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M771

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## A selected list of books about set theory

### Beginning

Roitman, Introduction to modern set theory  
Vaught, Set theory, an introduction  
Monk, Introduction to set theory  
Moschovakis, Notes on set theory  
Just and Weese, Discovering modern set theory I, II  
Hausdorff, Set theory

### Intermediate

Kunen, Set theory  
Jech, Set theory  
Handbook of mathematical logic, chapter on set theory  
Ciesielski, Set theory for the working mathematician

### Advanced

Bartoszynski and Judah, Set theory, on the structure of the real line  
Kanamori, The higher infinite  
Shelah, Proper forcing  
Shelah, Cardinal arithmetic  
Devlin, Constructibility  
Jech, The axiom of choice  
Fremlin, Consequences of Martin's axiom

## Prerequisites

The prerequisite for this course is Math 770, Introduction to Mathematical Logic. Usually the following topics are among those covered in 770:

Naive set Theory: cardinal and ordinal arithmetic, axioms of Zermelo-Fraenkel set theory with the axiom of choice.

First order logic: formulas, sentences, theories, structures, models, satisfaction, the compactness, completeness and incompleteness theorems.

What follows is excerpted from a manuscript which can be found on my home page.

## The Axioms of Set Theory

Here are some. The whole system is known as  $ZF$  for Zermelo-Fraenkel set theory. When the axiom of choice is included it is denoted  $ZFC$ . It was originally developed by Zermelo to make precise what he meant when he said that the well-ordering principle follows from the axiom of choice. Later Fraenkel added the axiom of replacement. Another interesting system is GBN which is Gödel-Bernays-von Neumann set theory.

### Empty Set:

$$\exists x \forall y (y \notin x)$$

The empty set is usually written  $\emptyset$ .

### Extensionality:

$$\forall x \forall y (x = y \iff \forall z (z \in x \iff z \in y))$$

Hence there is only one empty set.

### Pairing:

$$\forall x \forall y \exists z \forall u (u \in z \iff u = x \vee u = y)$$

We usually write  $z = \{x, y\}$ .

### Union:

$$\forall x \exists y (\forall z (z \in y \iff (\exists u (u \in x \wedge z \in u)))$$

We usually write  $y = \cup x$ .  $A \cup B$  abbreviates  $\cup\{A, B\}$ .  $z \subseteq x$  is an abbreviation for  $\forall u (u \in z \rightarrow u \in x)$ .

### Power Set:

$$\forall x \exists y \forall z (z \in y \iff z \subseteq x)$$

We usually write  $y = P(x)$ . For any set  $x$ ,  $x + 1 = x \cup \{x\}$ .

### Infinity:

$$\exists y (\emptyset \in y \wedge \forall x (x \in y \rightarrow x + 1 \in y))$$

The smallest such  $y$  is denoted  $\omega$ , so  $\omega = \{0, 1, 2, \dots\}$ .

### Comprehension Scheme:

$$\forall z \exists y \forall x [x \in y \iff (x \in z \wedge \theta(x))]$$

The comprehension axiom is being invoked when we say given  $z$  let

$$y = \{x \in z : \theta(x)\}.$$

The formula  $\theta$  may refer to  $z$  and to other sets, but not to  $y$ . In general given a formula  $\theta(x)$  the family  $\{x : \theta(x)\}$  is referred to as a *class*, it may not be a set. For example, the class of all sets is

$$\mathbf{V} = \{x : x = x\}.$$

Classes that are not sets are referred to as *proper classes*. Every set  $a$  is a class, since the formula “ $x \in a$ ” defines it. The comprehension axioms say that the intersection of a class and a set is a set. We use **boldface** characters to refer to classes.

1.1 Define  $X \cap Y$ ,  $X \setminus Y$ , and  $\cap X$  and show they exist.

1.2 The *ordered pair* is defined by

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}.$$

Show it exists. Show the  $\langle x, y \rangle = \langle u, v \rangle$  iff  $x = u$  and  $y = v$ .

1.3 The *cartesian product* is defined by

$$X \times Y = \{\langle x, y \rangle : x \in X \text{ and } y \in Y\}.$$

Show it exists.

