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M773
Computability Theory
Fall 2001

(9-5)
Def Turing machine, Turing computable functions.

(9-7)
Def Primitive recursive functions, Primitive recursive implies Turing computable.

1. Prove $f(n) = 2n$ is Turing computable by constructing an actual Turing machine.

(9-10)
Def Partial recursive functions, Partial recursive implies Turing computable. Started on converse.

2. Define n is square-free iff $n \geq 2$ and no m^2 divides n for $m \geq 2$. Let $S(n)$ be the sum of the first n square-free numbers. Prove S is a Primitive recursive function.

(9-12)
For any partial Turing computable f there exists primitive recursive g and R such that

$$f(\vec{n}) = g(\mu x\ R(x, \vec{n}))$$

3. Prove there is a total $f : \omega \to \omega$ such that the graph of f is a primitive recursive predicate but f is not primitive recursive.

(9-17)
Church-Turing Thesis, universal functions, halting problem, effective listing of all primitive recursive functions, a computable function which is not primitive recursive.

4. Prove that the halting problems for nonwritting Turing machines is decidable. A Turing machine m is nonwritting iff whenever $m(s, a) = (s', a', d)$ then $a = a'$. It is decidable whether or not m halts when given input $x =< x_1, \ldots, x_n >\in A^n$ started on x_1 in state s_0.

(9-19)

Padding Lemma, S-n-m Theorem,
computably enumerable=range of computable function = Σ^0_1,
computable=ce and co-ce, K not computable.

5. Show there exists a uniformly computable listing of all partial computable functions, $\{\psi_e(x) : e \in \omega\}$, which fails the padding lemma. Hint: Obtain a listing so that the empty function only occurs once, say as ψ_0.

(9-21)

every infinite ce set contains an infinite comp set, every inf ce set has a 1-1 enumeration, many-one reducibility, one-reducibility, K, W one-complete.

6. (a) Prove that every nonempty computably enumerable set has a primitive recursive enumeration.

   *(b) Prove or disprove: Every infinite computably enumerable set has a one-one primitive recursive enumeration.

(9-24)

(Myhill) 1-1 equivalence iff computable permutation, (Rogers) if ψ_e unif enum of partial comp fcns satisfies padding, s-1-1, then there is a comp permutation π with $\psi_e = \phi_{\pi(e)}$ all e. Rice's index theorem.

7. In Roger's Theorem prove that s-1-1 implies padding. Hint: Soare I-5.9.

(9-26)

Post's construction of a simple set, (Myhill) 1-1 equivalence to K is the same as creative.

8. Define $V_e = \{n :< e, n >\in V\}$. Prove or disprove:
   (a) $\exists V$ c.e. such $\{V_e : e \in \omega\}$ is the set of all computable sets.
   (b) $\exists V$ computable such $\{V_e : e \in \omega\}$ is the set of all computable sets.
   (c) $\exists V$ c.e. such $\{V_e : e \in \omega\}$ is the set of all nonempty c.e. sets.
   (d) $\exists f$ computable function such that for all e $\quad W_e \neq \emptyset$ implies $f(e) \in W_e$.
   (e) $\exists f$ partial computable such that for all e $\quad W_e \neq \emptyset$ implies $f(e) \downarrow\in W_e$.

(9-28)

Recursion Theorem, uniform recursion theorem, recursion theorem with parameters.

9. Prove

(a) $\exists^\infty e \;\; W_e = \{0, 1, 2, \ldots, e\}$

(b) Suppose $V \subseteq \omega$ is c.e. $\exists^\infty e \;\; W_e = V_e$ (where $V_e = \{n :< e, n >\in V\}$)

(c) there exists e_1, e_2 with $e_1 \neq e_2$ and $W_{e_1} = \{e_2\}$ and $W_{e_2} = \{e_1\}$

(10-1)

Def Turing reducible, Dekker deficiency set is simple and Turing equivalent to the set.

