_T (C \oplus T^\circ)$. But $T^\circ \leq_T A_s \leq_T C$ so $B \leq_T C$.
QED

Hmwk 18. (Mon 10-18) (a) Prove there exists $a, b \in \mathcal{D}$ with $o < a < b$ and not there exists c with either $o < c < a$ or $a < c < b$.

(b) (Extra Credit) Prove there exists $a, b \in \mathcal{D}$ with $o < a < b$ and ($c \leq b$ iff $c = 0$ or $c = a$ or $c = b$), for all $c \in \mathcal{D}$.

Definition 59 *The use of an oracle computation* $\{e\}^A(x)$ *written*

$$use(\{e\}^A(x))$$

is $n+1$ *where* n *is the maximum number for which the oracle for* A *is queried.*

Note that if $u = use(\{e\}^A(x))$ and $B \cap u = A \cap u$ then $\{e\}^A(x)$ and $\{e\}^B(x)$ are the same computation.

Theorem 60 *(Friedberg-Muchnik) There exists r.e. sets* A_0 *and* A_1 *such that* $A_0 \not\leq_T A_1$ *and* $A_1 \not\leq_T A_0$*.*

Proof
Our requirements are:
$R_{2e+i} \qquad \{e\}^{A_i} \neq A_{1-i}$
for each $e \in \omega$ and $i = 0, 1$.

The strategy for meeting this requirement is to attach a follower $x \in \omega$ to R_{2e+i} and then wait until $\{e\}_s^{A_{i,s}}(x) \downarrow= 0$. When this happens we put x into A_{1-i} and try to avoid injuring the computation $\{e\}_s^{A_{i,s}}(x)$. If we succeed then $\{e\}^{A_i}(x) = 0 \neq 1 = A_{1-i}(x)$. If we wait forever, then x is never put into A_{1-i} and so $A_{1-i}(x) = 0 \neq \{e\}^{A_i}(x)$. In either case the requirement R_{2e+i} is met. There are two possible successful outcomes for this strategy, either we wait forever or we act at some stage and then preserved the relevant computation.

Construction

Everything in the construction will be done effectively.

At each stage s of the construction we will have effectively constructed:

1. finite sets $A_{i,s}$ for $i = 0, 1$,
2. a follower $x = x_{q,s}$ for each R_q with $q < s$, and
3. a function f_s with domain s which is attempting to predicate the final outcomes of our strategy for each R_q with $q < s$.

At stage $s = 0$ put $A_{i,0} = \emptyset$ for $i = 0, 1$. Nobody has followers and f_s is the empty function.

At stage $s + 1$ look for the least $q = 2e + i < s$ such that

1. $f_s(q) =$'waiting' and

2. $\{e\}_s^{A_{i,s}}(x) \downarrow = 0$ with use less than s where $x = x_{q,s}$ is the follower of R_{2e+i}.

If we find such a q then we take the following actions:

1. Put x into A_{1-i}, i.e.,

$$A_{1-i,s+1} = A_{1-i,s} \cup \{x\}$$

2. Set $f_{s+1}(q) =$'acted'.

3. Reappoint followers for lower priority requirements, i.e. for each $q' > q$ with $q' < s+1$ put $x = \langle q', s+1 \rangle$ to be the follower of $R_{q'}$.

4. Make all lower priority requirements start over, i.e., for each $q' > q$ put $f_{s+1}(q') =$'waiting'.

We say that R_q acted at stage $s+1$. If there is no such q then we just continue to wait. In either case assign $x = (s, s+1)$ to be the follower of R_s and put $f_{s+1}(s) =$'waiting'.

This ends the stage and the construction.

Note that the sequence

$$(A_{s,0}, A_{s,1}, f_s, x_{q,s} \ : \ s \in \omega, \ q < s)$$

is recursive.

We put $A_i = \cup_{s \in \omega} A_{i,s}$. These are r.e. sets since $A_{i,s} \subseteq A_{i,s+1}$.

Verification

Claim. For each q

1. R_q acquires a permanent follower, i.e., there exist some stage s_0 such that for all $s > s_0$ the follower of R_q at stage s is that same as at stage s_0.

2. R_q is met, i.e, $\{e\}^{A_i} \neq A_{1-i}$

3. R_q acts at most finitely many times.

Proof
This is the main claim and it is proved by induction on q.

So suppose that (3) is true for all $q' < q$. Then there is a stage s_0 such that some $q' < q$ acted and no such $q' < q$ acts after stage s_0. Then the follower x_q of R_q appointed at stage s_0 is the permanent follower of R_q. Furthermore $f_{s_0}(q) =$'waiting'.

Suppose $q = 2e + i$. After stage s_0 there are two possibilities:
(a) for some $s > s_0$ we have that $\{e\}_s^{A_{i,s}}(x_q) \downarrow= 0$ with use less than s or
(b) not (a).

Suppose (a). In this case since no higher priority q' acts after stage s_0 then R_q will act. Hence x_q is put into A_{1-i}. Furthermore all other followers of lower priority requirements appointed now or at future stages will be larger than the use of the computation $\{e\}_s^{A_{i,s}}(x_q)$ (we assume that $s \leq \langle q', s\rangle$). Hence

$$\{e\}^{A_i}(x_q) \downarrow= 0 \neq 1 = A_{1-i}(x_q)$$

Suppose (b). In this case it must be that either

$$\{e\}^{A_i}(x_q) \uparrow \text{ or } \{e\}^{A_i}(x_q) \downarrow\neq 0.$$

In either case x_q is never put into A_{1-i} - this is because the possible followers of two distinct requirements are disjoint and no follower is used again for the same requirement. So $A_{1-i}(x_q) = 0 \neq \{e\}^{A_i}(x_q)$ and thus R_q is met.

So as we see R_q will act at most one more time after stage s_0 and so it acts only finitely many times. This proves the Claim and the Theorem.
QED

We say that R_q is injured when it is made to appoint new followers and start over. Hence, the terminology 'finite injury priority argument'.

Corollary 61 *There exists a set A which is r.e. and $0 <_T A <_T 0'$.*

Proof
Since 0 and $0'$ are $\leq_T$ comparable to every r.e. set it must be that both A_i from the Friedberg-Muchnik Theorem are strictly in between.
QED

Another way to prove that some r.e. degree is nontrivial is to construct a low simple set A. Since a simple set is not recursive we have that $0 <_T A$. Low means that $A' \equiv_T 0'$ so $A <_T 0'$ by Lemma 42.

Lemma 62 *(The Limit Lemma) Suppose $g \in \omega^\omega$, then*
$g \leq_T 0'$
iff
there exists $f : \omega \times \omega \to \omega$ recursive such that for all n

$$\lim_{s \to \infty} f(n, s) = g(n)$$

Proof
Suppose $g = \{e\}^{0'}$. Let $(0'_s \ : \ s \in \omega)$ be a recursive enumeration of $0'$, e.g., $0'_s = \{e < s \ : \ \{e\}_s(e) \downarrow\}$. Define

$$f(n, s) = \begin{cases} 1 & \text{if } \{e\}_s^{0'_s}(n) \downarrow \\ 0 & \text{otherwise} \end{cases}$$

Then $g(n) = \lim_{s \to \infty} f(n, s)$.

For the converse, suppose that $g(n) = \lim_{s \to \infty} f(n, s)$ where f is recursive. For each n using an oracle for $0'$ we can compute s_0 so that for every $s > s_0$ we have that $f(n, s) = f(n, s_0)$.

(Try $s_0 = 0$ and ask the oracle if the computation that searches for a change in f ever terminates. If yes, try $s_0 = 1$, etc. Continue incrementing s_0 until the oracle says that beyond this stage f does not change.)

It follows that $g(n) = f(n, s_0)$. Hence there is an algorithm with oracle $0'$ which computes g.
QED

Theorem 63 *There exists a low simple set A, i.e. $A' \equiv 0'$ and A is simple.*

Proof
We make A simple by a strategy that is suggested by the proof of the limit lemma, namely we would like to use

$$f(e, s) = \begin{cases} 1 & \text{if } \{e\}_s^{A_s}(e) \downarrow \\ 0 & \text{otherwise} \end{cases}$$

to show that $A' \leq_T 0'$. That is, $A'(e) = \lim_{s \to \infty} f(e, s)$. If $e \in A'$ then it is easy to see that $f(e, s) = 1$ for all sufficiently large s. The problem then is to make sure that if $f(e, s) = 1$ for infinitely many s, then $e \in A'$.

So we make the following requirements:

$N_e \quad (\exists^\infty s \ \ \{e\}_s^{A_s}(e) \downarrow) \to \{e\}^A(e) \downarrow$

In order to make sure that the set A is simple we have the following requirements:

$P_e \qquad (W_e \text{ infinite}) \rightarrow W_e \cap A \neq \emptyset$

The strategy for P_e is the same as for the Post Simple Set construction (Theorem 35), that is we wait for some $x \in W_{e,s}$ with $x > 2e$ and $A_s \cap W_{e,s} = \emptyset$ and put x into A_{s+1}.

The strategy for N_e is to wait until we see convergence and then try to prevent the computation from changing by restraining numbers less than the use of the computation from entering A.

The requirement P_e is positive since the strategy is try to put things into A while the requirement N_e is negative since it tries to keep things out of A.

Construction

At each stage in the construction we will have A_s and $r(e,s)$ for each e. We will always have that $r(e,s) = 0$ for $e \geq s$ so the function r is really a finite function.

Stage $s+1$. Look for the least $e < s$ such that

1. $W_{e,s} \cap A_s = \emptyset$
2. $\exists x > 2e$ with $x \in W_{e,s}$ and $x > r(e', s)$ for all $e' < e$.

For the least such e choose the least x as above and put $A_{s+1} = A_s \cup \{x\}$. We say in this case that P_e acted at stage $s+1$. If there is no such e put $A_{s+1} = A_s$.

Next we compute $r(e, s+1)$ for all $e < s+1$. If $\{e\}_s^{A_{s+1}}(e)\downarrow$, then put

$$r(e, s+1) = use(\{e\}_s^{A_{s+1}}(e)$$

otherwise put $r(e, s+1) = 0$.

This is the end of the construction. We let $A = \cup_{s \in \omega} A_s$ which is r.e.

Verification.

Claim.

1. P_e is met.
2. N_e is met.

3. $\lim_{s\to\infty} r(e,s) = r(e) < \infty$ exists.

Proof
We prove this by induction on e. Note that each P_e can act at most once, since after it acts W_e and A are no longer disjoint. Assume the claim is true for every $e' < e$.

(1) By induction we have some s_0 such that for all $s > s_0$ and $e' < e$ that $r(e', s) = r(e')$. Put

$$R = max\{r(e') \ : \ e' < e\}.$$

We can also choose s_0 so large that no $P_{e'}$ for $e' < e$ acts after stage s_0 since each $P_{e'}$ acts at most once. Suppose that W_e is infinite. It follows that at some stage $s > s_0$ there will be a $x \in W_{e,s}$ such that $x > 2e + R$. At stage $s + 1$ either $A_s \cap W_{e,s} \neq \emptyset$ or P_e will act. In either case P_e is met.

(2) Choose s_0 so that no $P_{e'}$ for $e' \leq e$ acts after stage s_0. This means that after stage s_0 no positive requirement can ever injure a computation of N_e. Hence if there is some $s_1 > s_0$ such that $\{e\}_{s_1}^{A_{s_1}}(e)\downarrow$ then no $x < use\{e\}_{s_1}^{A_{s_1}}(e)$ will ever enter A. It follows that this is the final computation and therefor $\{e\}^A(e)\downarrow$ with the same computation as at stage s.

(3) As above, either we never see convergence and then $r(e,s) = 0$ for all $s > s_0$ or we see convergence and then $r(e,s) = r(e,s_1)$ for all $s > s_1$.

This finishes the proof of the Claim and the Theorem.
QED

Hmwk 19. (Fri 10-22) (From Soare) A set A is auto-reducible iff there exists e such that for every x we have

$$\{e\}^{A\setminus\{x\}}(x)\downarrow = A(x).$$

Prove that there exists a A low r.e. set which is not auto-reducible.

We define

$$A_n = \{x \ : \ \langle n, x\rangle \in A\}$$

and

$$\oplus_{k\neq n} A_k = \{\langle k, x\rangle \in A \ : \ k < \omega \text{ and } k \neq n\}.$$

Theorem 64 *There exists an r.e. set A such that for every n*

$$A_n \not\leq_T \oplus_{k\neq n} A_k$$

Proof
This is a minor modification of the Friedberg-Muchnic argument (Theorem 60).

Our requirements are:
$R_{\langle e,n\rangle} \qquad \{e\}^{\oplus_{k\neq n} A_k} \neq A_n$
for $e, n \in \omega$. And the construction is nearly the same:
At stage $s+1$ look for the least $q = \langle e, n\rangle < s$ such that

1. $f_s(q) =$'waiting' and

2. $\{e\}_s^{\oplus_{k\neq n} A_{k,s}}(x) \downarrow = 0$ with use less than s where $x = x_{q,s}$ is the follower of R_q.

If we find such a q then we take the following actions:

1. Put
$$A_{s+1} = A_s \cup \{\langle n, x\rangle\}$$

2. Set $f_{s+1}(q) =$'acted'.

3. Reappoint followers for lower priority requirements, i.e. for each $q' > q$ with $q' < s+1$ put $x = \langle q', s+1\rangle$ to be the follower of $R_{q'}$.

4. Restart lower priority requirements, for each $q' > q$ put
$$f_{s+1}(q') = \text{'waiting'}.$$

Finally, assign $x = (s, s+1)$ to be the follower of R_s and $f_{s+1}(s) =$'waiting'.

The verification is virtually the same as in the Friedberg-Muchnic Theorem.
QED

Corollary 65 *Every recursive partially ordered set embeds into the r.e. degrees $\mathcal{R}$.*

Proof
Let $\mathbb{P} = (\omega, \trianglelefteq)$ be a partial order with $\trianglelefteq$ a recursive binary relation on ω. Define $J(p) = \{\langle q, x\rangle \in A \ : \ q \trianglelefteq p\}$ and let $j(p) = deg(J(p))$. Then

$$j : \mathbb{P} \to \mathcal{R}$$

is an order preserving embedding.
QED

Hmwk 20. (Mon 10-25) Prove there exists a recursive partial order $\mathbb{P}_0 = (\omega, \leq_0)$ such that every countable partial order $\mathbb{P}_1$ can be embedded into it, i.e., there exists a 1-1 mapping $j : \mathbb{P}_1 \to \mathbb{P}_0$ such that $p \leq_1 q$ iff $j(p) \leq_0 j(q)$.

It follows from this exercise that every countable partial order embeds into the r.e. degrees.

Hmwk 21. (Wed 10-27) Prove that for every creative set A there exist a set B which is r.e. and disjoint from A but cannot be separated from it by a recursive set. Prove that there exists disjoint r.e. sets A_0 and A_1 which are recursively inseparable but not creative.