1.4 A function is identified with its graph. For any sets  $X$  and  $Y$  we let  $Y^X$  be the set of all functions with domain  $X$  and range  $Y$ . Show this set exists.

1.5 Given a function  $f : A \mapsto B$  and set  $C \subseteq A$  the *restriction* of  $f$  to  $C$ , written  $f \upharpoonright C$  is the function with domain  $C$  and equal to  $f$  everywhere in  $C$ . Show that it exists.  $f''C$  is the set of all elements of  $B$  that are in the image of  $C$ . Show that it exists.

1.6 Prove  $\omega$  exists (i.e. that there does exist a smallest such  $y$ ). Prove for any formula  $\theta(x)$  if  $\theta(0)$  and  $\forall x \in \omega (\theta(x) \rightarrow \theta(x+1))$ , then  $\forall x \in \omega \theta(x)$ .

1.7 Suppose  $G : Z \rightarrow Z$ . Show that for any  $x \in Z$  there exists a unique  $f : \omega \rightarrow Z$  such that  $f(0) = x$  and for all  $n \in \omega$   $f(n+1) = G(f(n))$ .

1.8 Let  $(V, E)$  be a graph. Informally, two vertices in any graph are *connected* iff (either they are the same or) there is a finite path using the edges of the graph connecting one to the other. Use the preceding problem to formally define and prove that the relation  $x$  is connected to  $y$ , written  $x \sim y$ , exists and is an equivalence relation on  $V$ . Equivalence classes of  $\sim$  are called the *components* of the graph.

1.9 Let  $A$  and  $B$  be disjoint sets and  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be one-to-one functions. Consider the *bipartite graph*  $V$  which has vertices  $A \cup B$  and edges given by the union of the graphs of  $f$  and  $g$ , i.e., there is an edge between  $a \in A$  and  $b \in B$  iff either  $f(a) = b$  or  $g(b) = a$ . Describe what the finite components of  $V$  must look like as a subgraph of  $V$ . Describe the infinite components of  $V$ .

1.10 Define  $|X| = |Y|$  iff there is a one-to-one onto map from  $X$  to  $Y$ . We say  $X$  and  $Y$  have the same cardinality. Define  $|X| \leq |Y|$  iff there is a one-to-one map from  $X$  to  $Y$ . Define  $|X| < |Y|$  iff  $|X| \leq |Y|$  and  $|X| \neq |Y|$ . (Cantor-Schröder-Bernstein) Show that if  $|A| \leq |B|$  and  $|B| \leq |A|$  then  $|A| = |B|$ .

1.11 Show that  $|A| \leq |A|$ . Show that if  $|A| \leq |B|$  and  $|B| \leq |C|$  then  $|A| \leq |C|$ .

- 1.12 Show that  $|P(X)| = |\{0, 1\}^X|$ .
- 1.13 (Cantor) Show that  $|X| < |P(X)|$ .
- 1.14 Show that the class of sets,  $\mathbf{V}$ , is not a set.
- 1.15 Show that  $|A \times (B \times C)| = |(A \times B) \times C|$ .
- 1.16 Show that  $|A^{B \times C}| = |(A^B)^C|$ .
- 1.17 Show that if there is a function  $f : A \mapsto B$  that is onto, then  $|B| \leq |A|$ .<sup>1</sup>

A set is finite iff it can be put into one-to-one correspondence with an element of  $\omega$ . A set is *countable* iff it is either finite or of the same cardinality as  $\omega$ . A set is *uncountable* iff it is not countable.  $\mathbb{R}$  is the set of real numbers and we use  $\mathfrak{c} = |\mathbb{R}|$  to denote its cardinality which is also called the cardinality of the continuum. Below, you may use whatever set theoretic definitions of the integers, rationals and real numbers that you know. For example, you may regard the reals as either Dedekind cuts in the rationals, equivalence classes of Cauchy sequences of rationals, infinitely long decimals, or ?points on a line.

