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Math 771

5 a) Since κ is regular, there exists some $n \in \omega$
OK such that κ many of the F_α have cardinality n, so we may assume all F_α have the same size.
We now proceed by induction on n. If $n=1$, then either there exists $\beta < \kappa$ such that $F_\alpha = \{\beta\}$ for κ many β, or there is no such β. If β exists, let $\Sigma = \{\alpha : F_\alpha = \{\beta\}\}$. If no such β exists, then we can choose Σ so that every pair of its members are disjoint. Let $\Phi_\beta = \{F_\alpha : F_\alpha = \{\beta\}\}$ and form Σ by choosing for every nonempty Φ_β, one α s.t. $F_\alpha \in \Phi_\beta$. If fewer than κ many Φ_β are nonempty, then κ would be the union of fewer than κ many sets of cardinality less than κ, contradicting that κ is regular, thus $|\Sigma| = \kappa$.
Now assume the result is true for n. We will show it is true for $n+1$. If κ many of the F_α contain some $\beta < \kappa$, then we can use the inductive hypothesis on those sets to form Σ. If there is no such β, we can form Σ with empty root by transfinite induction. Let $1 \in \Sigma$, so $\alpha_1 = 1$ and at the $\gamma^{th} < \kappa$ stage choose F_{α_γ} s.t. $\alpha_\gamma > \alpha_\delta$ and $F_{\alpha_\gamma} \cap F_{\alpha_\delta} = \emptyset$ for all $\delta < \gamma$. Since κ is regular, F_{α_γ} is guaranteed to exist and so $|\Sigma| = \kappa$

b) Let $\kappa = \aleph_\omega^c$ and for all $\omega_n \le \alpha < \omega_{n+1}$, let $F_\alpha = \{n\}$. (If $\alpha = \omega$, let $F_\alpha = \{0\}$) Then any Δ system has root $\{n \in \omega\}$ or empty root. If Σ is a Δ system w/ empty root $|\Sigma| \le \aleph_0^c$ and if Σ has root n, then $|\Sigma| \le \aleph_n^c < \aleph_\omega^c = \kappa$.