Theorem 66 *(Sacks) Suppose* $0 <_T C \leq_T 0'$ *and* A *is r.e. Then there exists r.e. sets* A_0 *and* A_1 *such that*

1. *A is the disjoint union of A_0 and A_1,*
2. *$C \not\leq_T A_i$ for $i = 0, 1$, and*
3. *A_i is of low degree for $i = 0, 1$, i.e., $A_i' \equiv_T 0'$.*

Proof
By the limit lemma there exists a recursive function $g : \omega \times \omega \to 2$ such that for every n

$$C(n) = \lim_{s\to\infty} g(s, n).$$

To simplify notation let $C_s(n) = g(s, n)$.

Let $A = \{a_s \ : \ s \in \omega\}$ be a 1-1 recursive enumeration of A. If A is finite or even recursive the result is trivially true, so we don't have to worry about that case. We will achieve the splitting of A by simply putting a_s into exactly one of the two sets A_0 or A_1 at stage $s + 1$.

The lowness of the sets will be achieved by same requirements as in the low simple set proof:

$N_{e,i} \quad (\exists^\infty s \ \{e\}^{A_{i,s}}(e) \downarrow) \to \{e\}^{A_i}(e) \downarrow$

Our new requirements are for each $e \in \omega$ and $i = 0, 1$:

$R_{e,i} \quad \{e\}^{A_i} \neq C$

which we will write $R_q = R_{e,i}$ where $q = 2e + i$. If we meet each of these, then $C \not\leq_T A_i$ for $i = 0, 1$. For each q we will have two variables l_q and u_q

which are the length of agreement and the use of some computations. We will use u_q to satisfy both N_q and R_q.

We use the notation l_q^s and u_q^s to refer to the values of these variables at stage s. At stage $s = 0$ put $A_{i,s} = \emptyset$ and put $u_q = l_q = 0$.

Stage $s+1$.

Begin by computing the length of agreement l_q and the usage u_q for each $q < s+1$:

Suppose $q = 2e + i$.

(a) If $\{e\}_s^{A_{i,s}}(e)) \downarrow$, then:

$$u_q := \max\{u_q, use(\{e\}_s^{A_{i,s}}(e))\}.$$

(b) Next we adjust the length of agreement. There are two cases:

(1) For all $x \leq l_q$

$$\{e\}_s^{A_{i,s}}(x) \downarrow = C_s(x).$$

In this case we bump up the usage and increment l_q:

$$\begin{array}{rcl} u_q & := & \max\{\ u_q, use(\{e\}_s^{A_{i,s}}(x))\ :\ x \leq l_q\ \} \\ l_q & := & l_q + 1 \end{array}$$

(2) Not case (1). In this case we do not change l_q and u_q.

Now we take action. Find the least $q < s+1$ (if any) such that $a_s < u_q$. If $q = 2e + i$, then put a_s into the opposite set, A_{1-i}, i.e.,

$$A_{1-i,s+1} = A_{1-i,s} \cup \{a_s\}.$$

This means we protect the computations above from being injured.

If no such q exists, then put a_s into A_0. This ends the stage and the construction.

Now we verify that the construction works.

Claim. For each q

(1) R_q is met,

(2) $\lim_{s\to\infty} l_q^s = L_q < \infty$,

(3) $\lim_{s\to\infty} u_q^s = U_q < \infty$, and
(4) N_q is met.
Proof
In the case of (2) and (3) since our variables are nondecreasing this just means that at some stage they stop growing. The Claim is proved by induction on q. So suppose it is true for all $q' < q$ and let $R_q = R_{e,i}$

(1) For contradiction assume that R_q is not met, i.e.,

$$\{e\}^{A_i} = C.$$

Subclaim (a). $\lim_{s\to\infty} l_q^s = \infty$.

To see why this is true, note that for any x there will be some stage s_0 where $C_s\restriction x = C\restriction x$ for all $s > s_0$ and also $\{e\}^{A_i}\restriction x$ will be same computations as $\{e\}_{s_0}^{A_{i,s_0}}\restriction x$, i.e., the use of the oracle has settled down. After s_0 the variable l_q will be incremented until it is at least x, if it isn't already. This proves subclaim (a).

Now go to a stage s_0 such that

1. for all $s > s_0$ and for all $q' < q$ $u_{q'}^s = U_{q'}$ and
2. $a_s > \max\{U_{q'} \ : \ q' < q\}$ for all $s > s_0$.

Subclaim (b). If $s > s_0$ is a stage where l_q is incremented then

$$C(x) = \{e\}_s^{A_{i,s}}(x).$$

for any $x < l_q$

To see why this is true, note that u_q protects the computation $\{e\}_s^{A_{i,s}}(x)$ from ever changing since a_s is never beneath $u_{q'}$ for any higher priority $q' < q$. This means that

$$\{e\}_s^{A_{i,s}}(x) = \{e\}^{A_i}(x).$$

But we are assuming $\{e\}^{A_i} = C$. This proves subclaim (b).

Now we get a contradiction to our assumption that C is not recursive. To compute $C(x)$ search for a stage $s > s_0$ where $l_q > x$ and it has just been incremented. Then $C(x) = \{e\}_s^{A_{i,s}}(x)$.

This contradiction proves the main Claim part (1) that R_q is met.

(2) Since R_q is met there exists x such that either
(a) $\{e\}^{A_i}(x)\uparrow$ or
(b) $\{e\}^{A_i}(x)\downarrow\neq C(x)$.
Go to a stage s_0 such that

1. for all $s > s_0$ and for all $q' < q$ $u^s_{q'} = U_{q'}$,
2. $a_s > \max\{U_{q'} : q' < q\}$ for all $s > s_0$, and
3. $C_s(x) = C(x)$ for all $s > s_0$

It is impossible that at some stage $s > s_0$ where $l_q > x$ that l_q is incremented. This is because at s

$$\{e\}^{A_{i,s}}_s(x)\downarrow= C_s(x)$$

but u_q protects the computation $\{e\}^{A_{i,s}}_s(x)$ for the rest of the construction but then

$$\{e\}^{A_i}(x) = \{e\}^{A_{i,s}}_s(x) = C_s(x) = C(x)$$

which contradicts the choice of x.

(3) Note that u_q changes only when either l_q is incremented or when we see $\{e\}^{A_{i,s}}_s(e)$ converges. Hence if we go to a stage s_0 such that

1. for all $s > s_0$ and for all $q' < q$ $u^s_{q'} = U_{q'}$,
2. $a_s > \max\{U_{q'} : q' < q\}$ for all $s > s_0$, and
3. $l^s_q = L_q$ for all $s > s_0$

then u_q will change at most once more, after which it protects the computation $\{e\}^{A_{i,s}}_s(e)$ from changing and never changes again.

(4) The proof that N_q is met is the same as in the low simple set argument.

This ends the proof of the Claim and of the Sacks Splitting Theorem.
QED

Proposition 67 *Suppose $A = A_0 \cup A_1$ is a disjoint union of r.e. sets A_0 and A_1, then $A \equiv_T A_0 \oplus A_1$.*

Proof
Clearly $A = A_0 \cup A_1 \leq_m A_0 \oplus A_1$. To see that $A_i \leq A$, input x and first ask the oracle if $x \in A$. If yes, enumerate A_0 and A_1 until x shows up.
QED

Corollary 68 *(Friedberg Splitting) Every r.e. set which is not recursive is the disjoint union of two r.e. sets which are not recursive.*

Proof
Take $C = A$. Then $A_i \not\geq_T A$ but if either is recursive then by the Proposition we get a contradiction.
QED

Corollary 69 *For every $c \in \mathcal{D}$ if $o < c < o'$, then there exists $a \in \mathcal{R}$ with $a|c$.*

Proof
Let $A = 0'$. By the Theorem $A = A_0 \oplus A_1$ where $C \not\geq_T A_i$ for both $i = 0, 1$. But then at most one of the A_i can be $\leq_T C$, since otherwise

$$0' \equiv_T A_0 \oplus A_1 \leq_T C.$$

QED

Corollary 70 *There exists $a_0, a_1 \in \mathcal{R}$ such that*

$$(a_0 \vee a_1)' \neq a_0' \vee a_1'$$

Proof
By the Theorem there exists low r.e. sets A_i such that $A_0 \oplus A_1 \equiv_T 0'$. Hence

$$a_0' \vee a_1' = o' < o'' = (a_0 \vee a_1)'$$

QED

Corollary 71 *No r.e. degree is minimal, in fact, beneath any nontrivial r.e. degree is a nontrivial low r.e. degree.*

Proof
Given r.e. set A which is not recursive, let $C = A$ and then we have low r.e. sets A_0 and A_1 which split A and $A \not\leq_T A_i$. Then for each i we have that $0 <_T A_i <_T A$.
QED

Hmwk 22. (Fri 10-29) Define f is proper iff f is a partial recursive function and both the domain and range of f are nonrecursive subsets of ω. Prove that for every proper f that there exists proper f_0 and f_1 with f the disjoint union of f_0 and f_1.

Theorem 72 *(Lachlan, Yates) There exists a minimal pair of r.e. degrees, i.e. $a_0, a_1 \in \mathcal{R} \setminus \{o\}$ such that the only degree b with $b \leq a_0$ and $b \leq a_1$ is $b = o$.*

Proof

Requirements:

$P_{e,i}$ $\quad \psi_e \neq A_i$

N_{e_0,e_1} $\quad (\{e_0\}^{A_0} = \{e_1\}^{A_1} = B) \to B$ recursive.

Strategies:

For $P_{e,i}$ wait for $\psi_{e,s}(x) \downarrow= 0$ for some follower x and then put x into A_i.

For N_{e_0,e_1} restrain agreement to get (a) or (b):
(a) for some $l < \omega$ we have that $\{e_0\}^{A_0}\restriction l \downarrow= \{e_1\}^{A_1}\restriction l \downarrow$ and either
($\{e_0\}^{A_0}(l) \uparrow$ or $\{e_1\}^{A_1}(l) \uparrow$) or ($\{e_0\}^{A_0}(l) \downarrow\neq \{e_1\}^{A_1}(l) \downarrow$)
(b) $\{e_0\}^{A_0} = \{e_1\}^{A_1} = B$ and B is recursive by virtue of our restraining certain computations, that is, we can compute B by finding stages where we can be sure the approximate computation at that stage is the final one.

Outcomes:

For $P_{e,i}$ the outcomes are either to wait forever or to act at some time. We order them by { act < wait }.

For N_{e_0,e_1} the outcomes are either $l < \omega$ where l is the largest length of agreement which we see at a true stage or $\{\infty\}$ if the length of agreement has infinite limit. We use the ordering

$$\infty < \cdots < l+1 < l < \cdots < 2 < 1 < 0$$

because it is traditional to take limit infimums (rather than limsups) in the outcome tree to determine the truth path.

The outcomes are $\Lambda = \{\text{act,wait }\} \cup \{\infty\} \cup \omega$. The tree of outcomes is $\Lambda^{<\omega}$. At each stage s in the construction we will have recursively constructed $f_s \in \Lambda^s$ which is an approximation to the true path, i.e., the eventually correct outcomes.

If $\alpha \in \Lambda^n$ where $n = 2\langle e_0, e_1\rangle$ then α works on the requirement N_{e_0,e_1}. If $\beta \in \Lambda^n$ where $n = 2m+1$ and $m = 2e+i$, then β works on the requirement $P_{e,i}$.

Supplementary variables:

For each such β working on a positive requirement we have a restraint variable $R_\beta \in \omega$. Also for each such β we let

$$F_\beta = \{\langle \beta, x\rangle : x \in \omega\}$$

be the followers of β. These could be any pairwise disjoint family of uniformly recursive infinite subsets of ω.

For each α working on a negative requirement we have two variables l_α and u_α (length of agreement and the usage of some computations).

The Construction:

Stage $s = 0$. Put $A_{0,0} = A_{1,0} = \emptyset$ and $f_0 = \langle\rangle$, and put all supplementary variables, $R_\beta, l_\alpha, u_\alpha$ equal to zero.

Stage $s+1$. Given $A_{0,s}, A_{1,s}$, and $f_s \in \Lambda^s$ proceed as follows.

Action:

Look for the least $\beta \subsetneq f_s$ working on a positive requirement $P_{e,i}$ such that
(1) $f_s(|\beta|) =$‘wait’ and
(2) there exist $x > R_\beta$ with $x \in F_\beta$ and $x < s$ such that $\psi_{e,s}(x) \downarrow= 0$.

Put the least such x into A_i, i.e.,

$$A_{i,s+1} = A_{i,s} \cup \{x\}.$$

In this case we say that β and $P_{e,i}$ acted at stage $s+1$. If no such β exists, then no action is taken.

Update variables:

Define $f_{s+1}\upharpoonright n$ for $n \leq s+1$ by induction on n. At the same time we may update the supplementary variables for each $\gamma \subseteq f_{s+1}$.

Case $\beta = f_{s+1}\upharpoonright n$ where β is working on $P_{\hat{e},\hat{\imath}}$.

If $P_{\hat{e},\hat{\imath}}$ has acted at some stage $\leq s+1$ then put $f_{s+1}(n) =$'act'. Otherwise $f_{s+1}(n) =$'wait'.

Define R_β to be the maximum of the following sets:
(1)$\{u_\alpha : \alpha <_{lex} \beta\}$ where $\alpha <_{lex} \beta$ means that there exists k such that $\alpha\upharpoonright k = \beta\upharpoonright k$ and $\alpha(k) < \beta(k)$ in the ordering of outcomes.
(2) $\{u_\alpha : \alpha \subsetneq \beta$ and $\beta(|\alpha|) \neq \infty\}$.

Remarks. β preserves computations of αs which are lexicographically to its left because α's want β's to their right to respect their computations. β also respects computations directly below it except for those which β thinks will have an infinite length of agreement.

Case $\alpha = f_{s+1}\upharpoonright n$ and α is working on N_{e_0,e_1}.

We begin by asking:

Does $\{e_0\}_{s+1}^{A_0,s+1}(x)\downarrow = \{e_1\}_{s+1}^{A_1,s+1}(x)\downarrow$ for every $x \leq l_\alpha$?

If yes, we put $f_{s+1}(n) = \infty$ and we set:

$$\begin{aligned} u_\alpha &:= \max\{u_\alpha, use(\{e_i\}^{A_i,s+1}(x)) : x \leq l_\alpha, i = 0,1\} \\ l_\alpha &:= l_\alpha + 1 \end{aligned}$$

If no, we put $f_{s+1}(n) = l_\alpha$ and make no changes in the variables.

Remarks. If we see expansion in the length of agreement over what it was when last we set it, we guess optimistically that the length of agreement will expand forever. If we don't see this expansion, we pessimistically guess we will never see another expansion. (At least on the stages which go thru α.)