- 1.18 Show that the set of integers  $\mathbb{Z}$  is countable.
- 1.19 Show that the set of odd positive integers is countable.
- 1.20 Show that the set of points in the plane with integer coordinates is countable.
- 1.21 Show that the countable union of countable sets is countable.<sup>2</sup>
- 1.22 For any set  $X$  let  $[X]^{<\omega}$  be the finite subsets of  $X$ . Show that the set of finite subsets of  $\omega$ , which is written  $[\omega]^{<\omega}$ , is countable.
- 1.23 Show that if there are only countably many atomic sentences then the set of all propositional sentences is countable.
- 1.24 Show that the set of rationals  $\mathbb{Q}$  is countable.
- Remark: A clever mapping of the positive rationals to the positive integers is  $\frac{p_1^{k_1} \dots p_n^{k_n}}{q_1^{l_1} \dots q_m^{l_m}}$  is mapped to  $p_1^{2k_1} \dots p_n^{2k_n} \cdot q_1^{2l_1+1} \dots q_m^{2l_m+1}$ .
- 1.25 A number is *algebraic* iff it is the root of some polynomial with rational coefficients. Show that the set of algebraic numbers is countable.
- 1.26 Show that any nontrivial interval in  $\mathbb{R}$  has cardinality  $\mathfrak{c}$ .
- 1.27 Show that  $P(\omega)$  has cardinality  $\mathfrak{c}$ .
- 1.28 Show that the set of all infinite subsets of  $\omega$ , which is written  $[\omega]^\omega$ , has cardinality  $\mathfrak{c}$ .
- 1.29 Show that the cardinality of  $\mathbb{R} \times \mathbb{R}$  is  $\mathfrak{c}$ .

<sup>1</sup>This requires the axiom of choice. It is open if it is equivalent to AC.

<sup>2</sup>Do you think you needed to use the Axiom of Choice?

- 1.30 For any set  $X$  let  $[X]^\omega$  be the countably infinite subsets of  $X$ . Show that  $|\mathbb{R}^\omega| = |[ \mathbb{R} ]^\omega| = \mathfrak{c}$ .<sup>3</sup>
- 1.31 Show that the cardinality of the set of open subsets of  $\mathbb{R}$  is  $\mathfrak{c}$ .
- 1.32 Show that the set of all continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$  has size  $\mathfrak{c}$ .
- 1.33 Show that  $\omega^\omega$  has cardinality  $\mathfrak{c}$ .

### Wellorderings

A linear order  $(L, \leq)$  is a *wellorder* iff for every nonempty  $X \subseteq L$  there exists  $x \in X$  such that for every  $y \in X$   $x \leq y$  ( $x$  is the minimal element of  $X$ ). For an ordering  $\leq$  we use  $<$  to refer to the strict ordering, i.e.  $x < y$  iff  $x \leq y$  and not  $x = y$ . We use  $>$  to refer to the converse order, i.e.  $x > y$  iff  $y < x$ .

- 2.1 Prove: Let  $(L, \leq)$  be any well-ordering and  $f : L \rightarrow L$  an increasing function ( $\forall x, y \in L$   $x < y \rightarrow f(x) < f(y)$ ). Then for every  $x$  in  $L$   $x \leq f(x)$ .
- 2.2 For two binary relations  $R$  on  $A$  and  $S$  on  $B$  we write  $(A, R) \simeq (B, S)$  iff there exists a one-to-one onto map  $f : A \rightarrow B$  such that  
for every  $x, y$  in  $A$   $(xRy \text{ iff } f(x)Sf(y))$ .  
Such a map is called an *isomorphism*. If  $(L_1, \leq_1)$  and  $(L_2, \leq_2)$  are well-orders and  $(L_1, \leq_1) \simeq (L_2, \leq_2)$  then show the isomorphism is unique.
- 2.3 Let  $(L, \leq)$  be a wellorder and for any  $a \in L$  let  $L_a = \{c \in L : c < a\}$ . Show that  $(L, \leq)$  is not isomorphic to  $(L_a, \leq)$  for any  $a \in L$ .
- 2.4 (Cantor) Show that for any two wellorders exactly one of the following occurs: they are isomorphic, the first is isomorphic to an initial segment of the second, or the second is isomorphic to an initial segment of the first.  
Hint: Let  $(L, \leq)$  and  $(K, \leq')$  be two wellorders. Let

$$G = \{ \langle a, b \rangle : (L_a, \leq) \simeq (K_b, \leq') \}$$

Show that  $G$  is the graph of an order preserving bijection whose domain is all of  $L$  or whose range is all of  $K$ .