10. Computable Skolem functions? Prove or disprove:

(a) Given a computable $R \subseteq \omega^2$ such that $\forall x \exists y \; R(x, y)$ there exists a computable f such that $\forall x \; R(x, f(x))$

(b) Given a computable $R \subseteq \omega^3$ such that $\forall x \exists y \forall z \; R(x, y, z)$ there exists a computable f such that $\forall x \forall z \; R(x, f(x), z)$

Hint: Think "Simple".

(10-3)

(Martin) A effectively simple implies $A \equiv_T K$. Def $A \oplus B$, A'.

11. Prove

(a) $A \leq_T A \oplus B$ and $B \leq_T A \oplus B$

(b) $A \oplus B \equiv_T B \oplus A$

(c) $(A \oplus B) \oplus C \equiv_T A \oplus (B \oplus C)$

(d) if $A \leq_T C$ and $B \leq_T C$, then $A \oplus B \leq_T C$

(e) if $A \leq_T \hat{A}$ and $B \leq_T \hat{B}$, then $A \oplus B \leq_T \hat{A} \oplus \hat{B}$

(10-5)

Def Turing jump, $A \equiv_T B$ implies $A' \equiv_T B'$, (Kleene-Post) there exists incomparable Turing degrees.

(10-8)

Kleene-Post. For every $a > 0$ there exists b incomparable to a. There exists a, b nonzero with infimum 0.

12. Prove

(a) there exists A such that for every n
A_n is not Turing reducible to $\cup\{A_m : m \neq n\}$

(b) There exists Turing degrees a_r for $r \in \mathbf{Q}$ such that
for all $r, s \in \mathbf{Q}$ $\;(r < s$ iff $a_r < a_s)$. Hint: use part (a).

(c)* Same as part (b) but also $a_r < 0'$ for all r.

(10-10)

(Kleene-Post-Spector) Given increasing degrees $\langle a_n : n \in \omega \rangle$ there are upper bounds b, c such that $d \leq b$ and $d \leq c$ implie for some n that $d \leq a_n$.

13. Show that for every nonzero degree a there is a no zero b such that 0 is the infimum of a and b.

14. Show that $0^{(\omega)}$ is not a minimal upper bound of $\{0^{(n)} : n \in \omega\}$.
    Hint: in theorem above get B, C computable in $0^{(\omega)}$.

(10-12)

(Friedberg) For every $B \geq_T 0'$ there exists A with $A' \equiv_T B$.

15. Prove there exists a degree $a > 0$ with $a' = 0'$.

(10-15)

(Spector) minimal degrees exist. (Sacks) minimal upper bounds exist.

16. Prove there exists a perfect tree T such that every branch thru T has minimal degree.

(10-17)

Def Σ_n^0 etc., simple closure properties, universal Σ_n^0 sets, limit lemma.

17. Find the natural arithmetic classes in which the following sets belong:
    (a) $\{e : W_e = \emptyset\}$
    (b) $\{e : W_e \text{ is simple }\}$
    (c) $\{e : W_e \equiv_T K\}$
    (d) $\{< e_1, e_2 >: W_{e_1} =^* W_{e_2}\}$ (equal mod finite)

(10-19)

$\forall^\infty = \exists\forall$, $A' \in \Sigma_1^0(A)$, $A \leq_T 0^{(n)}$ iff $A \in \Delta_{n+1}^0$, $0^{(n)} \in \Sigma_n^0$, FIN Σ_2^0-complete

18. Prove that A is Σ_4^0 iff there exists a computable $P(s, t, x)$ such that for every x

$$A(x) \equiv \forall^\infty s\, \forall^\infty t\, P(s, t, x)$$

(10-22)

TOT is Π^0_2-complete, REC is Σ^0_3-complete.