Verification.

We begin by defining the true path $f \in \Lambda^\omega$. We define $f\restriction n$ by induction on n. First let

$$T_n = \{s > n : f\restriction n \subseteq f_s\}$$

these are the true stages and note that $T_n \subseteq T_{n-1}$. The set T_n is a recursive set which (by induction) is infinite. Define $f(n)$ by

$$f(n) = \liminf_{s \in T_n} f_s(n).$$

If $\beta = f\restriction n$ is working on $P_{e,i}$, then $f(n) =$'act' if $P_{e,i}$ every acts, and otherwise $f(n) =$'wait', meaning we wait forever. In the case $\alpha = f\restriction n$ is working on a negative requirement $f(n)$ will be ∞ if there are infinitely many $s \in T_n$ in which the length of agreement l_α has been incremented and otherwise it will be the final value of l_α.

Claim. For each n the requirement that $f\restriction n$ is working on is met.
Proof

Case $f\restriction n = \beta$ is working on $P_{e,i}$.

If $f(n) =$'act', then for some x we put x into A_i at a stage s where we saw $\psi e, s(x) \downarrow= 0$. But then $A_i(x) = 1 \neq \psi_e(x)$.

If $f(n) =$'wait', let us first prove that R_β does not change at any stage $s \geq \min(T_n)$. We first note that for every $s > \min(T_n)$ that it is not true that $f_s <_{lex} \beta$. Why? Suppose $f_s\restriction k = \beta\restriction k$ and $f_s(k) < \beta(k)$. If $\beta(k) =$'wait' and $f_s(k)$='act', then we get a contradiction, since then β is not on the true path f. In the case of a negative requirement $\alpha = \beta\restriction k$ then $\beta(k) = l < \omega$ (since nothing is to the left of ∞), but this would mean that the true path would go to the left of β. It follows that for every $s \in T_n$ the variables $\{u_\alpha : \alpha <_{lex} \beta\}$ will be what they were at the stage $s = min(T_n)$. Similarly for any u_α with $\alpha \subseteq \beta$ and $\beta(|\alpha|) \neq \infty$ these variables will have also reached their maximum since u_α is only changed when l_α is incremented.

To see that $P_{e,i}$ is met in this case let R^*_β be this final value of R_β. Let $x \in F_\beta$ with $x > R^*_\beta$. It is not the case that $\psi_e(x) \downarrow= 0$, because if this ever happened then for some large enough stage $s \in T_n$ the worker β would have acted (either putting this or some smaller x into A_i. Since x is never put into A_i the requirement is met because $\psi_e(x) \neq 0 = A_i(x)$.

Case $f\lceil n = \alpha$ is working on N_{e_0,e_1}.

If $f(n) = l$, then for every $s \in T_{n+1}$ the length of agreement was less than $l+1$, i.e. for some $x \le l+1$ it was not true that:

$$\{e_0\}^{A_{0,s}}(x)\downarrow = \{e_1\}^{A_{1,s}}(x)\downarrow$$

otherwise we would have incremented l_α. It follows that

$$\neg(\{e_0\}^{A_0} = \{e_1\}^{A_1} = B)$$

and so N_{e_0,e_1} is satisfied.

If $f(n) = \infty$, then we claim that B is recursive. To see this suppose $s_1 < s_2$ are successive stages in T_{n+1}. Note that $\alpha = f_{s_1}\lceil n = f_{s_2}\lceil n$ and $f_{s_1}(n) = f_{s_1}(n) = \infty$. This means that l_α was incremented at each stage s_i, say $l-1$ to l at stage s_1 and l to $l+1$ at stage s_2. At stage s_1 before any action the two computations agreed:

$$\{e_0\}^{A_{0,s_1}}_{s_1}\lceil l\downarrow = \{e_1\}^{A_{1,s_1}}_{s_1}\lceil l\downarrow .$$

If $\beta \subseteq f_{s_1}$ is the node which acted at stage s_1 (if any), then it must be that $\alpha \subseteq \beta$ and $\beta(n) = \infty$. This action could destroy either the left side or ride side of this agreement but not both, since some x may be put into A_0 or A_1 but not both. The variable u_α is set to protect the surviving side in subsequent stages. At stages s with $s_1 < s < s_2$ any acting node β must be lexicographically to the right of $\alpha\hat{\ }\infty = f\lceil(n+1)$, i.e., $f\lceil(n+1) <_{lex} \beta$. But this means that $R_\beta \ge u_\alpha$ and so the action at stage s cannot damage the surviving side. At stage s_2 we increment l to $l+1$ which means that the destroyed side must have come back and equaled the surviving side. This means that for each $s \in T_{n+1}$:

$$\{e_0\}^{A_0}\lceil l^s_\alpha = \{e_0\}^{A_{0,s}}_s\lceil l^s_\alpha$$

i.e., the final computation is the computation we see at this stage. Hence to compute $B(x)$ search for a stage $s \in T_{n+1}$ such that $x < l_\alpha$ and then $B(x) = \{e_0\}^{A_{0,s}}_s(x)$. It follows that B is recursive. This proves the Claim and the minimal pair theorem.
QED

Hmwk 23. (Fri Nov 5) Put the low simple non-auto reducible set construction on a tree of outcomes. Prove the construction works. Show that there is no injury on the true path.

Theorem 73 *(Friedberg, Enumeration without repetition) There exists an r.e. set U such that*

1. *$\{U_e : e \in \omega\}$ is the set of all r.e. sets and*
2. *$U_{e_1} \neq U_{e_2}$ for all $e_1 \neq e_2$*

Proof

We will first construct an r.e. set V and then modify it to get U. The requirements are:

$R_e \quad \forall \hat{e} < e \ (W_{\hat{e}} \neq W_e) \rightarrow W_e = V_x$ for some unique x.

The strategy for meeting this requirement is to appoint a follower x. As long as it looks like $\forall \hat{e} < e \ (W_{\hat{e}} \upharpoonright x \neq W_e \upharpoonright x)$ keep enumerating W_e into V_x. Otherwise make it a disloyal follower and put it into the garbage. What do we do with V_x when x is a disloyal follower? We make it into an initial segment.

Definition 74 *$A \subseteq \omega$ is an initial segment iff $A = \emptyset$ or $A = \omega$ or there exists $n < \omega$ such that $A = [0, n] =^{def} \{i < \omega : 0 \leq i \leq n\}$.*

So our modified requirement is:

$R_e \quad$ If $\forall \hat{e} < e \ (W_{\hat{e}} \neq W_e)$ and W_e is not an initial segment, then $W_e = V_x$ for some unique x.

At stage $s + 1$ in our construction we have the following sets:

1. F_s the followers
2. a 1-1 mapping from F_s to ω which tells us that x is the follower of e, say $f_s(x) = e$
3. D_s the disloyal former followers
4. $(V_{x,s} : x \in F_s \cup D_s)$
5. a nondecreasing variable g_s keeping track of last initial segment assigned to a disloyal follower.

The sets F_s and D_s will be disjoint finite sets whose union is an initial segment.

Construction

Stage $s+1$

Let $s = \langle e, ?\rangle$. (So we visit each e infinitely often.)

If no follower is assigned to R_e, let $x = min(\overline{F_s \cup D_s})$ and assign x to be the follower of R_e. Put $F_{s+1} = F_s \cup \{x\}$ and end the stage.

If x is the follower of R_e and

1. $\forall \hat{e} < e$
$$W_{\hat{e},s+1} \cap [0.x] \neq (W_{e,s+1}) \cap [0,x]$$
2. $W_{e,s+1} \cap [0,x]$ is not an initial segment

then put $V_{x,s+1} = V_{x,s} \cup W_{e,s+1}$ and end the stage. Actually in this case $V_{x,s} \subseteq W_{e,s}$ so we could have said put $V_{x,s+1} = W_{e,s+1}$.

If x is the follower of R_e and either of those two conditions fails then

1. change x into a disloyal follower, i.e., $F_{s+1} = F_s \setminus \{x\}$ and $D_{s+1} = D_s \cup \{x\}$,
2. let g_{s+1} be the minimum $g > g_s$ such that $V_{e,s} \subseteq [0,g]$, and
3. permanently assign V_x to be $[0, g_{s+1}]$, i.e., set $V_{x,s+1} = [0, g_{s+1}]$ and never change V_x again.

End the stage.

Verification

Claim. The following are equivalent for any e:

1. For each e if W_e is not an initial segment of ω and $W_e \neq W_{\hat{e}}$ for each $\hat{e} < e$.
2. R_e obtains a permanent follower x and hence $V_x = W_e$.

Proof

Suppose condition 2 holds. Then R_e obtains a permanent follower x. Then for all stages $s+1$ after x is appointed and for which $s = \langle e, ?\rangle$, we have that $W_{e,s} \cap [0,x]$ is not an initial segment and $W_{e,s} \cap [0,x] \neq W_{\hat{e},s} \cap [0,x]$ for each $\hat{e} < e$. Condition (1) follows since there are infinitely many such stages.

Suppose that condition 1 holds. Choose y so that $W_e \cap [0, y]$ is not an initial segment and

$$W_e \cap [0, y] \neq W_{\hat{e}} \cap [0, y]$$

for every $\hat{e} < e$. Go to some stage s_0 where

$$W_{e,s_0} \cap [0, y] = W_e \cap [0, y]$$

and

$$W_{\hat{e},s_0} \cap [0, y] = W_{\hat{e}} \cap [0, y]$$

for every $\hat{e} < e$. If R_e has no permanent follower then infinitely many followers are appointed to it. Hence some follower $x > y$ will be appointed after stage s_0. But such a follower will always remain loyal.
QED

Let $D = \cup_{s \in \omega} D_s$ be the set of disloyal followers. Then $\overline{D}$ is the set of permanent followers.
Claim.

1. $\{V_x \, : \, x \in \overline{D}\}$ is the set of r.e. sets which are not initial segments.

2. There exist a recursive set G such that

$$\{[0, n] \, : \, n \in G\} = \{V_x \, : \, x \in D\}.$$

3. $V_x \neq V_{x'}$ unless $x = x'$.

Proof

Part (1) follows from the first Claim.

For Part (2), since the sequence g_s is non-decreasing we see that

$$G = \{g_s \, : \, s \in \omega\}$$

is recursive.

For Part (3) note that there are two types of V_x. If x is a permanent follower of some R_e and then $V_x = W_e$ where W_e is not an initial segment and W_e is distinct from each $W_{\hat{e}}$. Or x is a disloyal follower at some stage $s+1$ and then $V_x = [0, g_{s+1}]$. Since the sequence g_s is bumped up each time it is used we see that the V_x for disloyal followers are distinct finite initial segments. This proves Claim.

QED

Let us show how to modify V to U to prove Friedberg's enumeration without repetition theorem. Note that V uniquely enumerates every r.e. set except ω, $\emptyset$, and the finite initial segments of the form $[0, n]$ where $n \notin G$. Let $\{x_n : 1 < n < \omega\}$ be a 1-1 recursive enumeration of $\overline{G}$. Now define U by $U_0 = \omega$, $U_2 = \emptyset$, $U_{2n} = [0, x_n]$ for $n > 1$, and $U_{2n+1} = V_n$.
QED

Hmwk 24. (Mon Nov 8)

(a) Prove there exists V r.e. such that
$\{V_e \ : \ e \in \omega\}$ = set of r.e. non-simple sets.

(b) Prove there exists U r.e. such that
$\{U_e \ : \ e \in \omega\}$ = set of r.e. non-simple sets and $U_{e_1} \neq U_{e_2}$ unless $e_1 = e_2$.

Definition 75 *Coding finite sets. For* $D \subseteq \omega$ *let* $x = \sum_{n \in D} 2^n$. *Write* $D_x = D$.

Definition 76 $(D_x \ : \ x \in R)$ *is a strong array iff* R *is an infinite recursive set and for every* $x, y \in R$ *we have* $D_x \cap D_y = \emptyset$ *whenever* $x \neq y$.

Definition 77 *A set* $A \subseteq \omega$ *is hypersimple iff* A *is r.e. ,* $\overline{A}$ *is infinite, and for every strong array* $(D_x \ : \ x \in R)$ *there exists* $x \in R$ *such that* $D_x \subseteq A$.

Proposition 78 *(Post)*

(1) Hypersimple implies simple.

(2) There is a simple set which is not hypersimple.

(3) There is a hypersimple set.

Proof

(1) If A is not simple, then there exists an infinite recursive set $R \subseteq \overline{A}$. Then $\{D_{2^x} \ : \ x \in R\}$ witnesses that A is not hypersimple.

(2) In Post's original construction of a simple set A (see Theorem 35) we constructed a simple set A by waiting until there was some $x \in W_{e,s}$ with $x > 2e$ and $W_{e,s} \cap A_s = \emptyset$ and then putting x into A. The reason that $\overline{A}$ was infinite was because for every e we had that $|[0, 2e] \cap A| \leq e$. This means that for every a we have that

$$[a, 4a] \cap \overline{A} \neq \emptyset$$

because $[a, 4a]$ is 3/4 of the interval $[0, 4a]$. So define $a_0 = 5$ and $a_{n+1} = 4a_n + 1$. Take x_n so that $D_{x_n} = [a_n, 4a_n]$ and note that $D_{x_n} \cap \overline{A} \neq \emptyset$ for each n so the recursive set $R = \{x_n \; : \; n < \omega\}$ witnesses that A is not hypersimple.
(3) This is a consequence of the following proposition, although originally Post gave a construction similar to his construction of a simple set.
QED

Proposition 79 *(Dekker) Deficiency sets are hypersimple.*

Proof
See Theorem 37. Suppose that $A = \{a_s \; : \; s \in \omega\}$ is a 1-1 recursive enumeration of A and A is not recursive. Define

$$D = \{s \; : \; \exists t > s \quad a_t < a_s\}.$$

As we saw before $A \equiv_T D$ and D is simple. A similar proof will show that D is hypersimple.

Suppose for contradiction that there exists a strong array $(D_x \; : \; x \in R)$ such that $D_x \cap \overline{D} \neq \emptyset$ for every $x \in R$.

Now we get a contradiction by showing that A is computable.

Input u. Find an $x \in R$ such that

$$u < \min\{a_s \; : \; s \in D_x\}.$$

Such an x exists, since a_s is a 1-1 enumeration and the D_x are pairwise disjoint. But now at least one of $t \in D_x$ is not deficient, so for all $s > t$ we have $a_s > a_t$. Hence $u \in A$ iff $u = a_s$ for some $s \leq \max D_x$.
QED

Hmwk 25. (Wed Nov 10) Define A to be bdd-hypersimple iff A is r.e. , $\overline{A}$ is infinite, and for every strong array $(D_x \; : \; x \in R)$ such that there exists $N < \omega$ such that $|D_x| \leq N$ for all $x \in R$, there exists $x \in R$ such that $D_x \subseteq A$. Prove that bdd-hypersimple is equivalent to simple.

