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Set Theory '10

OK

④. <u>Prove that if</u>

$\underset{\sim}{b} > \omega_1$ and $\{a_\alpha \in [\omega]^\omega : \alpha < \omega_1\}$ are almost disjoint <u>then</u> there exists $a \in [\omega]^\omega$ with $a \cap a_\alpha$ finite for every $\alpha < \omega_1$.

define a family, $\mathcal{F}$, of fns so that for $\omega \leq \alpha < \omega_1$
$f_\alpha(n) \in a_\alpha \cap a_n$.
Since $a_\alpha \cap a_n$ is finite, we define $f_\alpha(n)$ to be the <u>greatest</u> in $a_\alpha \cap a_n$.
since this family is size ω_1 & $\underset{\sim}{b} > \omega_1$,
get a bounding fn. g so that
$g >^* f_\alpha$ for all $\omega \leq \alpha < \omega_1$.

next define a set $X = \{x_0, x_1, \ldots\}$
so that $x_n \in a_n$
and x_n is the least in a_n so that
$x_n > g(n)$.

note that we may have assumed
$a_0, a_1, \ldots$ (the first ω sets in the original family)
were completely disjoint by throwing away
$a_n \setminus \bigcup_{i<n} a_i \cap a_n$
(the a_n would still be infinite since) is a finite union
of finite stuff.
So $|X| = \omega$.

also note that

$|X| \cap a_n = X_n$ for all $n < \omega$

and

$|X| \cap a_\alpha$ =is finite since there is an m, after which $X_m > g(m) > a_m \cap a_\alpha$ for all subsequent steps.

Picture:

$$\begin{array}{ccccc}
\cdots & a_m \cap a_\alpha, & & a_{m+1} \cap a_\alpha & \cdots \\
& \Downarrow & & \Downarrow & \\
\cdots & f_\alpha(m) & , & f_\alpha(m+1) & \cdots \\
& \wedge & & \wedge & \\
\cdots & g(m) & , & g(m+1) & \\
& \wedge & & \wedge & \\
\cdots & X_m & , & X_{m+1} & \cdots
\end{array}$$

So X is almost disjoint from the original family $\square$.