19. Prove there is **no** Δ^0_2-complete set A (i.e. A is Δ^0_2 and every Δ^0_2 is many-one reducible to A)

Hint: Consider $B = \{e : \varphi_e(e) \downarrow \text{ and } \varphi_e(e) \notin A\}$.

20. Prove of disprove:

(a) there exists a total $f \leq_T 0''$ such that for all e, if W_e is computable, then $W_{f(e)} = \overline{W_e}$.

(b) there exists a total $f \leq_T 0^{(3)}$ such that for all e, if W_e is computable, then $W_{f(e)} = \overline{W_e}$.

(10-24)

COF is complete Σ^0_3, $\exists\forall\exists = \forall^\infty\exists$, started first priority argument: low simple set.

(10-26)

(Friedberg-Muchnik) There are Turing incomparable c.e. sets.

21. Prove that SIMP$= \{\ e : W_e$ is simple$\}$ is a complete Π^0_3 set.

Hint: Like the proof for COF but also let W_e kick the e^{th} marker at most once to make A meet W_e if W_e infinite.

(10-29)

Every countable poset embedds into the computably enumerable degrees.

(10-31)

Friedberg Splitting Theorem, Corollaries of the Sack's Splitting Theorem.

22. (Trachtenbrock see p.121 2.5)

Define A is autoreducible iff there exists e such that for all x,

$$\{e\}^{A \setminus \{x\}}(x) \downarrow = \psi_A(x)$$

Prove

(a) $\forall B\ \ \exists A \equiv_m B$ such that A is autoreducible.

(b) there exist a ce A which is not autoreducible.

(c) there exist a low ce A which is not autoreducible.

(d)* there exist a ce $A \equiv_T K$ which is not autoreducible?

(11-2)

Sacks Splitting Theorem.

(11-5)

(Friedberg) Unique effective enumeration of all ce sets. Dekker deficiency set is hypersimple.

23. * Prove there is a partial computable ψ such that for every e there is a unique i such that $\varphi_e = \psi_i$.

(11-7)

Equivalent definitions of hypersimple:
for any computable sequence $\exists^\infty k$ with $[n_k, n_{k+1}) \subseteq A$,
if $\overline{A} = \{b_0 < b_1 < \cdots\}$ then for every computable f $\exists^\infty n$ $f(n) < b_n$.

(Nerode) $A \leq_{tt} B$ iff Turing reducible by machine which always converges for any oracle iff Turing recucible by a computable time bounded oracle machine.

24. Prove or disprove:
(a) there exists V ce such that $\{V_e : e \in \omega\}$ = set of simple sets.
(b) there exists V ce such that $\{V_e : e \in \omega\}$ = set of ce nonsimple sets.
(c) there exists V ce such that $\{V_e : e \in \omega\}$ = set of ce coinfinite sets.
(d) there exists V ce such that $\{V_e : e \in \omega\}$ = set of cofinite sets.

Extra credit: In case its true, show that there is a unique enumeration as in Friedberg's Theorem.

25. Prove that $\leq_{tt}$ is transitive.

(11-9)

(Post) If $K \leq_{tt} A$, then A not hypersimple. Example of a simple A with $K \leq_{tt} A$. Equivalent definitions of hyperhypersimple.

26. Prove that for every A ce with $\overline{A}$ infinite there exists a hypersimple $B \supseteq A$.

(11-12)

Examples of hypersimple but not hyperhypersimple sets, Maximal set is hyperhypersimple, (Friedberg) Maximal sets exist.

(11-14)

There exists a maximal set $M \equiv_T K$. Example of h^2-simple set which is not maximal: $M_1 \cap M_2$ where M_i maximal and $M_1 \neq^* M_2$.

27. Let A be h^2-simple and $f : \omega \to A$ a computable 1-1 onto enumeration. Prove that $B = \{f(n) : n \in A\}$ is h^2-simple but not maximal.