Definition 80 *For any set $A \subseteq \omega$ such that $\overline{A}$ is infinite define $g_A \in \omega^\omega$ by $g_A(n)$ is the $(n+1)^{th}$ element of $\overline{A}$, i.e.,*

$$\overline{A} = \{g_A(0) < g_A(1) < \cdots < g_A(n) < \cdots\}$$

Proposition 81 *For any r.e. set A with $\overline{A}$ infinite the following are equivalent:*

1. *A is hypersimple.*
2. *For any recursive increasing sequence $n_k < n_{k+1}$ there are infinitely many k with $[n_k, n_{k+1}) \subseteq A$.*
3. *For any recursive $f \in \omega^\omega$ there are infinitely many k such that $f(k) < g_A(k)$.*

Proof

(1) → (2). This is clear since if $D_{x_k} = [n_k, n_{k+1})$, then $R = \{x_k : k < \omega\}$ is a strong array. There are infinitely many since $R(l) =^{def} \{x_k : k > l\}$ is a strong array for any l.

(2) → (3). Given a recursive f construct a recursive sequence $n_{k+1} > n_k$ with the property that $f(n_k + 1) < n_{k+1}$ for each k. For any k such that $[n_k, n_{k+1}) \subseteq A$ note that $\overline{A} \cap [0, n_{k+1}) \subseteq [0, n_k)$ and so $g_A(n_k+1) = (n_k+1)^{th}$ element of $\overline{A}$ must be greater than n_{k+1}. Hence $f(n_k + 1) < g_A(n_k + 1)$.

(3) → (1). Suppose A is not hypersimple and hence there exists a strong array $(D_x : x \in R)$ such that $D_x \cap \overline{A} \neq \emptyset$ for all $x \in A$. Let $\{x_n : n \in \omega)$ be a 1-1 recursive enumeration of R and define

$$f(n) = 1 + \max(\cup_{m \leq n} D_{x_m})$$

Then $|\overline{A} \cap [0, f(n))| > n$ and so f dominates g_A.
QED

Hmwk 26. (Fri 11-12) Prove that for every r.e. set $A \subseteq \omega$ if $\overline{A}$ is infinite, then there exists a hypersimple set $B \supseteq A$.

Consider propositional logic with the set of atomic letters

$$\{P_n : n \in\}.$$

For any propositional sentence ψ and subset $A \subseteq \omega$ define

$$A \models \psi$$

inductively by

$$A \models P_n \text{ iff } n \in A$$

$$A \models \neg\psi \text{ iff } \text{ not } A \models \psi$$

$$A \models (\psi \vee \theta) \text{ iff } (A \models \psi \text{ or } A \models \theta)$$

and so forth for the other logical symbols.

By coding symbols as elements of ω and thinking of sentences as strings of symbols or finite sequences of elements of ω, we identify the set of propositional sentences with a recursive subset of ω, SENT. The details of this coding are left to the reader.

The following notion is known as truth-table (tt) reducibility.

Definition 82 *$A \leq_{tt} B$ iff there exists a recursive sequence*

$$(\theta_n \in SENT \ : \ n \in \omega)$$

such that for all $n \in \omega$

$$n \in A \textit{ iff } B \models \theta_n$$

Note: It is easy to see that $A \leq_{tt} C$ and $B \leq_{tt} C$ implies $(A \cap B) \leq_{tt} C$ and $\overline{A} \leq_{tt} C$. Hence the family of sets which are truth-table reducible to C is closed under finite boolean combinations. It is easy to see that $\leq_m$-reducible is stronger than $\leq_{tt}$, and $\leq_{tt}$ is stronger than $\leq_T$.

Proposition 83 *(Nerode) The following are equivalent:*

1. *$A \leq_{tt} B$.*

2. *There exist e with the property that*

$$\forall X \, \forall x \, \{e\}^X(x) \downarrow$$

   and $\{e\}^B = A$.

3. *There exists e and $f \in \omega^\omega$ recursive such that*

$$\forall x \, \{e\}^B_{f(x)}(x) \downarrow$$

   and $\{e\}^B = A$.

Proof

(1) $\rightarrow$ (2). Given $(\theta_n \ : \ n \in \omega)$ witnessing that $A \leq_{tt} B$, it is easy to construct an oracle machine e such that for any input x and oracle X that $\{e\}^X(x) \downarrow = 1$, if $X \models \theta_x$ and $\{e\}^X(x) \downarrow = 0$, if $X \models \neg\theta_x$.

(2) $\rightarrow$ (3). We show that the same e works. Input x and let

$$T_x = \{\sigma \in 2^{<\omega} \ : \ \{e\}^{\sigma}_{|\sigma|}(x) \uparrow\}.$$

The trees T_x are uniformly recursive in x. By Konig's tree lemma, since T_x has no infinite branch, it is finite. Therefor we can compute the least n such that for all $\sigma \in 2^n$ we have that $\sigma \notin T_x$. Put $f(x) = n$.

(3) $\rightarrow$ (1). Input x. Compute a use bound u_x so that for every possible computation $\{e\}^{?}_{f(x)}(x)$ the computation only asks about $i < u_x$. (Since it takes at least one step to ask the oracle anything there are at most $2^{f(x)}$ such simulations.)

Now define

$$t_x = \{R \subseteq [0, u_x] \ : \ \{e\}^{R}_{f(x)}(x) \downarrow = 1\}.$$

Define

$$\theta_x = \bigvee_{R \in t_x} (\bigwedge_{i \in R} P_i \ \wedge \ \bigwedge_{i \in [0,u_x] \setminus R} \ \neg P_i)$$

Then for any $x \in \omega$ we have that

$$x \in A \text{ iff } \{e\}^{B}_{f(x)}(x) \downarrow = 1 \text{ iff } B \cap [0, u_x] = R \in t_x \text{ iff } B \models \theta_x.$$

QED

Proposition 84 *(Post)*

1. *If A is simple, then $A <_m K$.*
2. *If A is hypersimple, then $A <_{tt} K$.*
3. *There exists a simple A with $A \equiv_{tt} K$.*

Proof

(1) If $K \leq_m A$ then A is creative and hence not simple. (See Theorem 33.)

(2) Since every r.e. set is many-one reducible to K it is enough to see that $K \leq_{tt} A$ implies A is not hypersimple.

Claim. Let $\Gamma = \{P_n : n \in A\}$. Then there exists a recursive list $(\rho_n : n < \omega)$ of propositional sentences such that for every n

1. $A \models \rho_n$ and

2. $\Gamma \cup \{\rho_m : m < n\} \not\vdash \rho_n$.

Proof
Since $\overline{K} \leq_{tt} A$ there exists a recursive function $\theta : \omega \to SENT$ such that $n \in \overline{K}$ iff $A \models \theta(n)$.

Now we effectively construct ρ_n as follows. Let

$$\Sigma_n = \{\rho : \Gamma \cup \{\rho_m : m < n\} \vdash \rho\}.$$

Note that Σ_n is recursively enumerable as a subset of SENT. Also $A \models \theta$ for every $\theta \in \Sigma_n$. It follows that $\theta^{-1}(\Sigma_n) \subseteq \overline{K}$ is r.e. By the S-n-m Theorem there exists a recursive function f such that

$$W_{f(n)} = \theta^{-1}(\Sigma_n)$$

and by the proof that K is creative we have that

$$f(n) \in \overline{K \cup \theta^{-1}(\Sigma_n)}.$$

Take $\rho_n = \theta(f(n))$.
QED

Let S_k be that set of all n such that the propositional letter P_n occurs in the sentence ρ_k, i.e., S_k is the support of ρ_k.

Claim. For any n let

$$m = \max\left(\bigcup\{S_k \;:\; k \leq 2^{2^{n+1}} + 1\}\right)$$

then $\overline{A} \cap [n, m) \neq \emptyset$.
Proof
Suppose not and assume that $[n, m) \subseteq A$. Let ρ_k^* be obtained from ρ_k by replacing all propositional letters P_i for $n < i < m$ by the letter P_n. Note that $\Gamma \vdash P_i$ for all these i and hence $\Gamma \vdash \rho_k^* \equiv \rho_k$ for every $k \leq 2^{2^{n+1}} + 1$. But there are at most $2^{2^{n+1}}$ logically inequivalent propositional sentences with atomic letters P_i for $i \leq n$ and so for some $k < l$ we have that $\rho_k^* \equiv \rho_l^*$. But this is a contradiction since then

$$\Gamma \vdash \rho_i \equiv \rho_j.$$

QED

Now it is an easy matter to construct a recursive sequence $n_k < n_{k+1}$ so that $\overline{A} \cap [n_k, n_{k+1}) \neq \emptyset$ for each k. Hence A is not hypersimple.

(3) Let B be any simple set which is not hypersimple. By Proposition 81 there exists a recursive increasing sequence $(n_k : k < \omega)$ such that for all k we have that $\overline{B} \cap [n_k, n_{k+1}) \neq \emptyset$. Now let

$$A = B \cup \bigcup_{k \in K} [n_k, n_{k+1})$$

A is simple because it is a superset of the simple set B. $\overline{A}$ is infinite because for each $k \in \overline{K}$ we have $\overline{A} \cap [n_k, n_{k+1}) \neq \emptyset$. We have that $K \leq_{tt} A$ because

$$k \in K \text{ iff } A \models \bigwedge\!\!\!\bigwedge_{n_k \leq i < n_{k+1}} P_i$$

QED

Definition 85 *V is a weak array iff V is r.e. and $V_x \cap V_y = \emptyset$ whenever $x \neq y$. As usual, $V_x = \{y : \langle x, y \rangle \in V\}$.*

Definition 86 *$A \subseteq \omega$ is hyperhypersimple iff A is re, $\overline{A}$ is infinite, and for every weak array V there exists x with $V_x \subseteq A$.*

Proposition 87 *For any $A \subseteq \omega$ for which A is r.e. and $\overline{A}$ is infinite the following are equivalent:*

1. *A is hyperhypersimple*
2. *for every infinite r.e. set B such that $W_x \cap W_y = \emptyset$ for all distinct $x, y \in B$ there exists $x \in B$ with $W_x \subseteq A$*
3. *for every weak array V there exists an infinite recursive set R such that $V_x \subseteq A$ for all $x \in R$*
4. *for every weak array V such that V_x is finite for all x there exists x such that $V_x \subseteq A$*

Proof

(1) iff (2) is true because the two types of arrays are the same.

(1) $\to$ (3), The sequence $(R_n = \{\langle n, m\rangle : m \in \omega\} : n < \omega)$ is a uniformly recursive partition of ω into infinite pieces. Take

$$U_n = \cup_{e \in R_n} V_e$$

Then U is weak array and so there exists n with $U_n \subseteq A$.

(4) $\to$ (1). Given a weak array V such that $V_e \cap \overline{A} \neq \emptyset$ for all e we find another weak array V^* such that V_e^* finite and $V_e^* \cap \overline{A} \neq \emptyset$ for all e. For each s define $V_{e,s}^* = V_{e,s_0+1}$ where s_0 is the largest $t \leq s$ such that $V_{e,t} \subseteq A_s$.
QED

Hmwk 27. (Mon 11-15) Prove

(a) If A is simple and B is simple, then $A \cap B$ is simple.

(b) If A is hypersimple and B is hypersimple, then $A \cap B$ is hypersimple.

(b) If A is hyperhypersimple and B is hyperhypersimple, then $A \cap B$ is hyperhypersimple.

Example 88 *There exists a hypersimple set A which is not hyperhypersimple.*

Proof
Let $B \subseteq \omega$ be any hypersimple set. Define $A \subseteq \omega$ by

$$A = \{\langle n, m\rangle : n \in B \text{ or } n \leq m\}.$$

A is not hyperhypersimple since each of the sets $V_n =^{def} \{\langle m, n\rangle \ : \ m \in \omega\}$ meets $\overline{A}$. To see that A is hypersimple suppose we are given a strong array $(D_n : n \in R)$. Let $pi(\langle m, n\rangle) = m$ be projection to the first coordinate. We can find an infinite recursive subset $S \subseteq R$ such that $(\pi(D_x \cap Q) : x \in S)$ are pairwise disjoint where $Q = \{\langle n, m\rangle : m < n < \omega\}$. Since B is hypersimple, there exists $x \in S$ with $\pi(D_x \cap Q) \subseteq B$ and hence $D_x \subseteq A$.
QED

Example 89 *Dekker deficiency sets are never hyperhypersimple.*

Proof

Let $A = \{a_s : s \in \omega\}$ be a one-one recursive enumeration of a non recursive set A. And $D = \{s : \exists t > s \;\; a_t < a_s\}$. We construct a weak array V to meet the requirements:

$R_x \qquad V_x \cap \overline{D} \neq \emptyset$

Stage s+1

Step (a). For any $x \leq s$ if R_x has a follower t such that $a_s < a_t$ then unappoint t so that now R_x has no follower.

Step(b). For the least x for which R_x has no follower, appoint s the follower of R_x and put $V_{x,s+1} = V_{x,s} \cup \{s\}$.

This ends the stage and the construction. Note that V is a weak array.

Claim. Each R_x obtains a permanent follower s and for this s we have $s \in V_x \cap \overline{D}$.

Proof

This is by induction on x. So after some sufficiently large stage s_0 no $y < x$ is appointed a new follower. Suppose for contradiction that R_x is appointed a new follower at stages $s_1, s_2, \ldots$ where $s_0 < s_1 < s_2 < \cdots$. Note that since higher priority requirements don't get new followers after s_0 each time R_x losses its follower it acquires the stage itself as its new follower. But this means that

$$a_{s_1} > a_{s_2} > a_{s_3} > \cdots$$

which is a contradiction.

QED

Definition 90 *$A \subseteq^* B$ iff $B \setminus A$ is finite.*

$A =^* B$ iff $A \subseteq^* B$ and $B \subseteq^* A$

$\forall^\infty$ means 'for all but finitely many'

$\exists^\infty$ means 'exists infinitely many'

Definition 91 *$M \subseteq \omega$ is maximal iff M is r.e. , $\overline{M}$ is infinite, and for every A r.e. if $M \subseteq A$ then $M =^* A$ or $A =^* \omega$.*

Proposition 92 *Maximal implies hyperhypersimple.*

Suppose V is a weak array such that $V_e \cap \overline{A} \neq \emptyset$ for all e. Define

$$B = A \cup \bigcup_{e<\omega} V_{2e}$$

then $A \neq^* B$ and $B \neq^* \omega$, so A is not maximal.
QED

Theorem 93 *(Friedberg) Maximal sets exist.*

Proof
We will construct the maximal set M as follows. We use the notation p_n for the n^{th} element of the complement of M, i.e.,

$$\overline{M} = \{p_0 < p_1 < p_2 < \cdots\}$$

Are requirements are

$$R_e \qquad (\forall^\infty n \;\; p_n \in W_e) \quad \text{or} \quad (\forall^\infty n \;\; p_n \notin W_e)$$

This guarantees that $M \cup W_e =^* \omega$ or $M \cup W_e =^* M$,

At stage s given M_s we let

$$\overline{M_s} = \{p_{0,s} < p_{1,s} < p_{2,s} < \cdots\}$$

The idea of this proof is called moving markers. We think of a marker labeled n with position $p_{n,s}$. As we slide the marker upward we put the uncovered numbers into M_s. In order to get $\overline{M}$ infinite we want each marker to eventually stop moving.

