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M771
Fall 2011

Homework problems are due in class one week from the day assigned (which is in parentheses).

1. (1-19 W) Prove that for ordinals if $0 < \alpha \leq \beta$, then there are unique γ, δ with $\beta = \alpha \cdot \gamma + \delta$ and $0 \leq \delta < \alpha$.

2. (1-19 W)[Kunen] Prove that if α, β are non-zero ordinals, then there is a largest ordinal which divides both of them. Here, δ divides α iff $\alpha = \delta\xi$ for some ξ.

3. (1-21 F) Prove that there are unboundedly many cardinals κ with $\aleph_\kappa = \kappa$.

4. (1-24 M) Prove that $[\aleph_\omega]^\omega$ has cardinality strictly larger than $\aleph_\omega$.

$$[X]^\kappa = \{A \subseteq X \ : \ |A| = \kappa\}.$$

5. (1-24 M) Assume the continuum hypothesis. Prove that $[\aleph_n]^\omega = \aleph_n$ for every n with $1 \leq n < \omega$.

6. (1-26 W) Suppose for all $n < \omega$ that $2^{\aleph_n} = \aleph_{n+1}$. Prove that $\aleph_\omega^\omega = 2^{\aleph_\omega}$.

7. (1-26 W) Suppose there exists α such that for all $n < \omega$ that $2^{\aleph_n} = \aleph_\alpha$. Prove that $2^{\aleph_\omega} = \aleph_\alpha$.

8. (1-28 F) Let INV stand for the property that every onto function has an inverse, i.e., if $f : A \to B$ is onto, then there exists $g : B \to A$ such that $f(g(b)) = b$ for every $b \in B$. Prove that INV is equivalent to AC (the axiom of choice).

9. (1-28 F) Prove there exists a pairwise disjoint family $\{X_\alpha \subseteq 2^\omega \ : \ \alpha < \mathfrak{c}\}$ of Bernstein sets.

10. (1-28 F) Prove there exists a family $\{X_\alpha \subseteq 2^\omega \ : \ \alpha < 2^\mathfrak{c}\}$ of Bernstein sets which are distinct, i.e., $X_\alpha \neq X_\beta$ whenever $\alpha \neq \beta$.

11. (1-31 M) Prove there exists a subset X of the plane which meets every line in exactly two points.

12. (2-4 F) Assume MA. Prove there exists $(f_\alpha \in \omega^\omega \ : \ \alpha < \mathfrak{c})$ such that
   (a) $\alpha < \beta < \mathfrak{c}$ implies $f_\alpha <^* f_\beta$
   (b) for every $g \in \omega^\omega$ there exists $\alpha < \mathfrak{c}$ such that $g <^* f_\alpha$

13. (2-7 M) Assume MA. Prove there exists an ultrafilter $\mathcal{U} \subseteq [\omega]^\omega$ such that for every $f \in \omega^\omega$ there exists $X \in \mathcal{U}$ such that $f \restriction X$ is either constant or one-to-one.

14. (2-9 W) For $x \in 2^\omega$ let $b_x = \{x \restriction n \ : \ n < \omega\}$ be the infinite branch thru $2^{<\omega}$ determined by x. Let $\mathcal{A} = \{b_x \subseteq 2^{<\omega} \ : \ x \in 2^\omega\}$ be the almost disjoint family of infinite paths.

Given $\mathcal{A}_0, \mathcal{A}_1 \subseteq \mathcal{A}$ with $\mathcal{A}_0 \cap \mathcal{A}_1 = \emptyset$ define the poset $\mathbb{P}$ as follows:

$(A_0, A_1) \in \mathbb{P}$ iff $A_0 \cap A_1 = \emptyset$ and there exists finite $F_0 \subseteq \mathcal{A}_0$ and $F_1 \subseteq \mathcal{A}_1$ such that $A_0 =^* \cup F_0$ and $A_1 =^* \cup F_1$.

