Set Theory HW 1

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OK

(1) $A$ is a $\mathcal{D}$-finite, infinite set.
$A_n = \{x : n \to A \mid x \text{ is } 1\text{-}1\}$
For $X \subseteq \omega \setminus \{0\}$, $A_X = \bigcup_{n \in X} A_n$

(a) $A_n$ is D-finite :
Suppose not. Let $f : \omega \xrightarrow{1-1} A_n$. Let $g : \omega \to A$ be defined by $g(m) = (f(k))(r)$ where $m = kn + r$ with $0 \le r < n$.
Since there are only finitely many $n$-tuples over a finite set and since $f$ is 1-1, we get $\forall n \, \exists m > n \, (\forall i \le n \, (g(i) \neq g(m)))$
Thus, if we let $h : \omega \to A$, $h(0) = g(0)$, $h(n+1) = g(m)$ where '$m$' is the least no. such that $g(m) \notin \{h(0), .., h(n)\}$,
then $h$ is 1-1, contradicting that $A$ is $\mathcal{D}$-finite

(b) $A_X$ is $\mathcal{D}$-finite :
Suppose not. Let $f : \omega \xrightarrow{1-1} A_X$. Let $g : \omega \to A$ be an enumeration of elements of $A$ occurring in the tuples $f(0), f(1), f(2), \dots$ in that order. Again, $\operatorname{rng}(g)$ is infinite since any 1-1 sequence over a finite set of size $n$ has length $\le n$ $\Rightarrow$ there are only finitely many such sequences. Let $h : \omega \to A$ be defined as in (a). Then $h : \omega \xrightarrow{1-1} A$ $\Rightarrow$ $A$ is not $\mathcal{D}$-finite $\Rightarrow \perp$.

(c) $\forall m, n \in \omega \setminus \{0\} \left[ \exists f : A_m \xrightarrow{1-1} A_n \iff m \le n \right]$

($\Leftarrow$) By induction on $m$, we show $\exists f : A_m \xrightarrow{1-1} A_{m+1}$.
Fix $m+1$ elements of $A$ say $\{b_0, b_1, \dots, b_m\}$.
Let $f((a_0, .., a_{m-1})) = (a_0, a_1, .., a_{m-1}, b)$ where $b$ is the least $i$ for which $b_i \neq a_j$ for $j = 0, 1, .. m-1$.
Then, $f$ is 1-1

($\Rightarrow$) Suppose $g: A_m \xrightarrow{1-1} A_n$

and suppose $m > n$.

Then, by ($\Leftarrow$) above get a 1-1 but not onto map

$f: A_n \to A_m$. But then $g \circ f$ is 1-1, not onto

map on $A_m \Rightarrow A_m$ is not $\mathcal{D}$-finite, a contradiction

(d) $X \subseteq Y \subseteq \omega \setminus \{0\}$

$\exists f: A_Y \xrightarrow{1-1} A_X \iff X = Y$.

($\Leftarrow$) Trivial.

($\Rightarrow$) Suppose $X \subsetneq Y$. Let $g: A_X \xrightarrow{1-1} A_Y$ be the identity map

Then $g$ is not onto.

If $f: A_Y \xrightarrow{1-1} A_X$ then $g \circ f: A_Y \xrightarrow{1-1} A_Y$ is not onto

$\Rightarrow A_Y$ is not $\mathcal{D}$-finite; contradiction.

(e) Fix a bijection $h: \mathbb{Q} \leftrightarrow \omega$ between rationals & $\omega$.

For a real $r$, let $X_r = \{h(x) : x \in \mathbb{Q} \text{ and } h(x) < r\}$.

Then, $r_1 < r_2 \Rightarrow X_{r_1} \subsetneq X_{r_2}$ as rationals are dense in $\mathbb{R}$.

Thus, $\{A_{X_r} : r \in \mathbb{R}\}$ is a collection of continuum many $\mathcal{D}$-finite, infinite sets with pairwise distinct cardinal types (i.e. there's no bijection between any two of them).

(2) (a) $(L,\le)$ is a linear ordering and $\forall X \subseteq L$ (either $X \cong L$ or $L\setminus X \cong L$ under the restriction of $\le$)

$\Rightarrow (L,\le)$ is either a well ordering or a reverse well ordering.

Assume $L$ is infinite.

case(1) $L$ has neither greatest nor least member.

subcase (1a) $<$ is not dense on $L$

Pick $a<b$ in $L$ such that $(a,b)=\emptyset$.

Let $X=(-\infty,a]$, $L\setminus X=[b,\infty)$

Then $X \not\cong L$, $L\setminus X \not\cong L$ contradiction

Subcase (1b) $<$ is dense on $L$.

Pick $a<b<c<d$. Put $X=[a,b)\cup(c,d)$.