Definition 94 *$\sigma \in 2^n$ is the n-state of x at stage s iff*

$$\textit{for all } e < n \qquad \sigma(e) = \begin{cases} 1 & \textit{if } x \in W_{e,s} \\ 0 & \textit{if } x \notin W_{e,s} \end{cases}$$

Two easy facts about the n-state are the following:

(1) Suppose $s_1 \leq s_2$,
$\sigma_1 \in 2^n$ is the n-state of x at stage s_1, and
$\sigma_2 \in 2^n$ is the n-state of x at stage s_2,
then $\sigma_1 \leq_{lex} \sigma_2$.

(2) For fixed n and x there is $\sigma \in 2^n$ such that σ is the n-state of x for all but finitely many stages s. We call this the final n-state of x.

Our strategy can be summarized simply as 'maximize the lexicographic order of the n-state of p_n'.

Stage $s+1$.
Find the least n (if any) such that there exists m with $n < m < s$ such that
if $\sigma \in 2^n$ is the n state of $p_{n,s}$ and
$\tau \in 2^n$ is the n state of $p_{m,s}$, then $\sigma <_{lex} \tau$.
For the least such n find the least m and shift the marker n to m:
Put $p_{n+i,s+1} = p_{m+i,s}$ for all $i < \omega$. Equivalently put

$$M_{s+1} = M_s \cup \{p_{j,s} : n \leq j < m\}$$

Otherwise as usual if there are no such n, m just go to the next stage with everything unchanged.

This ends the stage and the construction.

Claim. The markers eventually stop moving, i.e.,

$$\lim_{s \to \infty} p_{n,s} = p_n < \infty$$

Proof
This is proved by induction on n. Note that the only way the marker n moves is either that it is bumped up by some marker $m < n$ or it moves to a higher n-state. So consider some stage s_0 so that no marker $m < n$ moves after stage s_0. But it is impossible for p_n to change infinitely many times after this since its n-state would have to increase lexicographically infinitely many times. (Note that in between moves its n-state might also change without the marker moving but it can only increase if it doesn't move.)
QED

Claim. For each n there exists $\tau \in 2^n$ such that

$$\forall^\infty m \;\; \tau = \text{ the final } n\text{-state of } p_m.$$

Proof
Suppose not. Then there exists distinct $\tau_1, \tau_2 \in 2^n$ such that
$\exists^\infty m \;\; \tau_1 =$ the final n-state of p_m and
$\exists^\infty m \;\; \tau_2 =$ the final n-state of p_m.
Suppose $\tau_1 <_{lex} \tau_2$. Then we can choose m_1, m_2 with $n < m_1 < m_2$ and the final n-state of p_{m_i} is τ_i. This is a contradiction, since for some large enough stage $s_0 > m_2$ the markers p_j for $j \leq m_2$ have stopped moving and their final n-states are their states at stage s_0. But by the construction some marker $\leq p_{m_1}$ must move.

QED

This final claim proves the Theorem, since if $n = e + 1$ we have that $\tau(e) = 1$ implies $\forall^\infty m \;\; p_m \in W_e$

and

$\tau(e) = 0$ implies $\forall^\infty m \;\; p_m \notin W_e$

QED

Example 95 *There exists a hyperhypersimple set which is not maximal.*

Proof

First we note that it easy to get M_1 and M_2 maximal so that $M_1 \neq^* M_2$. Take any maximal set M and let $R \subseteq M$ to be an infinite recursive subset. Let $\pi : \omega \to \omega$ be a recursive bijection which takes R to $\overline{R}$. Let $M_1 = M$ and let $M_2 = \pi(M_1)$.

Now let $A = M_1 \cap M_2$. Then A is hyperhypersimple (see exercise) but not maximal since $A \subseteq M_1 \subseteq \omega$ and $A \neq^* M_1$ and $M_1 \neq^* \omega$.

QED

Remark. Yates noted that we can add to the maximal set construction an extra 'kick' to the p_e marker to ensure that $\{e\}(e) \downarrow$ iff $\{e\}_{p_e}(e)$. Then the maximal set constructed will be Turing equivalent to K.

Hmwk 28. (Wed 11-17) Suppose $A = \{a_n : n < \omega\}$ is a 1-1 recursive enumeration of a hyperhypersimple set A. Let $B = \{a_{a_n} : n < \omega\}$. Prove that B is hyperhypersimple but not maximal.

Hmwk 29. (Fri 11-19) An r.e. set $A \subseteq \omega$ is simple in R where R is an infinite recursive set iff $\overline{A} \cap R$ is infinite but contains no infinite r.e. subset. Is every r.e. set which is not recursive simple in some infinite recursive set? Hint: Consider a Friedberg splitting of a Maximal set.

Definition 96 *The lattice of r.e. sets is $\mathcal{E} = (r.e.sets, \subseteq)$. A subset $X \subseteq \mathcal{E}$ is definable iff there is a first order formula $\theta(v)$ in the language of $\subseteq$ such that*

$$X = \{A \in \mathcal{E} \;:\; \mathcal{E} \models \theta(A)\}.$$

Similarly for $X \subseteq \mathcal{E}^2$ or $X \subseteq \mathcal{E}^3$.

Example 97 *The following are definable in $\mathcal{E}$.*

1. $\{(A, B, C) \in \mathcal{E}^3 \;:\; A \cup B = C\}$

2. $\{(A, B, C) \in \mathcal{E}^3 \ : \ A \cap B = C\}$

3. $\{\emptyset\}$

4. $\{\omega\}$

5. *recursive sets*

   A is recursive iff $\mathcal{E} \models \exists B \ \ B \cap A = \emptyset$ *and* $B \cup A = \omega$

6. *r.e. but not recursive sets*

7. *infinite r.e. sets*

   A is infinite r.e. iff $\mathcal{E} \models \exists B \ \ B \subseteq A$ *and B is not recursive*

8. *finite sets*

9. *cofinite*

10. *simple sets*

11. *maximal sets*

Definition 98 *π is an automorphism of $\mathcal{E}$ iff $\pi : \mathcal{E} \to \mathcal{E}$ is a bijection such that for every $A, B \in \mathcal{E}$*

$$A \subseteq B \text{ iff } \pi(A) \subseteq \pi(B).$$

Note that for any first-order formula $\theta(v_1, \ldots, v_n)$ in the language of $\mathcal{E}$, i.e., $\subseteq$, that for any $\pi \in aut(\mathcal{E})$ and $A_1, \ldots, A_n \in \mathcal{E}$ we have that

$$\mathcal{E} \models \theta(A_1, \ldots, A_n) \text{ iff } \mathcal{E} \models \theta(\pi(A_1), \ldots, \pi(A_n))$$

Hence definable sets are closed under automorphisms.

Example 99 *If $A \in \mathcal{E}$, then $\{A\}$ is definable in $\mathcal{E}$ iff $A = \emptyset$ or $A = \omega$.*

Proof
If A is neither $\emptyset$ or ω, then we can choose $n, m < \omega$ such that $n \in A$ and $m \notin A$. Let $\pi : \omega \to \omega$ be the identity except $\pi(n) = m$ and $\pi(m) = n$. Define $\pi : P(\omega) \to P(\omega)$ by $\pi(A) = \{\pi(n) : n \in A\}$. Then since π is recursive it is clear that $\pi \in aut(\mathcal{E})$. But since

$$\pi(A) = (A \setminus \{n\}) \cup \{m\}$$

we see that $\{A\}$ is not closed under automorphisms and hence cannot be definable.
QED

Proposition 100 *1. For every $\pi \in aut(\mathcal{E})$ there exists a bijection $\hat{\pi}$ of ω such that $\pi(A) = \{\hat{\pi}(n) : n \in A\}$.*

2. Not every bijection $\pi : \omega \to \omega$ induces an automorphism of $\mathcal{E}$.

3. There are continuum many bijections $\pi : \omega \to \omega$ which induce an automorphism of $\mathcal{E}$.

Proof
(1) It is easy to see that the set of singletons

$$\{\{n\} : n \in \omega\} \subseteq \mathcal{E}$$

is definable in $\mathcal{E}$. Hence any automorphism $\pi : \mathcal{E} \to \mathcal{E}$ must permute the singletons. Define $\hat{\pi}(n)$ so that $\pi(\{n\}) = \{\hat{\pi}(n)\}$. But now for every $n \in \omega$

$$n \in A \text{ iff } \{n\} \subseteq A \text{ iff } \pi(\{n\}) \subseteq \pi(A) \text{ iff } \hat{\pi}(n) \in \pi(A)$$

Hence $\pi(A) = \{\hat{\pi}(n) : n \in A\}$.
(2) Take any bijection which maps the even integers to some non recursive infinite coinfinite set.
(3) Let M be a maximal set. Let $\pi : \omega \to \omega$ be any bijection such that $\pi \restriction M = id$. There are continuum many such bijections, one for each permutation of $\overline{M}$. But for any $A \in \mathcal{E}$ we have that $A \cap \overline{M}$ is finite or $A \cap \overline{M} = *\overline{M}$. But this gives us that $\pi(A) =^* A$. Similarly $\pi^{-1}(A) =^* A$.
QED
The following theorem shows that the family of hyperhypersimple sets is definable in $\mathcal{E}$.

Theorem 101 *(Lachlan) A is hyperhypersimple iff A is r.e. , $\overline{A}$ is infinite, and*

$$\mathcal{E} \models \forall B \supseteq A \;\; \exists C \supseteq A \;\; B \cap C = A \text{ and } B \cup C = \omega$$

Proof

Suppose A is not hyperhypersimple and V is a weak array such that $V_e \cap \overline{A} \neq \emptyset$ for all e. Define

$$B = A \cup \bigcup_{e \in \omega} (V_e \cap W_e)$$

Suppose for contradiction that C satisfies $B \cap C = A$ and $B \cup C = \omega$. Then for some e we have that $C = W_e$. Let $x \in V_e \cap \overline{A}$. If $x \in W_e$ then $x \in C \cap B$ but this contradicts $B \cap C = A$. If $x \notin W_e$ then $x \notin C$ and $x \notin B$ but this contradicts $B \cup C = \omega$.

Conversely suppose there exists B as above for which there is no C. We must show there is a weak array V such that $V_e \cap \overline{A} \neq \emptyset$ for all e. So let $B = \{b_s : s \in \omega\}$ be a 1-1 recursive enumeration of B and put $B_s = \{b_t : t < s\}$. Similarly, let A_s be a recursive enumeration of A.

We will construct $V_{e,s}$ pairwise disjoint subsets of B and meet the requirements:

$R_e \qquad V_e \cap \overline{A} \neq \emptyset$

We will carry along $g(e, s)$ a gate which we use to let elements into each V_e. At stage $s = 0$ as usual we put $V_{e,s} = \emptyset$.

Stage $s + 1$**.**
First define

$$g(e, s+1) = \begin{cases} g(e, s) & \text{if } V_{e,s} \cap \overline{A_s} \neq \emptyset \\ s+1 & \text{otherwise} \end{cases}$$

Look for the least $e < s$ such that $b_s \leq g(e, s+1)$ and put b_s into V_e, i.e.,

$$V_{e,s+1} = V_{e,s} \cup \{b_s\}$$

Claim. $\lim_{s\to\infty} g(e, s) = g(e) < \infty$ and R_e is met.
Proof
This is proved by induction on e. Choose s_0 so that for all $\hat{e} < e$ and $s > s_0$ we have that $g(\hat{e}, s) = g(\hat{e})$ and

$$b_s > \max\{g(\hat{e}) : \hat{e} < e\}$$

Suppose for contradiction that

$$\lim_{s\to\infty} g(e, s) = \infty$$

Define

$$C = A \cup \bigcup_{s \geq s_0} ([0, g(e, s+1)] \cap \overline{B_s}$$

Suppose $x \in \overline{A}$. Then we claim that

$$x \in C \text{ iff } x \in \overline{B}$$

This is a contradiction since then $C \cap B = A$ and $C \cup B = \omega$.

Suppose $x \in \overline{B}$. This implies that $x \in \overline{B_s}$ for all s. But if $g(e, s) \to \infty$ we have that $x \in C$.

Suppose $x \in C$. Then for some $s \geq s_0$ we have that $x \in [0, g(e, s)] \cap \overline{B_s}$ (since we are assuming $x \notin A$.) If $x \notin \overline{B}$ then $x \in B \setminus B_s$. Hence $x = b_t$ for some $t \geq s$. But notice that $b_t = x \leq g(e, s) \leq g(e, t+1)$. By our choice of s_0 we have that $b_t > g(\hat{e})$ for all $\hat{e} < e$ and so b_t will be put into V_e. But $x = b_t$ was assumed to be an element of $\overline{A}$. This means that $g(e, t)$ will never increase again which contradicts it going to ∞.

The reason R_e is met is because if $g(e, s)$ stops growing then eventually we stop putting b_s's into V_e. Hence V_e is finite and so it is impossible that $V_e \subseteq A$.

This proves the Claim and the Theorem.

QED

The following shows that the family of hypersimple sets is not definable in $\mathcal{E}$.

Theorem 102 *(Martin) There exists a hypersimple set A and $\pi \in aut(\mathcal{E})$ such that $\pi(A)$ is not hypersimple.*

Proof

We will construct the r.e. set A as usual by constructing a recursive increasing sequence A_s. We will construct a recursive sequence π_s of bijections of ω with the property that $\pi_s(n) = n$ for every $n \geq s$. So each π_s is really a finite permutation. π will be the limit of π_s.

Let $W^*_{e,s}$ be defined as follows:

$W^*_{e,s} = W_{e,s_0}$ where $s_0 \leq s$ is the largest $t \leq s$ with the property that for distinct $x, y \in W_{e,t}$ we have that $D_x \cap D_y = \emptyset$.

The list W^*_e automatically contains all strong arrays. Our requirements for this construction include:

$R_e \qquad W^*_e$ infinite $\to \exists x \in W^*_e \;\; F_x \subseteq A$

The strategy for making sure that $\overline{A}$ is a variant on the Post $2e$ strategy. At stage $s = 0$ in our construction we have $A_s = \emptyset$ and π_s the identity.