28. Let A_0, A_1 be a Friedberg splitting of a maximal set M, ie., M is the disjoint union of the A_i and each A_i is ce but not computable. Prove that A_0 is nowhere simple, ie. for any R computable if $R \cap \overline{A_0}$ is infinite, then it contains an infinite ce set.

(11-16)

Turing degree of a maximal set is high, a ce degree is high iff it contains a dominating function. Definable subsets of the lattice of ce sets, $\mathcal{E}$, eg. computable, finite, simple, maximal.

29. Suppose A is h^2-simple and $\overline{A} = \{b_0 < b_1 < \cdots\}$. let f be a computable function. Prove that $\forall^\infty n\ f(n) < b_n$. See p.212 - 1.11

(11-19)

(Lachlan) A is h^2-simple iff for all ce $B \supseteq A$ there exists ce $C \supseteq A$ with $B \cap C = A$ and $B \cup C = \omega$. Example of nontrivial automorphism of $\mathcal{E}$, i.e. π identity on maximal set.

30. (Martin p.198-5.5) Show there exists a nontrivial ce set with no maximal superset.

(11-21)

(Martin) There exists π an automorphism of $\mathcal{E}$ and a hypersimple set A such that $\pi(A)$ is not hypersimple.

31. Prove or disprove: Suppose $\pi : \omega \to \omega$ is a bijection and $\pi^{-1}(C) \in \mathcal{E}$ for every $C \in \mathcal{E}$. Then $\pi(C) \in \mathcal{E}$ for every $C \in \mathcal{E}$, ie. π gives an automorphism of $\mathcal{E}$.

Hint: Use a maximal set.

(11-26)

(Sacks) There exists a nontrivial high degree. (Shoenfield) Thickness Lemma.

(11-28)

True stages, window lemma. Sacks Jump Theorem.

(11-30)

Finish proof of Jump Theorem. Non-trivial L_n and H_n degrees, a ce set A such that $0^{(n)} <_T A^{(n)} <_T 0^{(n+1)}$ for all n.

(12-3)

Sacks Density Theorem.

(12-5)

Sacks Density Theorem (continued).

(12-7)

Final claim in proof of Sacks Density Theorem. Stated Robinson Jump Interpolation and derived some corollaries.

32. (Robinson) Prove that for any ce sets C, D with $D <_T C$ there exists ce sets A and B such that
    (a) A and B are $\leq_T$ incomparable, and
    (b) $D <_T A <_T C$ and $D <_T B <_T C$.

The last three problems can be proved as Corollaries of (possibly relativized versions of) Sack's Splitting, Robinson Jump Interpolation, and the Recursion Theorem.

33. Prove that if A is ce and has low degree, then there exists ce sets B and C such that
    (a) $A \leq_T B$ and $A \leq_T C$,
    (b) B and C are $\leq_T$ incomparable, and
    (c) B' and C' are $\leq_T$ incomparable

34. Prove that if A is ce and not computable there exists ce sets B and C such that
    (a) $B \leq_T A$ and $C \leq_T A$,
    (b) B and C are $\leq_T$ incomparable, and
    (c) $A' \equiv_T B' \equiv_T C'$.

35. Prove there exists a ce sets A and B such that for every n

(a) $A^{(n)} <_T 0^{(n+1)}$ and $B^{(n)} <_T 0^{(n+1)}$ and
(b) $A^{(n)} \oplus B^{(n)} \equiv_T 0^{(n+1)}$

(12-10)

Friedberg-Muchnik on a tree. Lachlan-Yates minimal pair strategy for a single negative requirement.

(12-12)

Minimal pairs using trees.

(12-14)

Minimal pair of high degrees.

Handout: Bootleg copy of Julia Knight, Chris Ash, Computable structures and the hyperarithmetical hierarchy, Chapters 4-5. The remaining time was spent proving

$$\Delta^1_1 = HYP$$

Some details were omitted.

The same material is covered in the first 30 pages of Gerald Sacks book, Higher recursion theory. And also somewhere in Hartley Rogers book.