Order $\mathbb{P}$ by $(A_0, A_1) \trianglelefteq (B_0, B_1)$ iff $A_0 \supseteq B_0$ and $A_1 \supseteq B_1$.
Prove that $\mathbb{P}$ has ccc.

15. (2-9 W) Assume MA. Prove that if $\mathcal{A}_0, \mathcal{A}_1 \subseteq \mathcal{A}$ with $\mathcal{A}_0 \cap \mathcal{A}_1 = \emptyset$ and $|\mathcal{A}_0 \cup \mathcal{A}_1| < |2^\omega|$, then there are disjoint a_0 and a_1 such that $b \subseteq {}^* a_0$ for every $b \in \mathcal{A}_0$ and $b \subseteq {}^* a_1$ for every $b \in \mathcal{A}_1$.

16. (2-11 F) Assume MA. Suppose $\mathcal{F} \subseteq \omega^\omega$ and $|\mathcal{F}| < |2^\omega|$. Prove there exists $g \in \omega^\omega$ such that for every $f \in \mathcal{F} \ \ \exists^\infty n \in \omega \ \ g(n) = f(n)$.

17. (2-14 M) Suppose $D \subseteq 2^\kappa$ is an open dense set. Prove there exists $E \subseteq D$ a dense open set which is countably supported, i.e., there exists a countable $S \subseteq \kappa$ such that for every $x, y \in 2^\kappa$ if $x \restriction S = y \restriction S$, then ($x \in E$ iff $y \in E$).

18. (2-14 M) For each $\alpha < \omega_1$ define

$$C_\alpha = \{x \in 2^{\omega_1} \ : \ \forall n \in \omega \ \ x(\alpha + n) = 0\}.$$

(a) Prove that C_α is closed nowhere dense in 2^{ω_1}.
(b) Prove that $\bigcup_{\alpha < \omega_1} C_\alpha$ is not meager in 2^{ω_1}.

19. (2-16 W) Suppose $A_\alpha \subseteq \alpha$ for α a countable limit ordinal has order type ω and is an unbounded subset of α. Prove that $(A_\alpha \ : \ \alpha^{lim} < \omega_1)$ contains an

uncountable Δ-system, i.e., there exists an uncountable set Σ of countable limit ordinals and a finite R such that $A_\alpha \cap A_\beta = R$ for every pair of distinct $\alpha, \beta \in \Sigma$.

20. (2-18 F) Prove that 2^ω is compact.

21. (2-21 M) Suppose that T is a Souslin tree and $D \subseteq T$ is dense open. Prove that there exists $\alpha < \omega_1$ such that $T_\alpha \subseteq D$.

22. (2-23 W) Let T be a tree of height ω_1 such that T_α is finite for every $\alpha < \omega_1$. Prove that T must contain a chain of size ω_1.

Bonus Problem: Show that a tree of height ω_2 and countable width must have a chin of size ω_2.

23. (2-25 F) Suppose $f : \omega_1 \to \omega_1$. Define

$$C_f = \{\alpha < \omega_1 \ : \ \forall \beta < \alpha \ f(\beta) < \alpha\}$$

Prove that C_f is a closed unbounded subset of ω_1 (club).
Prove that for every club $C \subseteq \omega_1$ there exists f with $C_f \subseteq C$.

Bonus Problem: Suppose M is structure in a countable language and M has universe ω_1. Prove that the set

$$\{\alpha < \omega_1 \ : \ M \cap \alpha \preceq M\}$$

is a club. The symbol $\preceq$ stands for elementary substructure.

24. (2-28 M) Assume CH. Given $\{A_\alpha \in [\omega_2]^{\leq \omega} \ : \ \alpha < \omega_2\}$ prove there exists a stationary $S \subseteq \omega_2$ such that $\{A_\alpha \ : \ \alpha \in S\}$ is a Δ-system.