Stage $s + 1$**.**
Given π_s and A_s. We say that $e < s$ requires attention iff

1. $\neg\exists n \in W^*_{e,s} \quad D_n \subseteq A_s$
2. $\exists x, y \in W^*_{e,s}$ such that
   (a) $x, y \notin A_s$
   (b) $\exists n \in W^*_{e,s} \;\; x \in F_n$
   (c) $e < x < y < s, \quad e < \pi_s(x), \quad e < \pi_s(y)$
   (d) i. e-state of x at stage s = e-state of y at stage s
   ii. e-state of $\pi_s(x)$ at stage s = e-state of $\pi_s(y)$ at stage s
   (e) $2x < \pi_s(y)$.

The action at this stage is the following. For the least $e < s$ (if any) which requires attention we choose the least x for which there is a y and then we choose the least y. For this choice $(e, x, y) = (e_s, x_s, y_s)$ we
(a) put x into A, $A_{s+1} = A_s \cup \{x_s\}$
(b) put $\pi_{s+1} = \pi_s \circ swap(x, y)$ where $swap(x, y)$ refers to the transposition which interchanges x and y.
As usual if there is no e which requires attention we do nothing and go onto the next stage.

This ends the construction. Let Q denote the stages s where action takes place at stage $s + 1$. Then

$$A = \{x_s : s \in Q\}$$

We define

$$\pi(u) = \lim_{s \to \infty} \pi_s(u)$$

although at this point we have not proved that this limit always exists. Note the pointwise limit of 1-1 functions must be 1-1 where it is defined.

Note that for $s \in Q$ we have that $\pi_{s+1}(x_s) = \pi_s(y_s)$. Since x_s enters A we have (by $2a$) that x_s will never be a x_t or y_t latter. It follows that $\pi(x_s) = \pi_{s+1}(x_s)$. Hence

$$B =^{def} \{\pi_{s+1}(x_s) : s \in Q\} = \{\pi(x_s) : s \in Q\}$$

is well defined and r.e.

Claim (1) for any n we have that $|B \cap [0, 2n]| \leq n$.
Proof
Note that (by $2e$) we have that $\pi(x_s) = \pi_s(y_s) > 2x_s$. Since each x_s is distinct the Claim follows.
QED

As we have seen before this implies that B is not hypersimple. (Proposition 78).

Claim (2) $\lim_{s\to\infty} \pi_s(u) = \pi(u) < \infty$ for every u.
Proof
Fix s_0 so that $A \cap [0, u] = A_{s_0} \cap [0, u]$. Now the only way that $\pi_{s+1}(u) \neq \pi_s(u)$ for some $s > s_0$ is if $u = x_s$ or $u = y_s$. But in either case since $x_s < y_s$ and x_s enters A we have A changes in the interval $[0, u]$ which is a contradiction.
QED

We don't know yet that π is onto.

Claim (3) For each e

(a) R_e is met.

(b) $\exists s_0\ \forall s > s_0 \quad e_s > e$

Proof
This is proved by induction on e.

(a) We may suppose by induction that there exists s_0 such that $e_s \geq e$ for all $s > s_0$. Suppose R_e is not met. Then W_e^* is infinite and for all $n \in W_e^*$ we have that $F_n \cap \overline{A} \neq \emptyset$. For each $n \in W_e^*$ define

$$x_n = \min(F_n \cap \overline{A})$$

Since the F_n are pairwise disjoint all of the x_n are distinct. Note there exist $\sigma, \tau \in 2^e$ such that

$\exists^\infty n \in W_e^* \quad \sigma =$ final e-state of x_n and $\tau =$ final e-state of $\pi(x_n)$.

Choose x_n and x_m such that

1. $n, m \in W_e^*$

2. $e < x_n < x_m$

3. $2x_n < \pi(x_m)$

4. σ is the final e-state of x_n and x_m, and

5. τ is the final e-state of $\pi(x_n)$ and $\pi(x_m)$.

Increase s_0 (if necessary) so that not only is $e_s \geq e$ for all $s \geq s_0$ but also so that

1. $n, m \in W^*_{e,s_0}$

2. $x_n < x_m < s_0$ and $\pi(x_n) < s_0$ and $\pi(x_m) < s_0$

3. $\pi_s(x_n) = \pi(x_n)$ and $\pi_s(x_m) = \pi(x_m)$ all $s > s_0$

4. σ is the e-state of x_n and x_m at stage s_0,

5. τ is the e-state of $\pi(x_n)$ and $\pi(x_m)$ at stage s_0 and

6. $A_{s_0} \cap [0, x_m] = A \cap [0, x_m]$

Recall that we chose $x_n, x_m \in \overline{A}$. It is easy to check that e requires attention at stage s_0 and x_n and x_m witness this fact. But this means that x_n or some smaller x enters A. But this contradicts the condition that A does not change below x_m.

(b) Suppose that $e_s \geq e$ for all $s > s_0$ and R_e is met. If W^*_e is infinite, then for some $x \in W^*_e$ we have that $F_x \subseteq A$. But this will be seen at some stage and so e will not require attention after that. If W^*_e is finite, then suppose that

$$\cup\{F_x \ : \ x \in W^*_e\} \subseteq [0, n].$$

After we reach a stage $s > s_0$ where $A_s \cap [0, n] = A \cap [0, n]$, then e will never again require attention because then A would change beneath n.
QED

Claim (4) π is onto.
Proof
Given z choose s_0 so that $e_s > z$ for all $s \geq s_0$. If $\pi_{s_0}(u) = z$, then u will never be either x_s or y_s for any $s \geq s_0$. This is because we required that $\pi_s(x_s) > e_s > z$ and $\pi_s(x_s) > e_s > e$. Hence $\pi(u) = z$.
QED

Claim (5)
(a) $\forall C \in \mathcal{E} \quad \pi(C) \in \mathcal{E}$

(b) $\forall C \in \mathcal{E} \quad \pi^{-1}(C) \in \mathcal{E}$

Proof

(a) Fix s_0 so that for all $s > s_0$ we have that $e_s > e$. Then we show that

$$\pi(W_e) = \bigcup_{s>s_0} \pi_s(W_{e,s})$$

To see this first suppose $y \in \pi(W_e)$. Then there exists $x \in W_e$ with $\pi(x) = y$ but for all sufficiently large s we have that $x \in W_{e,s}$ and $\pi_s(x) = \pi(x)$ and thus $y \in \pi_s(W_{e,s})$.

To see the other inclusion, suppose that $y \in \pi_s(W_{e,s})$ for some $s > s_0$. We claim that for every $t > s$ that $y \in \pi_t(W_{e,t})$. This is proved by induction on t. Suppose that $\pi_t(u) = y$ for some $u \in W_{e,t}$. Then $\pi_{t+1}(u) = \pi_t(u)$ unless $u = x_t$ or $u = y_t$ and then $\pi_{t+1}(x_t) = \pi_t(y_t)$ and $\pi_{t+1}(y_t) = \pi_t(x_t)$. But since x_t and y_t have the same e_t-type and $e_t > e$, if one is in $W_{e,t}$ so is the other. In either case we have that there exists $v \in W_{e,t+1}$ with $\pi_{t+1}(v) = y$. Now to see that $y \in \pi(W_e)$ suppose that $\pi(u) = y$ and choose sufficiently large $t > s_0$ such that $\pi_t(u) = \pi(u) = y$. Since π_t is a bijection and $y \in \pi_t(W_{e,t})$, it must be that $u \in W_{e,t}$ and hence $u \in W_e$.

(b) This is similar, except we use that $\pi_t(x_t)$ and $\pi_t(y_t)$ have the same e_t-type.

QED

Hmwk 30. (Wed 11-24) Prove that there exists a bijection $\pi : \omega \to \omega$ such that $\pi(A) \in \mathcal{E}$ for all $A \in \mathcal{E}$ but $\pi \notin aut(\mathcal{E})$. (Hint: use a maximal set.)

Definition 103 *For A and B predicates over subsets of ω or finite products of ω we define:*

$\Pi^0_0 = \Sigma^0_0 =$ *the recursive predicates.*

A is Σ^0_{n+1} iff there exists B which is Π^0_n and $A(x) \equiv \exists y \;\; B(x, y)$.

A is Π^0_{n+1} iff there exists B which is Σ^0_n and $A(x) \equiv \forall y \;\; B(x, y)$.

A is Δ^0_n iff A is Σ^0_n and A is Π^0_n.

Note that by DeMorgan's Laws

$$\Pi^0_n = \{\neg A : A \in \Sigma^0_n\}.$$

Δ^0_n is closed under complementation.

Proposition 104 *Suppose Γ is Σ_n^0, Π_n^0, or Δ_n^0. Then Γ is closed under $\leq_m$, i.e., $A \leq_m B \in \Gamma$ implies $A \in \Gamma$. This implies that if the predicate $B(x, y)$ is in Γ and f is a recursive function, then $A(x, y) \equiv B(x, f(x))$ is in Γ. Also, if $A, B \in \Gamma$, then $A \wedge B$ and $A \vee B$ are both in Γ. Finally, Γ predicates are closed under bounded quantification, e.g., $\exists u < x\ A(u, x, \ldots)$ and $\forall u < x\ A(u, x, \ldots)$.*

Proposition 105 *If $B(x, y)$ in Σ_n^0, then $A(x) \equiv \exists y\ \ B(x, y)$ is in Σ_n^0. If $B(x, y)$ in Π_n^0, then $A(x) \equiv \forall y\ \ B(x, y)$ is in Π_n^0.*

Proposition 106 $\Sigma_n^0 \cup \Pi_n^0 \ \subseteq\ \Delta_{n+1} \ =\ \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$

Definition 107 *We say that A is universal for Γ iff*

$$\Gamma = \{B\ :\ \exists x\ \ B = A_x\}.$$

We say that A is m-complete for Γ iff

$$\Gamma = \{B\ :\ B \leq_m A\}$$

Note that universal for Γ implies m-complete for Γ. Also, the complement of a set universal for Γ is universal for $\widetilde{\Gamma}$ and the same for m-completeness.

Proposition 108 *For each $n > 0$ there is a universal Σ_n^0 set.*

Proposition 109 *For each $n > 0$ we have $Red(\Sigma_n^0)$, $Sep(\Pi_n^0)$, $\neg Sep(\Sigma_n^0)$, and $\neg Red(\Pi_n^0)$.*

Proof
See definitions 20. We first show $Red(\Sigma_n^0)$. Let

$$A(x) \equiv \exists y\ R(x, y) \quad \text{and} \quad B(x) \equiv \exists y\ S(x, y)$$

where R and S are Δ_n^0. Reduce A and B by

$$A_0(x) \equiv \exists y\ (R(x, y) \wedge \forall z < y\ \neg S(z, x))$$

and

$$B_0(x) \equiv \exists y\ (S(x, y) \wedge \forall z \leq y\ \neg R(z, x))$$

Since $Red(\Gamma) \rightarrow Sep(\widetilde{\Gamma})$ Proposition 22, it follows that $Sep(\Pi_n^0)$ holds.

To see $\neg Sep(\Sigma^0_n)$, first construct a doubly universal pair A and B. These are Σ^0_n sets such that for every pair C and D of Σ^0_n sets there exists a u with $C = A_u$ and $D = B_u$. To get A and B let U be a universal Σ^0_n set. Then define

$$A = \{(\langle x, y\rangle, z) \ : \ \langle x, z\rangle \in U\}$$

and

$$B = \{(\langle x, y\rangle, z) \ : \ \langle y, z\rangle \in U\}$$

then $u = \langle x, y\rangle$ codes the pair U_x and U_y. Now applying reduction to A and B we get $A^0 \subseteq A$ and $B^0 \subseteq B$. Note that this simultaneously reduces all cross sections A_u and B_u. Assuming for contradiction that separation holds, let C be Δ^0_n such that $A^0 \subseteq C$ and $B^0 \subseteq \overline{C}$. We get a contradiction since, then C would be a universal Δ^0_n set. This is because if P is Δ^0_n then there exists u with $A_u = P$ and $B_u = \overline{P}$. But the reduction followed by separation can't effect the u cross section, so $C_u = P$.
QED

Hmwk 31. (Mon 11-29) Prove there does not exist a universal Δ^0_n set.

Lemma 110 *$A \subseteq \omega$ is Π^0_2 iff there exists P recursive such that*

$$A(x) \textit{ iff } \exists^\infty s \ \ P(s, x)$$

Proof
($\leftarrow$) $\exists^\infty s \ \ P(s, x)$ iff $\forall t \ \exists s > t \ P(s, x)$
($\rightarrow$) Suppose

$$A(x) \text{ iff } \forall n \ \exists m \ R(n, m, x)$$

where R is Δ^0_1. Define $P \subseteq \omega^{<\omega} \times \omega$ by

$$P(\sigma, x) \text{ iff } \forall i < |\sigma| \ \ [R(i, \sigma(i), x) \text{ and } \forall j < i \ \neg R(i, j, x)]$$

QED

Theorem 111 *(Post) Suppose $A \subseteq \omega$. Then A is Δ^0_2 iff $A \leq_T 0'$*

Proof
Suppose A is Δ^0_2. Then by Lemma 110 there exists recursive $P(u, x)$ and $Q(v, x)$ such that

$$A(x) \equiv \exists^\infty u \ \ P(u, x)$$

$$\neg A(x) \equiv \exists^\infty v \;\; Q(v,x)$$

Now define $g(x,s)$ as follows. Input x,s and let u_s be the maximum $u \leq s$ such that $P(u,x)$ (zero if no such u). Similarly define v_s to be the maximum $v \leq s$ such that $Q(v,x)$. Define

$$g(x,s) = \begin{cases} 1 & \text{if } u_s \geq v_s \\ 0 & \text{if } u_s < v_s \end{cases}$$

It is easy to check that

$$A(x) = \lim_{s \to \infty} g(x,s)$$

and so by the Limit Lemma 62 we have that $A \leq_T 0'$.

Conversely if $A \leq_T 0'$ then by the Limit Lemma we have g recursive such that

$$A(x) = \lim_{s \to \infty} g(x,s)$$

but then

$$A(x) \equiv \forall^\infty s \;\; g(x,s) = 1 \equiv \exists^\infty s \;\; g(x,s) = 1$$

so A is Δ^0_2.
QED

Lemma 112 *(1) $A \subseteq \omega$ is $\Sigma^0_1(B)$ iff $A \leq_m B'$.*
(2) A is $\Delta^0_2(B)$ iff $A \leq_T B'$.

Proof
A is $\Sigma^0_1(B)$ iff there exists a predicate $R \leq_T B$ such that

$$A(x) \text{ iff } \exists y \;\; R(x,y)$$

(1) is just a relativization of the standard result that $0'$ is Σ^0_1-m-complete.
(2) is just the relativization of Post's Theorem 111.
QED

Theorem 113 *(Post)*
(1) $A \leq_T 0^{(n)}$ iff A is Δ^0_{n+1}.
(2) $0^{(n)}$ is an m-complete Σ^0_n-set.