### Axiom of Choice

(AC) *Axiom of Choice*: For every family  $F$  of nonempty sets there exists a choice function, i.e. a function  $f$  with domain  $F$  such that for every  $x$  in  $F$ ,  $f(x) \in x$ .

(WO) *Well-ordering Principle*: Every nonempty set can be well ordered.

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<sup>3</sup>See previous footnote.

(TL) *Tuckey's Lemma*: Every family of sets with finite character has a maximal element. A family of sets  $F$  has finite character iff for every set  $X$ ,  $X \in F$  iff for every finite  $Y \subseteq X$ ,  $Y \in F$ . (MP) *Maximality Principle*: Every family of sets closed under

the unions of chains has a maximal member.

(ZL) *Zorn's Lemma*: Every family of sets contains a maximal chain.

3.1 Show that ZL implies MP.

3.2 Show that MP implies TL.

3.3 Show that TL implies AC.

3.4 (Zermelo) Show that AC implies WO.

Hint: Given  $X$  use AC to get a function  $f : P(X) \mapsto X$  such that  $f(A) \in X \setminus A$  for all  $A$  a proper subset of  $X$ . Call a well ordering  $(L, \leq)$  respectful iff  $L \subseteq X$  and for every  $a \in L$   $f(L_a) = a$ . Show that the union of all respectful well orderings is a well ordering of  $X$ .

3.5 Given a nonempty family  $\mathcal{F}$  let  $<$  be a strict well-ordering of  $\mathcal{F}$ . Say that a chain  $\mathcal{C} \subseteq \mathcal{F}$  is greedy iff for every  $a \in \mathcal{F}$  if

$$\{b \in \mathcal{C} : b < a\} \cup \{a\}$$

is a chain, then either  $a \in \mathcal{C}$  or  $b < a$  for every  $b \in \mathcal{C}$ . Show that the union of all greedy chains is a maximal chain. Conclude that WO implies ZL.

## Ordinals

A set  $X$  is *transitive* iff  $\forall x \in X (x \subseteq X)$ . A set  $\alpha$  is an *ordinal* iff it is transitive and strictly well ordered by the membership relation (define  $x \leq y$  iff  $x \in y$  or  $x = y$ , then  $(\alpha, \leq)$  is a wellordering). We also include the empty set as an ordinal. For ordinals  $\alpha$  and  $\beta$  we write  $\alpha < \beta$  for  $\alpha \in \beta$ . The first infinite ordinal is written  $\omega$ . We usually write

$$0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \dots, n = \{0, 1, \dots, n-1\}, \dots, \omega = \{0, 1, 2, \dots\}$$

4.1 Show: If  $\alpha$  is an ordinal then so is  $\alpha + 1$ . (Remember  $\alpha + 1 = \alpha \cup \{\alpha\}$ .) Such ordinals are called *successor ordinals*. Ordinals that are not successors are called *limit ordinals*.

4.2 Show: If  $\alpha$  is an ordinal and  $\beta < \alpha$ , then  $\beta$  is an ordinal and  $\beta \subseteq \alpha$  and  $\beta = \{\gamma \in \alpha : \gamma \in \beta\}$ .

**Axiom of Regularity:**

$$\forall x (x \neq \emptyset \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x)))$$

Another way to say this is that the binary relation  $R = \{(u, v) \in x \times x : u \in v\}$  has a minimal element, i.e., there exist  $z$  such that for every  $y \in x$  it is not the case that  $zRy$ . Note: a minimal element is not the same as a least element.