Then $X\not\cong L$ and $L\setminus X \not\cong L$.

Thus, $L$ must have a greatest or least element (say it has a greatest element)

Let $\kappa = \aleph(L)$ so that $\kappa$ doesn't inject into $L$.

By induction on $\alpha<\kappa$, construct $\{x_\alpha : \alpha<\kappa\}$ as follows:

$x_0 =$ <- least member of $L$.

Put $X=\{x_0\}$. Then $L\setminus X \cong L \Rightarrow x_0$ has a $\le$-successor in $L$. Call it $x_1$. Suppose $\underbrace{\{x_\alpha : \alpha<\lambda\}}_{X_\lambda}$ has been constructed such that

$(X_\lambda,\le)\cong\lambda$. Then, if $X_\lambda \cong L$ we'll be done. Otherwise $L\setminus X_\lambda \cong L$ in which case we set $x_\lambda$ to be the $\le$-least element in $L\setminus X_\lambda$. The construction terminates at some $\alpha<\kappa$ since $\kappa$ cannot be injected into $L$. This shows that $(L,\le)$ is a well ordering.

The case when $(L,\le)$ has a largest element will imply that $\le$ is a reverse well order on $L$ in an analogous way.

2(b) $\forall \alpha \left[ \forall X \subseteq \alpha \, (X \cong \alpha \text{ or } \alpha \setminus X \cong \alpha) \iff \exists \beta \, (\alpha = \omega^\beta) \right]$

($\Rightarrow$) Use Cantor normal form of $\alpha$, say

$$\alpha = \omega^{\beta_1} \cdot l_1 + \omega^{\beta_2} \cdot l_2 + \cdots + \omega^{\beta_n} l_n$$

where $\beta_1 > \beta_2 \cdots > \beta_n$ and $l_1, \ldots, l_n \in \omega \setminus \{0\}$.

If $n \geq 2$, considering $X = \omega^{\beta_1} l_1$ we get $X \not\cong \alpha$ & $\alpha \setminus X \not\cong \alpha$.

So $n = 1$ and $\alpha = \omega^{\beta_1} \cdot l_1$. If $l_1 \geq 2$, then for $X = \omega^{\beta_1} \cdot (l_1 - 1)$ we get $X \not\cong \alpha$ and $\alpha \setminus X \not\cong \alpha$. So $l_1 = 1$ and $\alpha$ must be an ordinal power of $\omega$.

($\Leftarrow$) By induction on $\beta$.

<u>At $\beta = 0, 1$</u> : Trivial.

<u>At $\beta + 1$</u> : $\alpha = \omega^{\beta+1} = \omega^\beta \cdot \omega = \underbrace{\omega^\beta + \omega^\beta + \cdots}_{\omega \text{ terms}} = \sup_{n \in \omega} \omega^\beta \cdot n$

Let $X \subseteq \omega^{\beta+1}$. Let $X_n = X \cap \underbrace{[\omega^\beta \cdot (n+1) \setminus \omega^\beta \cdot n]}_{= \alpha_n \text{ (say)}}$.

Then $X_n \subseteq \alpha_n$ & $\alpha_n \cong \omega^\beta$

$\Rightarrow$ One of $X_n$, $\alpha_n \setminus X_n$ has order type $\omega^\beta$

Let $A = \{n \in \omega : X_n \cong \omega^\beta\}$

$B = \{n \in \omega : \alpha_n \setminus X_n \cong \omega^\beta\}$

$A \cup B = \omega \Rightarrow$ One of them is infinite, say $A$ (WLOG)

Then, for each $n$, $\omega^\beta \cdot n$ embeds into $X \Rightarrow X \cong \omega^{\beta+1}$

<u>At limits</u> : $\alpha = \omega^\lambda = \sup_{\beta < \lambda} \omega^\beta$, $\lambda$ is a limit ordinal

Let $X \subseteq \omega^\lambda$. $X_\beta = X \cap \omega^\beta$, $\beta < \lambda$. Then, one of $X_\beta$, $\omega^\beta \setminus X_\beta$ is order isomorphic to $\omega^\beta$.

Put $A = \{\beta < \lambda : X_\beta \cong \omega^\beta\}$

$B = \{\beta < \lambda : \omega^\beta \setminus X_\beta \cong \omega^\beta\}$.

One of $A$, $B$ is cofinal (unbounded) in $\lambda$, say $A$ (WLOG).

Claim: $X \cong \omega^\lambda$

If not, $X$ embeds into $\omega^\beta$ for some $\beta < \lambda$.

$\Rightarrow$ If $\beta' > \beta$ & $\beta' < \lambda$, then $\beta' \notin A$ (otherwise $X$ has a copy of $\omega^{\beta'}$.) But then $A$ is bounded in $\lambda$: contradiction.