Proof
(1) for $n = 2$:
    $A \leq_T 0''$ iff $A \leq_T (0')'$ iff A is $\Delta^0_2(0')$.
    A is $\Delta^0_2(0')$ iff there exists $R_1, R_2 \leq_T 0'$ such that

$$A(x) \text{ iff } \exists n \, \forall m \; R_1(n, m)$$

$$\neg A(x) \text{ iff } \exists n \, \forall m \; R_2(n, m)$$

but since $R_1, R_2 \leq_T 0'$ iff R_1 and R_2 are Δ^0_2, we have that
    A is $\Delta^0_2(0')$ iff A is Δ^0_3.
(2) for $n = 2$:
    $0''$ is $\Sigma^0_1(0')$ and m-complete for $\Sigma^0_1(0')$. But $\Sigma^0_1(0')$ is Σ^0_2. This is because B is $\Sigma^0_1(0')$ iff there exists $R \leq_T 0'$ such that

$$B(x) \text{ iff } \exists y \;\; R(x, y)$$

But $R \leq_T 0'$ iff R is Δ^0_2. Hence B is Σ^0_2 iff B is $\Sigma^0_1(0')$.

The proofs for $n > 2$ are analogous.
QED

Hmwk 32. (Wed 12-1) Prove there does not exist A which is m-complete for Δ^0_2.

Proposition 114 *$EMP =^{def} \{e \; : \; W_e = \emptyset\}$ is Π^0_1-m-complete.*

Proof

$$e \in EMP \text{ iff } \forall x, s \;\; x \notin W_{e,s}$$

so EMP is Π^0_1. Let A be Π^0_1, then there is R recursive so that

$$A(x) \text{ iff } \forall y \;\; R(x, y).$$

Using S-n-m Theorem get f recursive so that for every x

$$W_{f(x)} = \{y \; : \; \neg R(x, y)\}$$

Then $A(x)$ iff $f(x) \in EMP$.
QED

Proposition 115 *$TOT =^{def} \{e : W_e = \omega\}$ is Π^0_2-m-complete.*
$FIN =^{def} \{e : W_e$ is finite$\}$ is Σ^0_2-m-complete.

Proof

$$e \in TOT \text{ iff } \forall x\, \exists s\; x \in W_{e,s}$$

$$e \in FIN \text{ iff } \exists x\, \forall y, s\; (y \in W_{e,s} \rightarrow y < x)$$

so TOT is Π^0_2 and FIN is Σ^0_2. Now suppose that A is Π^0_2 we show that

$$(A, \overline{A}) \leq_m (TOT, FIN)$$

which simultaneously shows that TOT is Π^0_2-m-complete and FIN is Σ^0_2-m-complete. Suppose

$$A(x) \text{ iff } \exists^\infty s\;\; P(s, x)$$

where P is Δ^0_1. Using S-n-m find a recursive function f so that

$$W_{f(x)} = \{t : \exists s > t\;\; P(s, x)\}$$

Hence $A(x) \rightarrow W_{f(x)} = \omega$ while $\neg A(x) \rightarrow W_{f(x)}$ is finite.
QED

Proposition 116 *$COF =^{def} \{e : \overline{W_e}$ is finite $\}$ is Σ^0_3-m-complete.*

Proof

$$e \in COF \text{ iff } \exists n\, \forall m > n\, \exists s\;\; m \in W_{e,s}$$

Now suppose that A is Σ^0_3. Then there exists P which is Δ^0_1 such that

$$A(x) \text{ iff } \exists n\;\; \exists^\infty m\;\; P(n, m, x)$$

Input x and describe the r.e. set B_x by using a moving marker construction similar to the construction of a maximal set but simpler. At any stage s we have that

$$\overline{B_{x,s}} = \{p_{0,s} < p_{1,s} < p_{2,s} < \cdots\}$$

We look for the least $n < s$ (if any) such that $P(n, s, x)$ and bump the n^{th} marker, i.e., enumerate $p_{n,s}$ into B_x, i.e., $B_{x,s+1} = B_{x,s} \cup \{p_{n,s}\}$. Note that if $A(x)$ is true then there exist n so that the n^{th} marker is bumped infinitely often and so B_x is cofinite. On the other hand if $\neg A(x)$, then each marker eventually stops moving and so B_x is coinfinite.

By the usual S-n-m argument we can find a recursive function f so that $B_x = W_{f(x)}$ for all x and so

$$A(x) \text{ iff } f(x) \in COF$$

QED

Proposition 117 *$REC =^{def} \{e : W_e$ is recursive $\}$ is Σ_3^0-m-complete.*

Proof

$$e \in REC \text{ iff } \exists e' \ \ (W_e \cup W_{e'} = \omega \text{ and } W_e \cap W_{e'} = \emptyset)$$

and $W_e \cup W_{e'} = \omega$ is Π_2^0 and $W_e \cap W_{e'} = \emptyset$ is Π_1^0. To see that it is m-complete, use a moving marker argument as above. Just add an additional reason to bump the e^{th} marker to make sure that if B_x is coinfinite, then for each e

$$\psi_e(e)\downarrow \ \rightarrow \ \ \psi_{e,p_e}(e)\downarrow$$

This guarantees that if B_x is coinfinite, then $K \leq_T B_x$.
QED

Hmwk 33. (Fri 12-3)

(a) Let A be an infinite r.e. set. Let

$$Q_A = \{e \ : \ W_e = A\}$$

Prove that Q_A is Π_2^0-m-complete.

(b) Let A be a finite nonempty set. Prove that

$$Q_A = \{e \ : \ W_e = A\}$$

is $D(\Sigma_1^0)$-m-complete, where

$$D(\Sigma_1^0) = \{A \cap \overline{B} \ : \ A, B \in \Sigma_1^0\}.$$

Lemma 118 *Suppose A is Σ_{k+1}^0 then there exists B Π_k^0 such that*

$$A(x) \textit{ iff } \exists y \ \ B(x,y) \textit{ iff } \exists ! y \ \ B(x,y)$$

Proof
Suppose

$$A(x) \text{ iff } \exists y \;\; P(x,y)$$

where P is Π^0_k. Then

$$A(x) \text{ iff } \exists y \;\; (P(x,y) \land \forall z < y \, \neg P(x,z)$$

Define

$$C(x,y) \text{ iff } \forall z < y \, \neg P(x,z)$$

In case $k+1=1$ then C is Δ^0_1. In case $k+1>1$ then since C is Σ^0_k we have by induction a Π^0_{k-1} predicate D so that

$$C(x,y) \text{ iff } \exists u \; D(x,y,u) \text{ iff } \exists ! u \; D(x,y,u)$$

Hence

$$A(x) \text{ iff } \exists y \, \exists u \;\; (P(x,y) \land D(x,y,u)) \text{ iff } \exists ! y \, \exists ! u \;\; (P(x,y) \land D(x,y,u))$$

so taking $B(x, \langle y,u \rangle) \equiv P(x,y) \land D(x,y,u)$ does the trick.
QED

Proposition 119 *(a) A is Π^0_3 iff there exists B which is Δ^0_1 such that*

$$A(u) \equiv \exists^\infty s \, \forall n \; B(s,n,u)$$

(b) A is Π^0_4 iff there exists B which is Δ^0_1 such that

$$A(x) \equiv \exists^\infty s \, \exists^\infty t \; B(s,t,x)$$

Proof
(a) Suppose

$$A(u) \equiv \forall x \, \exists y \, \forall z \; R(x,y,z,u)$$

where R is Δ^0_1. Define

$$Q(x,u) \equiv \exists y \, \forall z \; R(x,y,z,u)$$

Then by Lemma 118 there is a C which is Π^0_1 and

$$Q(x,u) \equiv \exists y \;\; C(x,y,u) \equiv \exists ! y \;\; C(x,y,u)$$

Hence

$$A(u) \equiv \forall x \exists! y \;\; C(x, y, u)$$

$$A(u) \equiv \exists^{\infty} \sigma \in \omega^{<\omega} \;\; \forall i < |\sigma| \;\; C(i, \sigma(i), u)$$

Note that $\forall i < |\sigma| \;\; C(i, \sigma(i), u)$ is Π^0_1 and so there is B recursive so that

$$\forall n \; B(\sigma, n, u) \equiv \forall i < |\sigma| \;\; C(i, \sigma(i), u)$$

(b) Suppose

$$A(u) \equiv \forall x \; \exists y \; R(x, y, u)$$

where R is Π^0_2. By Lemma 118 applied to $\exists y \;\; R(x, y, u)$ we may assume that

$$A(u) \equiv \forall x \; \exists! y \; R(x, y, u)$$

Hence

$$A(u) \equiv \exists^{\infty} \sigma \; \forall i < |\sigma| \;\; R(i, \sigma(i), u)$$

but the predicate

$$Q(\sigma, u) \equiv \forall i < |\sigma| \;\; R(i, \sigma(i), u)$$

is Π^0_2 so there exists a recursive B so that

$$Q(\sigma, u) \equiv \exists^{<\infty} \tau \;\; B(\sigma, \tau, u)$$

Hence

$$A(u) \equiv \exists^{\infty} \sigma \; \exists^{\infty} \tau \;\; B(\sigma, \tau, u)$$

QED

Hmwk 34. (Mon 12-6)

Let $PTIME = \{e \;:\; \psi_e$ runs in polynomial time $\}$, i.e., there exists a polynomial $p(x)$ such that $\psi_e(x)$ halts in less than $p(x)$ steps for every x. Prove that $PTIME$ is Σ^0_2-m-complete.

Hmwk 35. (Wed 12-8) For each e let $Q_e = \{\frac{n}{m+1} \;:\; \langle n, m \rangle \in W_e\} \subseteq \mathbb{Q}$. Define

$\Omega = \{e \;:\; Q_e$ is order isomorphic to $\omega\}$.

Prove that Ω is Π^0_3-m-complete.

Definition 120 *$A \subseteq \omega^\omega$ is Σ^1_1 iff there exists a recursive $R \subseteq \omega^{<\omega} \times \omega^{<\omega}$ such that*

$$x \in A \equiv \exists y \in \omega^\omega \ \ \forall n \in \omega \ \ R(x\restriction n, y\restriction n).$$

Similarly $A \subseteq \omega$ is Σ^1_1 iff there exists a recursive $R \subseteq \omega \times \omega^{<\omega}$ such that

$$k \in A \equiv \exists y \in \omega^\omega \ \ \forall n \in \omega \ \ R(k, y\restriction n).$$

Π^1_1 sets are the complements of Σ^1_1 sets and $\Delta^1_1 = \Pi^1_1 \cap \Sigma^1_1$.

We can give similar definitions of Σ^1_1 and Σ^0_n and Π^0_n for $\mathcal{X}$ any finite product $\mathcal{X} = \prod_{i<N} X_i$ where each X_i is either ω or ω^ω.

Proposition 121

1. *$\Pi^0_1 \subseteq \Sigma^1_1$*
2. *If $A \subseteq \mathcal{X} \times \omega^\omega$ is Σ^1_1 then B is Σ^1_1 where*

$$B(x) \text{ iff } \exists y \ \ A(x,y)$$

3. *If A and B are Σ^1_1 then $A \wedge B$ and $A \vee B$ are Σ^1_1.*
4. *If $A \subseteq \omega \times \mathcal{X}$ is Σ^1_1 then both*
   - *(a) $B(x) \equiv \exists n \in \omega \ \ A(n,x)$ and*
   - *(b) $C(x) \equiv \forall n \in \omega \ \ A(n,x)$*

   are Σ^1_1.

Proof
(1) trivial
(2) Suppose $\mathcal{X} = \omega^\omega$ and

$$A(x,y) \equiv \exists z \ \ \forall n \ \ R(x\restriction n, y\restriction n, z\restriction n)$$

define

$$R^*(\sigma,\tau) \text{ iff } R(\sigma,\tau_0,\tau_1) \text{ where } \tau(i) = \langle \tau_0(i), \tau_1(i) \rangle$$

Then

$$B(x) \equiv \exists u \ \ \forall n \ \ R^*(x\restriction n, u\restriction n)$$

(3) Suppose

$$A(x) \equiv \exists y \ \ C(x,y)$$

$$B(x) \equiv \exists z \;\; D(x,z)$$

where C and D are Π^0_1. Then

$$A(x) \vee B(x) \;\;\equiv\;\; \exists w \;\; (C(x,w) \vee D(x,w))$$

and

$$A(x) \wedge B(x) \;\;\equiv\;\; \exists y \, \exists z \;\; (C(x,y) \wedge D(x,z))$$

(4a) Suppose

$$A(n,x) \equiv \exists y \;\; \forall m \;\; R(n, x\restriction m, y\restriction m)$$

Define

$$R^*(x\restriction n, y\restriction n) \text{ iff } R(y(0), x\restriction(n-1), y^*\restriction(n-1)) \text{ where } y^*(i) = y(i+1)$$

Then

$$B(x) \equiv \exists n \; A(n,x) \equiv \exists y \;\; \forall m \;\; R^*(x\restriction m, y\restriction m)$$

(4b) Suppose

$$A(n,x) \equiv \exists y \;\; \forall m \;\; R(n, x\restriction m, y\restriction m)$$

Define

$R^*(x\restriction m, z\restriction m)$ iff $R(i, x\restriction j, y_i\restriction j)$ for each $\langle i,j\rangle < m$ and $y_i(j) = z(\langle i,j\rangle)$.
Then

$$C(x) \equiv \forall n \; A(n,x) \equiv \exists z \;\; \forall m \;\; R^*(x\restriction m, z\restriction m)$$

QED

Proposition 122 *Universal Σ^1_1 sets exists, hence $\Sigma^1_1 \neq \Pi^1_1$.*

Proof
Let $U \subseteq \omega \times \mathcal{X} \times \omega^\omega$ be a universal Π^0_1 set for subsets of $\mathcal{X} \times \omega^\omega$, then

$$V(n,x) \equiv \exists y \;\; A(n,x,y)$$

is Universal Σ^1_1.
QED

Theorem 123 *(Tennenbaum) There exists a recursive linear order $(\omega, \trianglelefteq)$ which is isomorphic to $\omega + \omega^*$ with the property that every nonempty r.e. subset of ω has a $\trianglelefteq$-least and $\trianglelefteq$-greatest element.*

Proof
Note that ω^* stands for reverse ω or equivalently the order type of the negative integers. Let

$$L = \{x \in \omega \ : \ |\{y : y \lhd x\}| < \omega\} \text{ and } R = \{x \in \omega \ : \ |\{y : x \lhd y\}| < \omega\}$$

In our construction we make sure that $\omega = L \cup R$ and each is infinite. At stage s we assume that we have (effectively) determined the finite linear order $\unlhd\upharpoonright(s \times s)$ and just decide where to put the new element, s, of

$$s + 1 = \{0, 1, 2, \ldots, s\}.$$

Our requirements are:

R_e $\qquad W_e$ infinite $\rightarrow W_e \cap L \neq \emptyset$ and $W_e \cap R \neq \emptyset$.

