Set Theory

Ashutosh Kumar

(3) $\neg CH \iff \exists \{A_\alpha \subseteq \omega_1 \mid \alpha < \omega_2\}$ s.t. $\forall \alpha < \omega_2 \, (|A_\alpha| \geq \omega)$ and $\forall \alpha < \beta < \omega_2 \, (|A_\alpha \cap A_\beta| < \omega)$

OR

proof: ($\Rightarrow$) Let $\{A_\alpha \subseteq \omega \mid \alpha < 2^\omega\}$ be an almost disjoint (a.d.) family of subsets of ω, of size continuum.

$\neg CH \Rightarrow 2^\omega \geq \omega_2$. So $\{A_\alpha \mid \alpha < \omega_2\}$ is also an a.d. family over ω_1, of size ω_2.

($\Leftarrow$) Notice that there must be some countable ordinal $\gamma < \omega_1$, such that ω_2-many A_α's meet γ on an infinite set. Since otherwise, put $F_\gamma = \{\alpha < \omega_2 \mid |A_\alpha \cap \gamma| \geq \omega\}$ for $\gamma < \omega_1$. We have $\forall \gamma < \omega_1 \, (|F_\gamma| < \omega_2)$ and $\bigcup_{\gamma < \omega_1} F_\gamma = \omega_2$ which contradicts the regularity of ω_2.

So fix such an ordinal $\gamma_0 < \omega_1$ and consider the family $\{B_\alpha = A_\alpha \cap \gamma_0 \mid \alpha \in F_{\gamma_0}\}$. This is a family of size ω_2 and its members are pairwise a.d. & therefore distinct

$\Rightarrow 2^{|\gamma_0|} \geq \omega_2 \Rightarrow 2^\omega \geq \omega_2$.