25. (3-2 W) Assume $\diamondsuit$. Prove there exists $(A_\alpha \ : \ \alpha < 2^{\omega_1})$ such that each A_α is a stationary subset of ω_1 and $A_\alpha \cap A_\beta$ is countable whenever $\alpha \neq \beta$.

26. (3-7 M) Let $\alpha > \omega$ be a limit ordinal. Prove that $(V_\alpha, \in)$ is a model of ZC, i.e., ZFC minus the replacement axioms.

27. (3-9 W) For κ an uncountable regular cardinal, define H_κ to the family of all sets X whose transitive closure has cardinality less than κ, $|TC(X)| < \kappa$. Prove that $(H_\kappa, \in)$ is a model of $ZFC \setminus$Powerset Axiom.

28. (3-11 F) Prove that the following are equivalent:

(a) x is an ordinal

(b) x and every element of x is transitive

(c) x is transitive and $(x, \in)$ is a strict linear order

29. (3-21 M) (Ashutosh Kumar) Say that a limit ordinal α is weakly-reflecting iff for any sentence θ if $V_\alpha \models \theta$, then there are unboundedly many $\beta < \alpha$ such that $V_\beta \models \theta$.

(a) Prove that the set $\{\alpha < \omega_1 \ : \ \alpha \text{ is weakly-reflecting}\}$ contains a closed unbounded set.

(b) Show that ZC+weakly-reflecting schema does not imply ZFC.

30. (3-23 W) Assume ZFC+V=L. Prove that $L_{\omega_1} = H_{\omega_1}$.

31. (3-25 F) Assume ZFC+V=L. Suppose that $Q \subseteq \omega_1$ is a stationary set. Prove that $\diamondsuit(Q)$ holds. This means that there exists $(S_\alpha \subseteq \alpha \ : \ \alpha \in Q)$ such that for every $X \subseteq \omega_1$

$$\{\alpha \in Q \ : \ S_\alpha = X \cap \alpha\}$$

is stationary.

32. (3-28 M) Assume there is a standard model of ZFC. Prove that there is a model of ZFC + Con(ZFC) + there is no standard model of ZFC.

33. (3-30 W) Assume that $\mathbb{P}$ is a poset in M a transitive standard model of ZFC and $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-filter. Prove that the following are equivalent:

1. G is $\mathbb{P}$-generic over M.

2. For every $A \in M$ if

$$M \models A \subseteq \mathbb{P} \text{ is a maximal antichain in } \mathbb{P}$$

   then $G \cap A \neq \emptyset$.

3. For every $E \subseteq \mathbb{P}$ with $E \in M$ there exists $p \in G$ such that either $p \in E$ or no $q \leq p$ is in E.

34. (4-1 F) Given $\mathbb{P}$-names τ and σ construct a $\mathbb{P}$-name ρ such that for every G $\mathbb{P}$-generic over M

$$\rho^G = \tau^G \times \sigma^G$$

Do not use $\Vdash$ in your construction.

35. (4-4 M) Suppose $\mathcal{F}$ is a family of subsets of $\mathbb{P}$. Define

$$D = \{p \in \mathbb{P} \ : (\forall E \in \mathcal{F} \ \ E \text{ is dense beneath } p) \text{ or } (\exists E \in \mathcal{F}\, \forall q \leq p \ \ q \notin E)\}$$

Prove that D is dense in $\mathbb{P}$.

36. (4-4 M) Suppose G is $\mathbb{P}$-generic over M and $p \in \mathbb{P}$. Prove that either $p \in G$ or there exists $q \in G$ with $p \perp q$.

37. (4-4 M) Suppose $G \subseteq \mathbb{P} \in M$ satisfies

1. $\forall p, q \in G \quad p$ and q are compatible
2. $\forall p \leq q$ if $p \in G$ then $q \in G$
3. $G \cap D \neq \emptyset$ for all $D \in M$ which are dense in $\mathbb{P}$

Prove that G is a $\mathbb{P}$ filter, i.e., $\forall p, q \in G \ \ \exists r \in G$ with $r \leq p$ and $r \leq q$.

