Chris Deviley
Math 771

OK

16. Let $\alpha > \omega$ be a limit ordinal.

Claim: $(V_\alpha, \in) \models ZFC -$ replacement

Extensionality: Since V_α is transitive,

if $x, y \in V_\alpha$ and $\forall z \in V_\alpha (z \in x \leftrightarrow z \in y)$

then $\text{pred}(V_\alpha, x, \in) = \text{pred}(V_\alpha, y, \in)$

and $\text{pred}(V_\alpha, x, \in) = x$, $\text{pred}(V_\alpha, y, \in) = y$ since V_α is transitive

$\Rightarrow x = y$.

Foundation: Since $V_\alpha \subset WF$, if $x \in V_\alpha$ and $\exists y \in V_\alpha$ s.t. $y \in x$

then take $z = \min\{w \in V_\alpha \cap x\}$, where $w < v$ iff $\text{rank}(w) < \text{rank}(v)$

then $z \in x$ and $\neg \exists w \in V_\alpha$ s.t. $w \in x \wedge w \in z$

Comprehension: Let $\varphi(x, y, \bar{w})$ be a formula and fix $y, \bar{w} \in V_\alpha$.

Consider $\{x \in y : \varphi^{V_\alpha}(x, y, \bar{w})\} = A$

then $\text{rank}(A) = \sup\{\text{rank}(x) + 1 : x \in A\}$
$\leq \text{rank}(y) + 1$

and since $y \in V_\alpha \Rightarrow \text{rank}(y) < \alpha$, and α is a limit
$\Rightarrow \text{rank}(y) + 1 < \alpha$

$\Rightarrow A \in V_\alpha$.

Pairing: Let $x, y \in V_\alpha$ and consider $\{x,y\}$

$$\text{rank}(\{x,y\}) = \sup\{\text{rank}(z)+1 : z \in \{x,y\}\}$$

assume wlog, that $\text{rank}(x) \geq \text{rank}(y)$

since $x \in V_\alpha \Rightarrow \text{rank}(x) < \alpha$

$\Rightarrow \text{rank}(x)+1 < \alpha$, since α limit

So, $\{x,y\} \in V_\alpha$.

Union: Let $\mathcal{F} \in V_\alpha$ and consider $\bigcup\mathcal{F}$

since $\mathcal{F} \in V_\alpha \Rightarrow \text{rank}(\mathcal{F}) < \alpha$

$\Rightarrow \forall x \in \mathcal{F}\ \forall y \in x\ \text{rank}(y) < \text{rank}(x) < \text{rank}(\mathcal{F}) < \alpha$

So, $\text{rank}(\bigcup\mathcal{F}) = \sup\{\text{rank}(y)+1 : y \in \bigcup\mathcal{F}\} < \alpha$ since α is a limit.

Infinity: Since α is a limit ordinal, and $\alpha > \omega$

$\text{rank}(\omega) = \omega < \alpha \Rightarrow \omega \in V_\alpha$.

Powerset: Let $x \in V_\alpha$ and consider $\{y : y \subset x\}$

since $y \subset x \Rightarrow \text{rank}(y) \leq \text{rank}(x) < \alpha$

So, since α is a limit ordinal

$\Rightarrow \exists \beta^{\text{suc}} < \alpha$ s.t. $\text{rank}(y) \leq \text{rank}(x) = \beta$

$\Rightarrow \{y : y \subset x\} \in V_{\beta+1} \subset V_\alpha$

Choice: Let $A \in V_\alpha$

Then in WF, $\exists R \subset A^2$ s.t. R well-orders A, and $R \subset V_\alpha$ since pairing holds. And since $\text{rank}(R) \leq \text{rank}(A^2) = \sup\{\text{rank}(x)+1 : x \in A^2\} < \alpha$
$\Rightarrow R \in V_\alpha$.

16. continued

<u>Claim</u>: $(V_{\omega+\omega}, \in) \not\models ZFC$

enough to show $(V_{\omega+\omega}, \in) \not\models$ replacement

Notice, since $\mathrm{rank}(\omega) < \mathrm{rank}(\omega+\omega)$, $\omega \in V_{\omega+\omega}$

Fix a bijection $f: \omega \xrightarrow{\sim} \omega+\omega$ and

let $\varphi(x,y) =$ "$f(x) = y$"

then $\forall x \in \omega\ \exists! y\ \varphi(x,y)$, since $\varphi(x,y)$ is the graph of a function

So, if replacement held,

$\{y : \exists x \in \omega\ \varphi(x,y)\} \in V_{\omega+\omega}$.

but $\mathrm{rank}(\{y : \exists x \in \omega\ \varphi(x,y)\}) = \omega+\omega \Rightarrow$ not in $V_{\omega+\omega}$ $\lightning$.

So, $(V_{\omega+\omega}, \in) \not\models ZFC$. qed.

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17. Let κ be an uncountable regular cardinal. OK

<u>Claim</u>: $(H_\kappa, \in) \models ZFC - \text{powerset}$

<u>Extensionality</u>: Let $x, y \in H_\kappa$ and $x \in y \in H_\kappa$

then since $x \in y \Rightarrow tc(x) \subset tc(y)$
$\Rightarrow |tc(x)| \leq |tc(y)| < \kappa$
$\Rightarrow x \in H_\kappa$.
So, H_κ is transitive.
So, if $x \in H_\kappa$, $\text{pred}(H_\kappa, x, \in) = x$
$\Rightarrow$ if $y \in H_\kappa$ and $\forall z \in H_\kappa (z \in x \longleftrightarrow z \in y)$
$\Rightarrow x = y$.

