A.Miller
M771
Fall 2010

Homework problems are due in class one week from the day assigned (which is in parentheses).

1. (1-22 F) (Tarski) Suppose A is an infinite Dedekind finite set. For $n \in \omega^+$ define $A_n = \{s : n \to A \ : \ s \ 1-1\}$. Prove that

(a) A_n is Dedekind finite.

(b) For $X \subseteq \omega$ the set $A_X = \bigcup_{n \in X} A_n$ is Dedekind finite.

(c) There is a 1-1 map of A_n into A_m iff $n \leq m$.

(d) If $X \subseteq Y \subseteq \omega^+$ and there is a 1-1 map of A_X into A_Y, then $X = Y$.

(e) Show that if there is an infinite Dedekind finite cardinal then there are continuum many distinct infinite Dedekind finite cardinals.

2. (1-22 F) (a) Suppose L is a linear order and for every $X \subseteq L$ either $X \simeq L$ or $\overline{X} \simeq L$. (Overline denotes the complement.)

(b) Suppose α is ordinal. Show that (1) and (2) are equivalent.

(1) $X \simeq \alpha$ or $\overline{X} \simeq \alpha$ for every $X \subseteq \alpha$.

(2) There is a β with $\alpha = \omega^\beta$.

3. (1-27 W) Prove that the following are equivalent:

(1) There exists infinite sets $(A_\alpha \subseteq \omega_1 : \alpha < \omega_2)$ which are pairwise almost disjoint, i.e., $A_\alpha \cap A_\beta$ is finite whenever $\alpha < \beta < \omega_2$.

(2) CH is false.

4. (2-1 M) (Solomon) Prove that if $\mathfrak{b} > \omega_1$ and $\{a_\alpha \in [\omega]^\omega : \alpha < \omega_1\}$ are almost disjoint, then there exists $a \in [\omega]^\omega$ with $a \cap a_\alpha$ finite for every $\alpha < \omega_1$.

5. (2-5 F) (a) Let $\kappa > \omega$ be a regular cardinal. Prove that for every $(F_\alpha \in [\kappa]^{<\omega} \ : \ \alpha < \kappa)$ there exists $\Sigma \in [\kappa]^\kappa$ such that $(F_\alpha \in [\kappa]^{<\omega} \ : \ \alpha \in \Sigma)$ is a Δ-system. (b) Show that this fails for singular κ.

6. (2-8 M) Assume MA_{ω_1}. Prove that there exists $(A_\alpha \in [\omega_1]^{\omega_1} \ : \ \alpha < \omega_2)$ such that $A_\alpha \cap A_\beta$ is finite for every $\alpha < \beta < \omega_2$.

7. (2-10 W) Prove that MA_{ω_1} is equivalent to:

For every ccc $\mathbb{P}$ and $(E_\alpha \subseteq \mathbb{P} : \alpha < \omega_1)$ there exists a $\mathbb{P}$-filter G such that for every $\alpha < \omega_1$ either $G \cap E_\alpha$ is nonempty or there exists $p \in G$ such that no $q \leq p$ is in E_α.

8. (2-12 F) Assume MA_{ω_1}. Prove there exists an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ of size ω_1 such that for every $\mathcal{A}_0 \subseteq \mathcal{A}$ there exists $b \subseteq \omega$ such that $a \subseteq^* b$ for all $a \in \mathcal{A}_0$ and $a \subseteq^* \overline{b}$ for all $a \in \overline{\mathcal{A}_0}$.

9. (2-17 W) Let $T \subseteq \omega^{<\omega_1}$ be a Souslin subtree.

(a) Suppose $f : T \to \mathbb{R}$ satisfies

$$\forall s, t \in T \ \ (s \subseteq t \to f(s) \leq f(t))$$

Prove that the range of f is countable.

(b) Suppose $f : T \to \mathbb{R}$ is continuous. This means that for any $s \in T_\lambda$ for λ limit, $f(s \restriction \alpha_n)$ converges to $f(s)$ for any sequence α_n converging up to λ. Prove that the range of f is countable.

10. (2-19 F) Suppose $(C_\alpha \ : \ \alpha < \omega_1)$ is a ladder system. Show that for every $\Sigma \in [\omega_1]^{\omega_1}$ there exists $\Gamma \in [\Sigma]^{\omega_1}$ such that $(C_\alpha \ : \ \alpha \in \Gamma)$ is a Δ-system.

11. (2-19 F) (MA_{ω_1}) Suppose $(C_\alpha \ : \ \alpha < \omega_1)$ is a ladder system. Prove that for every $(f_\alpha : C_\alpha \to \omega \ : \ \alpha < \omega_1)$ there exists $F : \omega_1 \to \omega$ such that $F \restriction C_\alpha =^* f_\alpha$ for every $\alpha < \omega_1$.

12. (2-22 M) Suppose $T \subseteq (\omega + \omega^*)^{<\omega_1}$ is a subtree which is an Aronszajn tree and which has the property that for every $s \in T$ all immediate successors of s are in T. Let $(T, \trianglelefteq)$ be the following ordering:
$s \triangleleft t$ iff either

1. there exists $\alpha < |s|, |t|$ such that $s \restriction \alpha = t \restriction \alpha$ and $s(\alpha) < t(\alpha)$ or
2. $s \subset t$ with $|s| = \alpha < |t|$ and $t(\alpha) \in \omega^*$.

Prove:

(a) $(T, \trianglelefteq)$ is a dense linear order.

(b) Neither ω_1 nor ω_1^* order embed into $(T, \trianglelefteq)$.