- 4.3 Show  $\alpha$  is an ordinal iff  $\alpha$  is transitive and linearly ordered by the membership relation.
- 4.4 For any ordinals  $\alpha$  and  $\beta$  show that  $\alpha \cap \beta = \alpha$  or  $\alpha \cap \beta = \beta$ . Show any two ordinals are comparable, i.e., for any two distinct ordinals  $\alpha$  and  $\beta$  either  $\alpha \in \beta$  or  $\beta \in \alpha$ .
- 4.5 The union of set  $A$  of ordinals is an ordinal, and is  $\sup(A)$ .
- 4.6 Show that the intersection of a nonempty set  $A$  of ordinals is the least element of  $A$ , written  $\inf(A)$ . Hence any nonempty set of ordinals has a least element.
- 4.7 Prove transfinite induction: Suppose  $\phi(0)$  and  $\forall \alpha \in \mathbf{ORD}$  if  $\forall \beta < \alpha \phi(\beta)$ , then  $\phi(\alpha)$ . Then  $\forall \alpha \in \mathbf{ORD} \phi(\alpha)$ .

#### Replacement Scheme Axioms:

$$\forall a ([\forall x \in a \exists! y \psi(x, y)] \rightarrow \exists b \forall x \in a \exists y \in b \psi(x, y))$$

The formula  $\psi$  may refer to  $a$  and to other sets but not to  $b$ . Replacement says that for any function that is a class the image of a set is a set. If  $\mathbf{F}$  is a function, then for any set  $a$  there exists a set  $b$  such that for every  $x \in a$  there exists a  $y \in b$  such that  $\mathbf{F}(x) = y$ .

- 4.8 (von Neumann) Let  $(L, \leq)$  be any well-ordering. Show that the following is a set:

$$\{(x, \alpha) : x \in L, \alpha \in \mathbf{ORD}, \text{ and } (L_x, \leq) \simeq \alpha\}.$$

Show that every well ordered set is isomorphic to a unique ordinal.

Let  $\mathbf{ORD}$  denote the class of all ordinals.

#### *Transfinite Recursion:*

If  $\mathbf{F}$  is any function defined on all sets then there exists a unique function  $\mathbf{G}$  with domain  $\mathbf{ORD}$  such that for every  $\alpha$  in  $\mathbf{ORD}$   $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \upharpoonright \alpha)$ .

This is also referred to as a transfinite construction of  $\mathbf{G}$ .

- 4.9 Suppose  $\mathbf{F} : \mathbf{V} \rightarrow \mathbf{V}$ , i.e., a class function. Define  $g$  (an ordinary set function) to be a good guess iff  $\text{dom}(g) = \alpha \in \mathbf{ORD}$ ,  $g(0) = \mathbf{F}(\emptyset)$ , and  $g(\beta) = \mathbf{F}(g \upharpoonright \beta)$  for every  $\beta < \alpha$ . Show that if  $g$  is a good guess, then  $g \upharpoonright \beta$  is good guess for any  $\beta < \alpha$ .
- 4.10 Show that if  $g$  and  $g'$  are good guesses with the same domain, then  $g = g'$ .
- 4.11 Show that for every  $\alpha \in \mathbf{ORD}$  there exists a (necessarily unique) good guess  $g$  with domain  $\alpha$ .
- 4.12 (Fraenkel) Prove transfinite recursion.
- 4.13 Explain the proof of WO implies ZL in terms of a transfinite construction.

For an example, consider the definition of  $V_\alpha$  for every ordinal  $\alpha$ . Let  $V_0 = \emptyset$ ,  $V_{\alpha+1} = P(V_\alpha)$  (power set) for successor ordinals, and for limit ordinals  $V_\lambda = \cup_{\beta < \lambda} V_\beta$ . Thus if we define  $\mathbf{F}$  as follows:

$$\mathbf{F}(x) = \begin{cases} P(x(\alpha)) & \text{if } x \text{ is a function with domain } \alpha + 1 \in \mathbf{ORD} \\ \cup_{\alpha < \lambda} x(\alpha) & \text{if } x \text{ is a function with domain limit ordinal } \lambda \\ P(\emptyset) & \text{if } x = \emptyset \\ -\pi & \text{otherwise} \end{cases}$$

then  $\mathbf{G}(\alpha) = V_\alpha$ .