We assume at stage s in our construction that some requirements R_e, say $e \in F_s \subseteq s$, have followers $l_e < s$ and $r_e < s$ which satisfy:
if $e < e'$ and $e, e' \in F_s$, then $l_e \lhd l_{e'} \lhd r_{e'} \lhd r_e$.
At stage $s + 1$ we look for the smallest $e < s$ (if any) such that

1. $e \notin F_s$ (or equivalently R_e has no followers)
2. there exists $l, r \in W_{e,s}$ such that for every $e' < e$ with $e' \in F_s$ we have that
$$l_{e'} \lhd l \lhd r \lhd r_{e'}$$

For the smallest such e and smallest such pair l, r we appoint $l = l_e$ and $r = r_e$ the followers of R_e and put

$$F_{s+1} = \{e' < e \ : \ e' \in F_s\} \cup \{e\}$$

i.e., we unappoint all followers for $e' > e$. If there is no such e we do not change any followers.

In either case, we put s into the ordering $\unlhd\upharpoonright(s \times s)$ in the first gap above all the l_e for $e \in F_{s+1}$ (and therefore, below all the r_e for $e \in F_{s+1}$.)

Claim. For each e if W_e is infinite, then R_e obtains permanent followers l_e and r_e and is met.
Proof

Suppose the Claim is true for all $e' < e$. Suppose s_0 is a large enough stage so that no $e' < e$ acts after stage s_0. Let e_0 be the maximum element of F_{s_0} below e. Then since $s > s_0$ are put between l_{e_0} and r_{e_0} and W_e is infinite, it must be that some followers are appointed to R_e if it doesn't already have them. These followers are permanent.
QED

Since infinitely many W_e are infinite and hence acquire permanent followers, it must be that L and R are infinite and therefore the order type we construct is $\omega + \omega^*$.
QED

Corollary 124 *(Jockusch) There exists a recursive function $f : [\omega]^2 \to 2$ such that there is no infinite recursive $H \in [\omega]^\omega$ such that $f \upharpoonright [H]^2$ is constant.*

Proof
Define

$$f(x,y) = \begin{cases} 1 & \text{if } x < y \to x \lhd y \\ 0 & \text{if } x < y \to y \lhd x \end{cases}$$

QED

Definition 125 *$T \subseteq \omega^{<\omega}$ is a well-founded tree iff*
(a) $\forall \sigma, \tau \;\; \sigma \subseteq \tau \in T \to \sigma \in T$
(b) T has no infinite branch, i.e., $[T] = \emptyset$ where

$$[T] =^{def} \{x \in \omega^\omega \; : \; \forall n \;\; x \upharpoonright n \in T\}.$$

Definition 126 *(Kleene-Brouwer ordering) For $\sigma, \tau \in \omega^{<\omega}$*

$$\sigma <_{KB} \tau \textit{ iff } \sigma \supsetneq \tau \textit{ or } \exists n < \min(|\sigma|, |\tau|) \;\; \sigma \upharpoonright n = \tau \upharpoonright n \textit{ and } \sigma(n) < \tau(n)$$

$$\sigma \leq_{KB} \tau \textit{ iff } \sigma <_{KB} \tau \textit{ or } \sigma = \tau$$

Proposition 127 *$\leq_{KB}$ is a recursive linear ordering of $\omega^{<\omega}$.*

Theorem 128 *(Kleene-Brouwer) Given a tree $T \subseteq \omega^{<\omega}$*
T is well-founded iff $(T, \leq_{KB})$ is a well-ordering.

Proof
Suppose that T is not well-founded and $x \in [T]$. Then for each n

$$x \upharpoonright (n+1) <_{KB} x \upharpoonright n$$

and so $(T, \leq_{KB})$ is not a well-ordering.

Conversely, suppose that $(T, \leq_{KB})$ is not a well-ordering and $(\sigma_n \in T \ : n < \omega)$ is $<_{KB}$-descending, i.e.,

$$\sigma_{n+1} <_{KB} \sigma_n.$$

Then an easy induction produces $x \in \omega^\omega$ with the property that

$$\forall n \ \forall^\infty m \quad x \upharpoonright n \subseteq \sigma_m.$$

It follows that $x \in [T]$ and so T is not well-founded.
QED

Definition 129 *For $T \subseteq \omega^{<\omega}$ a tree and α an ordinal we define $T_\alpha \subseteq T$ as follows:*

(a) $\sigma \in T_0$ iff $\sigma \in T$ and $\forall n \ \ \sigma n \notin T$. (Terminal nodes of T.)

(b) $\sigma \in T_\alpha$ iff $\sigma \in T$ and $\forall n \ \ (\sigma n \in T \rightarrow \sigma n \in T_{<\alpha})$.

(c) $T_{<\alpha} =^{def} \cup_{\beta<\alpha} T_\beta$.

Definition 130 *For $\sigma \in T$*

(a) $rank_T(\sigma) = \alpha$ where α is the smallest ordinal with $\sigma \in T_\alpha$.

(b) $rank_T(\sigma) = \infty$ if there is no such α.

Proposition 131 *For $T \subseteq \omega^{<\omega}$ a tree, T is well-founded iff $rank_T(\langle\rangle) < \infty$, i.e., its an ordinal.*

Proof
Note that if $rank_T(\sigma) = \infty$, then there exists n such that $rank_T(\sigma n) = \infty$. Hence, $rank_T(\langle\rangle) = \infty$ implies that T has an infinite branch. On the other hand if $rank_T(\sigma) < \infty$, then for every n with $\sigma n \in T$ we have that

$$rank_T(\sigma n) < rank_T(\sigma)$$

Hence T cannot have an infinite branch.
QED

Definition 132 *$c : T \to \omega$ is a hypcode iff $T \subseteq \omega^{<\omega}$ is a recursive well-founded tree and c is partial recursive map with domain T. Given a hypecode c we define the sets $H(c,\sigma)$ as follows by induction on the rank of σ. Fix $U \subseteq \omega \times \mathcal{X}$ a universal Σ^0_1 set.*

(a) for $\sigma \in T_0$ a terminal node of T

$$H(c,\sigma) = U_{c(\sigma)}$$

(b) for $\sigma \in T$ not terminal and $c(\sigma) = 0$

$$H(c,\sigma) = \cup_{n,\sigma n \in T} H(c,\sigma n)$$

(c) for $\sigma \in T$ not terminal and $c(\sigma) > 0$

$$H(c,\sigma) = \cap_{n,\sigma n \in T} H(c,\sigma n)$$

$A \subseteq \mathcal{X}$ is hyperarithmetic (HYP) iff there exists a hypcode c and

$$A = H(c) =^{def} H(c, \langle\rangle).$$

Proposition 133 *$HYP \subseteq \Delta^1_1$.*

Proof
$x \in H(c)$ iff there exists $f : T \to \{0,1\}$ such that

1. $\forall \sigma \in T_0$
$$f(\sigma) = 1 \text{ iff } x \in U_{c(\sigma)}$$
2. $\forall \sigma \in T \setminus T_0$ if $c(\sigma) = 0$ then
$$f(\sigma) = 1 \text{ iff } \exists n \ \ (\sigma n \in T \ \ \wedge \ \ f(\sigma n) = 1)$$
3. $\forall \sigma \in T \setminus T_0$ if $c(\sigma) > 0$ then
$$f(\sigma) = 1 \text{ iff } \forall n \ \ (\sigma n \in T \to f(\sigma n) = 1)$$
4. $f(\langle\rangle) = 1$

It is easy to check that $1-4$ are all arithmetic predicates and so $H(c)$ is Σ^1_1. To see that the complement of $H(c)$ is also Σ^1_1 just note that
$x \notin H(c)$ iff there exists $f : T \to \{0,1\}$ such that

1,2,3, and

4′. $f(\langle\rangle) = 0$.

QED

Theorem 134 *(Kleene-Souslin)*

Suppose A and B are disjoint Σ^1_1 sets. Then they can be separated by a hyperarithmetic set C. Hence $HYP = \Delta^1_1$.

Proof

To simplify the notation we assume that $A, B \subseteq \omega^\omega$ although essentially the same proof will work for $A, B \subseteq \omega$ or any $\mathcal{X}$. Since A, B are Σ^1_1 there are recursive trees

$$T^A, T^B \subseteq \cup_{n<\omega}\omega^n \times \omega^n$$

such that

$$x \in A \text{ iff } \exists y \ \ \forall n \ \ (x{\restriction}n, y{\restriction}n) \in T^A$$

$$x \in B \text{ iff } \exists z \ \ \forall n \ \ (x{\restriction}n, z{\restriction}n) \in T^B$$

The fact that A and B are disjoint implies that it is impossible to find (x, y, z) such that $(x{\restriction}n, y{\restriction}n) \in T^A$ and $(x{\restriction}n, z{\restriction}n) \in T^B$ for all n. This tells us how to find our recursive well-founded tree T.

Given $\rho \in \omega^{<\omega}$ we determine a triple $trip(\rho) = (\sigma, \tau_1, \tau_2)$ by the rule that $\sigma(i) = \rho(3i)$, $\tau_1(i) = \rho(3i+1)$, and $\tau_2(i) = \rho(3i+2)$. We take the natural length functions, namely

- $|\sigma| = |\tau_1| = |\tau_2| = n$ if $|\rho| = 3n$,
- $|\sigma| = n+1, \ \ |\tau_1| = |\tau_2| = n$ if $|\rho| = 3n+1$, and
- $|\sigma| = |\tau_1| = n+1, \ \ |\tau_2| = n$ if $|\rho| = 3n+2$.

Now we define the recursive well-founded tree $T \subseteq \omega^{<\omega}$ and hypcode $c : T \to \omega$ as follows:

1. for $\rho \in \omega^{<\omega}$ with length $|\rho| = 3n+2$ and $trip(\rho) = (\sigma, \tau_1, \tau_2)$ if
   (a) $(\sigma{\restriction}n, \tau_1{\restriction}n) \in T^A$,

(b) $(\sigma \upharpoonright n, \tau_2) \in T^B$, and

(c) $(\sigma, \tau_1) \notin T^A$,

then ρ is a terminal node of T and put $c(\rho) = n_0$ where

$$U_{n_0} = \emptyset.$$

2. for $\rho \in \omega^{<\omega}$ with length $|\rho| = 3(n+1)$ and $trip(\rho) = (\sigma, \tau_1, \tau_2)$ if

   (a) $(\sigma, \tau_1) \in T^A$,

   (b) $(\sigma \upharpoonright n, \tau_2 \upharpoonright n) \in T^B$, and

   (c) $(\sigma, \tau_2) \notin T^B$,

   then ρ is a terminal node of T and put $c(\rho) = n_1$ where

$$U_{n_1} = [\sigma] =^{def} \{x \in \omega^\omega \ : \ \sigma \subseteq x\}.$$

3. For any other ρ we put ρ into T iff it is a proper subset of a terminal node of T. For these ρ we put $c(\rho) = 0$ if $|\rho| = 3n$ or $|\rho| = 3n + 1$ and put $c(\rho) = 1$ if $|\rho| = 3n + 2$.

Now given $trip(\rho) = (\sigma, \tau_1, \tau_2)$ define the following sets:

$$A_\rho = \{x \in [\sigma] \ : \ \exists y \supseteq \tau_1 \ \ \forall n \ \ (x \upharpoonright n, y \upharpoonright n) \in T^A\}$$

$$B_\rho = \{x \in [\sigma] \ : \ \exists z \supseteq \tau_2 \ \ \forall n \ \ (x \upharpoonright n, z \upharpoonright n) \in T^B\}$$

To finish the proof we verify the following:

Claim. For each $\rho \in T$ let $trip(\rho) = (\sigma, \tau_1, \tau_2)$ then

$$A_\rho \subseteq H(c, \rho) \subseteq [\sigma]$$

and

$$B_\rho \subseteq [\sigma] \setminus H(c, \rho)$$

Proof

Case ρ a terminal node of T.

Note that in case 1 of the definition of T, we have that A_ρ is the empty set and $c(\sigma)$ is a code for the empty set and so its OK. In case 2 of the definition of T, we have that B_ρ is the empty set and $c(\sigma)$ is a code for $[\sigma]$ and so its OK.

Case $|\rho| = 3n$ and ρ not terminal.
Note that for nonterminal nodes ρ we have that for every k that $\rho k \in T$. In this case $trip(\rho k) = (\sigma k, \tau_1, \tau_2)$.

$$A_{\rho k} = [\sigma k] \cap A_\rho$$

$$B_{\rho k} = [\sigma k] \cap B_\rho$$

and by induction

$$A_\rho = \cup_{k<\omega} A_{\rho k} \subseteq \cup_{k<\omega} H(c, \rho k) =^{def} H(c, \rho) \subseteq [\sigma]$$

($c(\rho) = 0$, so we take unions)

$$B_\rho = \cup_{k<\omega} B_{\rho k} \subseteq \cup_{k<\omega}([\sigma k] \setminus H(c, \rho k)) = [\sigma] \setminus H(c, \rho)$$

The last equality holds because each $H(c, \rho k) \subseteq [\sigma k]$ and $([\sigma k] : k < \omega)$ is a partition of $[\sigma]$.

Case $|\rho| = 3n + 1$ and ρ not terminal.
In this case $trip(\rho k) = (\sigma, \tau_1 k, \tau_2)$, and also $c(\rho) = 0$, i.e., we take unions. Note that for every k that $B_{\rho k} = B_\rho$ since neither σ nor τ_2 change. Also, by the definition of A_ρ note that

$$A_\rho = \cup_{k<\omega} A_{\rho k}.$$

Now by inductive hypothesis we have that

$$A_\rho = \cup_{k<\omega} A_{\rho k} \subseteq \cup_{k<\omega} H(c, \rho k) =^{def} H(c, \rho)$$

$$B_\rho \subseteq [\sigma] \setminus H(c, \rho k)$$

for every k so

$$B_\rho \subseteq [\sigma] \setminus H(c, \rho)$$

as was to be proved.

Case $|\rho| = 3n+2$ and ρ not terminal.
In this case $trip(\rho k) = (\sigma, \tau_1, \tau_2 k)$, and $c(\rho) = 1$, i.e., take intersections. Note that for every k that $A_{\rho k} = A_\rho$ since neither σ nor τ_1 change. Now by inductive hypothesis we have that

$$A_\rho \subseteq \cap_{k<\omega} H(c, \rho k) =^{def} H(c, \rho)$$

$$B_\rho = \cup_{k<\omega} B_{\rho k} \subseteq \cup_{k<\omega} [\sigma] \setminus H(c, \rho k) = [\sigma] \setminus H(c, \rho)$$

as was to be proved.

This proves the Claim. However since $A_{\langle\rangle} = A$ and $B_{\langle\rangle} = B$ the Theorem follows.
QED