38. (4-8 F) Suppose $\mathbb{P} \in M$ and $\mathbb{P}$ is everywhere non-trivial:

$$\forall p \in \mathbb{P}\ \exists q, r \leq p \ \ q \perp r.$$

Prove that $G \notin M$ for every G $\mathbb{P}$-generic over M.

39. (4-11 M) Suppose G is $\mathbb{P}$-generic over M and for all κ, γ ordinals in M if $f \in \kappa^\gamma \cap M[G]$ there exists $g \in M$ such that $M \models g : \gamma \to [\kappa]^\omega$ and $\forall \alpha < \gamma \ \ f(\alpha) \in g(\alpha)$. Prove that there exists $p \in G$ such that

$$M \models \mathbb{P}_{\leq p} \text{ has ccc.}$$

40. (4-15 F) Suppose M is a countable transitive standard model satisfying

$$M \models ZFC + \neg CH + \mathbb{P} = 2^{<\omega_1}$$

and G is $\mathbb{P}$-generic over M. Prove that

(a) $M \cap \mathcal{P}(\omega) = M[G] \cap \mathcal{P}(\omega)$ and
(b) $M[G] \models CH$.

41. (4-18 M) In a countable transitive model M of ZFC+GCH define $\mathbb{P}$ by: $p \in \mathbb{P}$ iff there exists $\alpha < \omega_1$ such that $p = (S_\beta \subseteq \beta \; : \; \beta < \alpha)$.
Define $p \leq q$ iff $\alpha_p \geq \alpha_q$ and $S^q_\beta = S^p_\beta$ for all $\beta < \alpha_q$.
Prove that if G is $\mathbb{P}$-generic over M then
(a) M and $M[G]$ have the same cardinals and
(b) $M[G] \models \diamondsuit$

42. (4-20 W) Suppose that M is a countable transitive model of ZFC,

$$M \models \mathbb{P} = Fn(\omega_2, 2, \omega)$$

G is $\mathbb{P}$-generic over M, and $x \in \mathcal{P}(\omega) \cap M[G]$ is definable in $M[G]$ with parameters from M. Prove that $x \in M$.

43. (4-25 M) Suppose that M is a countable transitive model of ZFC and $\alpha \in ORD^M$. Prove that there is a symmetry model $\mathcal{N}$ with $M \subseteq \mathcal{N} \subseteq M[G]$ and
(a) $\mathcal{N} \models ZF + \neg AC$
(b) $V^{\mathcal{N}}_\alpha = V^M_\alpha$

44. (4-27 W) Suppose that $\mathcal{N}$ is Cohen Basic model for $\neg AC$ and $X \in \mathcal{N}$ is the Dedekind-finite dense subset of 2^ω. Prove that $\mathcal{N}$ satisfies:

$$\forall Y \subseteq X \; \exists U \subseteq 2^\omega \text{ open such that } \quad Y =^* X \cap U$$

where $=^*$ means equal mod finite.

45. (4-28 F) Let $\mathcal{N}$ be same model as in (44), i.e., $M \subseteq \mathcal{N} \subseteq M[G]$ where $X = \{x_n \; : \; n < \omega\}$ and x_n is the n^{th} column of G. Working in $\mathcal{N}$ define

$$\mathbb{P}_0 = \{s \; : \; \exists n < \omega \;\; s : n \to X \text{ is one-to-one } \}$$

where $s \trianglelefteq t$ iff $s \supseteq t$. Working in $M[G]$ define $G_0 = \{s_n \; : \; n < \omega\}$ where $s_n : n \to X$ is defined by $s_n(i) = x_i$. Prove that G_0 is $\mathbb{P}_0$-generic over $\mathcal{N}$ and $\mathcal{N}[G_0] = M[G]$.