(c) If $X \subseteq \mathbb{R}$ and X order embeds into $(T, \trianglelefteq)$, then X is countable.

13. (2-26 F) For $A \subseteq \omega_1$ define the following linear order type,

$$L_A = \sum_{\alpha < \omega_1} L_A^\alpha.$$

$$L_A^\alpha = \begin{cases} \eta & \text{if } \alpha \notin A \\ 1+\eta & \text{if } \alpha \in A \end{cases}$$

Here η stands for the order type of the rationals. Prove

$$L_A \simeq L_B \text{ iff } (A \Delta B) \text{ is nonstationary.}$$

14. (3-1 M) Assume $\diamondsuit$. Prove there exists a rigid Souslin tree.

15. (3-3 W) Suppose there exists $(S_\alpha \subseteq \alpha : \alpha < \omega_1)$ such that for every $X \subseteq \omega_1$ there exists a limit ordinal $\alpha < \omega_1$ with $X \cap \alpha = S_\alpha$. Prove that there exists a diamond sequence.

16. (3-5 F) Let $\alpha > \omega$ be a limit ordinal. Prove that $(V_\alpha, \in)$ is a model of ZFC minus the replacement axiom. Prove that $(V_{\omega+\omega}, \in)$ does not model ZFC.

17. (3-8 M) Let κ be an uncountable regular cardinal. Prove that $(H_\kappa, \in)$ is a model of ZFC minus the power set axiom.

18. (3-10 W) Let ZC=ZFC minus the replacement axioms. Prove that (ZC plus the reflection schema) implies the replacement axioms.

19. (3-12 F) (Kumar) Prove that the Mostowski collapse of OD is HOD, if

$$f(x) = \{f(y) : y \in x \cap OD\}$$

then $HOD = \{f(x) : x \in OD\}$.

20. (3-15 M) Let M, N be standard models of ZFC with same ordinals and sets of ordinals. Prove that $M = N$.

21. (3-17 W) Suppose that $M \subseteq N$ are standard models of ZF and $\kappa, \lambda \in M$. Prove that if $N \models \kappa \to (\omega)^{<\omega}_\lambda$, then $M \models \kappa \to (\omega)^{<\omega}_\lambda$.

22. (3-19 F) (Kumar) Say that a limit ordinal α is weakly-reflecting iff for any sentence θ if $V_\alpha \models \theta$, then there are unboundedly many $\beta < \alpha$ such that $V_\beta \models \theta$. Prove that the set $\{\alpha < \omega_1 \ : \ \alpha \text{ is not weakly-reflecting}\}$ is not stationary. Hence ZC+weakly reflecting does not imply ZFC.

23. (3-22 M) Suppose that κ is a strongly inaccessible cardinal. Prove that V_κ is a model of ZFC. Prove that $V_\kappa = H_\kappa$.

24. (3-22 M) Prove that the least κ (if it exists) such that V_κ is a model of ZFC is a strong limit cardinal with countable cofinality.

25. (3-24 W) Assume ZFC+V=L. Prove that for any regular κ that $L_\kappa = H_\kappa$.

26. (3-26 F) Assume ZFC+V=L. For any regular cardinal $\kappa > \omega$ let

$$Q_\kappa = \{\alpha < \omega_1 \ : \ L_\alpha \equiv L_\kappa\}.$$

Prove that Q_κ is unbounded in ω_1 and Q_κ is stationary iff $\kappa = \omega_1$.

27. (4-6 M) Assume ZFC+V=L. Let κ be an uncountable regular cardinal. Let N be the elements of L_κ which are definable in L_κ without parameters. Prove that N is a countable elementary substructure of L_κ. Let α be minimal such that $L_\alpha \equiv L_\kappa$. Prove that $Th(L_\kappa) \notin L_\alpha$.

28. (4-16 F) (M a countable standard transitive model of ZFC.) Suppose G is $\mathbb{P}$-generic over M and for every $f \in M[G]$ if $M[G] \models f : \kappa \to \gamma$ where κ and γ are ordinals, there exists $g \in M$ such that $M \models g : \kappa \to [\gamma]^\omega$ with $f(\alpha) \in g(\alpha)$ for every $\alpha < \kappa$. Prove that there exists $p \in G$ such that $M \models \mathbb{P}_{\leq p}$ is ccc.

29. (4-23 F) Suppose $M \models \mathbb{P}_1 \subseteq \mathbb{P}$ is a dense suborder. Prove

1. If G_2 is $\mathbb{P}_2$-generic over M, then $G_1 = G_2 \cap \mathbb{P}_1$ is $\mathbb{P}_1$-generic over M.
2. If G_1 is $\mathbb{P}_1$-generic over M, then $G_2 = \{p \in \mathbb{P}_2 : \exists q \in G_1 \ \ q \leq p\}$ is $\mathbb{P}_2$-generic over M.
3. In either case, $M[G_1] = M[G_2]$.

30. (4-23 F) Prove that if $\mathbb{P}$ is a countable poset which is everywhere non-trivial ($\forall p \in \mathbb{P} \ \exists q_1, q_2 \leq p \ \ q_1 \perp q_2$), then $\omega^{<\omega}$ is isomorphic to a dense subset of $\mathbb{P}$.

31. (4-26 M) Suppose there is a standard model of ZFC. Prove there exists a countable standard model M such that

$$M \models ZFC + \forall n < \omega \ \ ZFC_n \text{ has a standard model}$$

but $M \models$"ZFC does not have a standard model".