<u>Foundation</u>: Notice $H_\kappa \subset V_\kappa$, since $\forall x \in H_\kappa$, $\text{rank}(x) < \kappa$
So, $H_\kappa \subset WF$.

If $x \in H_\kappa$ and $\exists y \in H_\kappa$ s.t. $y \in x$,

take $z = \min\{w \in H_\kappa \cap x\}$, $w_0 < w_1$ iff $\text{rank}(w_0) < \text{rank}(w_1)$

then $z \in x$ and $\neg \exists w \in H_\kappa$ s.t. $w \in x \wedge w \in z$.

<u>Comprehension</u>: Let $\varphi(x, y, \bar{w})$ be a formula and fix $y, \bar{w} \in H_\kappa$.
Consider $\{x \in y : \varphi^{H_\kappa}(x, y, \bar{w})\}$
Since $y \in H_\kappa \Rightarrow |tc(y)| < \kappa$
$\Rightarrow |tc(\{x \in y : \varphi^{H_\kappa}(x, y, \bar{w})\})| \leq |tc(y)| < \kappa$
$\Rightarrow \{x \in y : \varphi^{H_\kappa}(x, y, \bar{w})\} \in H_\kappa$.

Pairing: Let $x, y \in H_\kappa$ and consider $\{x,y\}$.

$$tc(\{x,y\}) = \{x,y\} \cup \bigcup\{tc(z) : z \in \{x,y\}\}$$

So, $|tc(\{x,y\})| = |tc(x) \cup tc(y)|$

and since $x, y \in H_\kappa \Rightarrow |tc(x)|, |tc(y)| < \kappa$,

$$|tc(\{x,y\})| < \kappa$$

$$\Rightarrow \{x,y\} \in H_\kappa.$$

Union: Let $\mathcal{F} \in H_\kappa$ and consider $\bigcup\mathcal{F}$

Since $|tc(\mathcal{F})| < \kappa$, if $A \in \mathcal{F}$

$$|tc(A)| < |tc(\mathcal{F})| < \kappa$$

So, $|tc(\bigcup\mathcal{F})| < \kappa$

$\Rightarrow \bigcup\mathcal{F} \in H_\kappa.$

Replacement: Let $\varphi(x, y, A, \bar{w})$ be a formula and fix $A, \bar{w} \in H_\kappa$.

Assume $\forall x \in A \; \exists! y \in H_\kappa \; \varphi^{H_\kappa}(x, y, A, \bar{w})$

and consider $\{y : \exists x \in A \; \varphi^{H_\kappa}(x, y, A, \bar{w})\} = B$

since each $y \in B$ has $|tc(y)| < \kappa$, $B \subset H_\kappa$.

So, $tc(B) = B \cup \bigcup\{tc(y) : y \in B\}$

and $|B|, |tc(y)| < \kappa$ for all $y \in B$

$\Rightarrow |tc(B)| < \kappa$, since κ is regular.

$\mathbb{I}$a

Infinity: Since $\kappa > \omega$ and $tc(\omega) = \omega < \kappa$

$\Rightarrow \omega \in H_\kappa$.

Choice: Let $A \in H_\kappa$. Since $H_\kappa \subset V_\kappa \in WF$ and $(WF, \in) \models$ Choice,

$\exists R \subset A^2$ that well orders A in WF

Since pairing holds, $R \subset H_\kappa$

and since $A \in H_\kappa \Rightarrow |tc(A)| < \kappa$, and $|A| = |A^2|$

$\Rightarrow |tc(A^2)| < \kappa$

$\Rightarrow |tc(R)| < \kappa$

$\Rightarrow R \in H_\kappa$.

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(19) Consider the Mostowski Collapse of OD, OR

$$OD \xrightarrow[F]{\sim} M$$

Notice that $M \subset OD$, since if $m \in M$, then $F^{-1}(m) \in OD$

$\Rightarrow \exists \varphi(v, \vec{\alpha}), \vec{\alpha} \in ON$ s.t.

$$\{F^{-1}(m)\} = \{y : \varphi(y, \vec{\alpha})\}$$

Let $\Theta(x,y)$ be the graph of $F^{-1} : M \to OD$ and define $\psi(x, \vec{\alpha}) = \exists! y\,(\varphi(y, \vec{\alpha}) \wedge \Theta(x,y))$

Then, since F^{-1} is an isomorphism, there is exactly one such x

$\Rightarrow \{m\} = \{x : \psi(x, \vec{\alpha})\} \Rightarrow m \in OD$.

Claim: $M = HOD$

$M \subseteq HOD$: if $m \in M$, $tc(m) \subset M$, since M is transitive

$\Rightarrow tc(m) \subset OD \Rightarrow m \in HOD$

$HOD \subseteq M$: if $s \in HOD$, $tc(s) \subset OD$

$\Rightarrow F(s) \in M$ and $tc(F(s)) \subset OD$

$\Rightarrow F(s) \in HOD \cap M$

$\Rightarrow F^{-1}(F(s)) \in M \cap OD$

$\Rightarrow s \in M$

Therefore, $M = HOD$. qed.