- 4.14 Show that if  $\alpha \leq \beta$ , then  $V_\alpha \subseteq V_\beta$  and if  $\alpha < \beta$ , then  $V_\alpha \in V_\beta$ . Show that each  $V_\alpha$  is transitive.
- 4.15 Show that every set is included in a transitive set. Show that for every transitive set  $x$  there exists an ordinal  $\alpha$  such that  $x \in V_\alpha$ , i.e.,  $\mathbf{V} = \cup_{\alpha \in \mathbf{ORD}} V_\alpha$ .

*Ordinal arithmetic:* ( $\alpha, \beta$  are ordinals, and  $\lambda$  is a limit ordinal.)

Addition:

$$\alpha + 0 = \alpha$$

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1$$

$$\alpha + \lambda = \sup\{\alpha + \beta : \beta < \lambda\}$$

Multiplication:

$$\alpha 0 = 0$$

$$\alpha(\beta + 1) = (\alpha\beta) + \alpha$$

$$\alpha\lambda = \sup\{\alpha\beta : \beta < \lambda\}$$

Exponentiation:

$$\alpha^0 = 1$$

$$\alpha^{\beta+1} = \alpha^\beta \alpha$$

$$\alpha^\lambda = \sup\{\alpha^\beta : \beta < \lambda\}$$

So for example the addition function  $+: \mathbf{ORD} \times \mathbf{ORD} \mapsto \mathbf{ORD}$  exists by transfinite recursion. For each  $\alpha \in \mathbf{ORD}$  define a function  $\mathbf{F}_\alpha$  on all sets as follows:

$$\mathbf{F}_\alpha(g) = g(\beta) + 1 \text{ if } g \text{ is a map with domain an ordinal } \beta + 1,$$

$$\mathbf{F}_\alpha(g) = \sup\{g(\gamma) : \gamma < \lambda\} \text{ if } g \text{ is a map with domain a limit ordinal } \lambda, \text{ and}$$

$$\mathbf{F}_\alpha(g) = \alpha \text{ otherwise.}$$

Hence for each  $\alpha$  we have a unique  $\mathbf{G}_\alpha : \mathbf{ORD} \mapsto \mathbf{ORD}$  which will exactly be  $\mathbf{G}_\alpha(\beta) = \alpha + \beta$ . Since each  $\mathbf{G}_\alpha$  is unique we have defined the function  $+$  on all pairs of ordinals.

Note that more intuitively  $\alpha + \beta$  is the unique ordinal isomorphic to the well-order  $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$  ordered lexicographically. Similarly  $\alpha \cdot \beta$  is the unique ordinal isomorphic to the well-order  $\beta \times \alpha$  ordered lexicographically. Exponentiation is much harder to describe.

- 4.16 Show that  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ .

- 4.17 For any ordinals  $\alpha$  and  $\beta > 0$  show there exists unique ordinals  $\gamma$  and  $\delta$  such that  $\alpha = \beta\gamma + \delta$  and  $\delta < \beta$ .
- 4.18 Show for any  $\beta > 0$  there exists unique ordinals  $\gamma_1, \dots, \gamma_n, d_1, \dots, d_n$  such that  $\gamma_1 > \gamma_2 > \dots > \gamma_n$ ;  $0 < d_1, d_2, \dots, d_n < \omega$  and

$$\beta = \omega^{\gamma_1} d_1 + \omega^{\gamma_2} d_2 + \dots + \omega^{\gamma_n} d_n$$

This is called *Cantor normal form*.

### Cardinal Arithmetic

An ordinal  $\kappa$  is a *cardinal* iff for every  $\alpha < \kappa$ ,  $|\alpha| < |\kappa|$ . The cardinality of a set  $A$  is the least cardinal  $\kappa$ ,  $|A| = |\kappa|$  and we write  $|A| = \kappa$ . The  $\alpha^{th}$  uncountable cardinal is written either  $\aleph_\alpha$  or  $\omega_\alpha$ .

- 5.1 Show that for any ordinal  $\alpha$  the cardinal  $\aleph_\alpha$  exists.
- 5.2 Is there a cardinal such that  $\aleph_\alpha = \alpha$ ?

For cardinals  $\kappa$  and  $\gamma$  we define  $\kappa\gamma$  to be the cardinality of the cross product and  $\kappa + \gamma$  to be the cardinality of the union of  $A$  and  $B$  where  $A$  and  $B$  are disjoint and  $|A| = \kappa$  and  $|B| = \gamma$ .

- 5.3 Let  $\kappa$  be an infinite cardinal. Define the *lexicographical order*  $\leq_l$  on  $\kappa \times \kappa$  by  $(x, y) \leq_l (u, v)$  iff  $x < u$  or  $(x = u$  and  $y \leq v)$ . Define  $\leq'$  on  $\kappa \times \kappa$  by  $(x, y) \leq' (u, v)$  iff  $\max\{x, y\} < \max\{u, v\}$  or  $[(\max\{x, y\} = \max\{u, v\}$  and  $(x, y) \leq_l (u, v)]$ . Show that  $(\kappa, \leq) \simeq (\kappa \times \kappa, \leq')$ .
- 5.4 Show that for infinite cardinals  $\kappa$  and  $\gamma$ ,  $\kappa + \gamma = \kappa\gamma = \max\{\kappa, \gamma\}$ .
- 5.5 Show that for any infinite cardinal  $\kappa$  the union of  $\kappa$  many sets of cardinality  $\kappa$  has cardinality  $\kappa$ .

The *cofinality* of an infinite limit ordinal  $\beta$ ,  $cf(\beta)$ , is the least  $\alpha \leq \beta$  such that there is a map  $f : \alpha \rightarrow \beta$  whose range is unbounded in  $\beta$ .

- 5.6 For  $\lambda$  a limit ordinal show the following are all equivalent:
- $\alpha$  is the minimum ordinal such that  $\exists X \subseteq \lambda$  unbounded in  $\lambda$  such that  $(X, \leq) \simeq (\alpha, \leq)$ .
  - $\alpha$  is the minimum ordinal such that  $\exists f : \alpha \rightarrow \lambda$  such that  $f$  is one-to-one, order preserving, and the range of  $f$  is cofinal (unbounded) in  $\lambda$ .
  - $cf(\lambda) = \alpha$
- 5.7 For  $\kappa$  an infinite cardinal show that  $cf(\kappa) = \alpha$  iff  $\alpha$  is the minimum cardinal such that  $\kappa$  is the union of  $\alpha$  many sets of cardinality less than  $\kappa$ .

- 5.8 Let  $\alpha$  and  $\beta$  be limit ordinals and suppose  $f : \alpha \rightarrow \beta$  is strictly increasing and cofinal in  $\beta$ . Show  $cf(\alpha) = cf(\beta)$ .
- 5.9  $\kappa$  is *regular* iff  $cf(\kappa) = \kappa$ .  $\kappa$  is *singular* iff  $cf(\kappa) < \kappa$ . Show that for any limit ordinal  $\beta$ ,  $cf(\beta)$  is a regular cardinal.
- 5.10  $\kappa^+$  is the least cardinal greater than  $\kappa$ . Show that for any infinite cardinal  $\kappa$ ,  $\kappa^+$  is a regular cardinal.
- 5.11 For  $\alpha$  a limit ordinal show that  $cf(\aleph_\alpha) = cf(\alpha)$ .

$\kappa^\gamma$  is the set of all functions from  $\gamma$  to  $\kappa$ , but we often use it to denote its own cardinality.  $\kappa^{<\gamma}$  is  $\bigcup\{\kappa^\alpha : \alpha < \gamma\}$ , but we often use it to denote its own cardinality, i.e.,  $\kappa^{<\gamma} = |\kappa^{<\gamma}|$ . **Note in the section all exponentiation is cardinal exponentiation.**

- 5.12 Show that  $|\kappa^{<\omega}| = |[ \kappa ]^{<\omega}| = \kappa$  for any infinite cardinal  $\kappa$ .  $[ \kappa ]^{<\omega}$  is the set of finite subsets of  $\kappa$ .
- 5.13 Show that  $\aleph_\omega < \aleph_\omega^\omega$ . Show that for any cardinal  $\kappa$ ,  $\kappa < \kappa^{cf(\kappa)}$ .
- 5.14 (König) Show that  $cf(2^\kappa) > \kappa$ .