

1. (1-27) Let $(L, \leq)$ be a wellorder and for any $a \in L$ let $L_a = \{c \in L : c < a\}$.
Suppose $f : P(X) \rightarrow X$ is such that $f(A) \in X \setminus A$ for all A a proper subset of X .
Call a well ordering $(L, \leq)$ respectful of f iff $L \subseteq X$ and for every $a \in L$ $f(L_a) = a$.
Prove that for any two respectful well-orderings $(A, \leq_A)$ and $(B, \leq_B)$ that either
 $A = B$ or
 $A_x = B$ for some $x \in A$ or
 $B_x = A$ for some $x \in B$
and in any of these cases $\leq_A$ and $\leq_B$ agree on $A \cap B$.
2. (2-3) Define $0! = 1$, $(\alpha + 1)! = \alpha! \cdot (\alpha + 1)$, and $\gamma! = \sup_{\alpha < \gamma} \alpha!$ for limit γ .
Prove that: ε_0 is the least $\alpha > \omega$ such that $\alpha! = \alpha$
where $\varepsilon_0 = \sup_{n \in \omega} \tau_n$, where $\tau_0 = \omega$ and $\tau_{n+1} = \omega^{\tau_n}$.
3. (2-8) Prove there exists a family $\mathcal{B}$ with $|\mathcal{B}| = 2^c$ such that each $X \in \mathcal{B}$ is a Bernstein set.
4. (2-8) Prove that for any limit ordinal $\alpha > \omega$ that V_α is model of Z+AC (Zermelo set theory plus the axiom of choice).
5. (2-8) Prove in Z (no replacement) that
For any set X and $G : X \rightarrow X$ and $x_0 \in X$ there exists a unique $f : \omega \rightarrow X$ such that $f(0) = x_0$ and for all $n \in \omega$ we have that $f(n+1) = G(f(n))$.

Extra Credit (no time limit)

Does Z prove that for every set X there exists a transitive set $Y \supseteq X$?

6. (2-15) (Goodstein) The complete expansion of a positive integer in a base is gotten by writing everything possible including exponents and exponents of exponents etc. as powers of the base. For example the number 36 written in complete base two is: $2^{(2^2+1)} + 2^2$. The same number in complete base 3 is: $3^3 + 3^2$.

Let a_n be a sequence described as follows. Given a_k calculate a_{k+1} by writing a_k in base k then substitute $k+1$ for every k , then subtract one. For example:

$$a_2 = 36 = 2^{(2^2+1)} + 2^2 \text{ implies } a_3 = 3^{(3^3+1)} + 3^3 - 1 = 2.2876 \dots \times 10^{13}$$

or for example if $a_6 = 6^4 + 2 \cdot 6^3 = 1728$, then

$$a_7 = (7^4 + 2 \cdot 7^3) - 1 = 7^4 + 7^3 + 6 \cdot 7^2 + 6 \cdot 7^1 + 6 = 3086$$

Show that given any pair of positive integers n and m , if we let $a_n = m$ then for some $k > n$ we get $a_k = 0$.

7. (2-15) (Bukovsky, Hechler) Suppose λ is a singular limit cardinal and for some γ we have $2^\alpha = \gamma$ for a cofinal set of $\alpha < \lambda$. Prove that $2^\lambda = \gamma$.

8. (2-22) Show that $V_\kappa \models \text{ZFC}$ for any strongly inaccessible cardinal κ .

9. (2-22) Show that the least ordinal κ (assuming one exists) for which $V_\kappa \models \text{ZFC}$ is an uncountable cardinal with countable cofinality.

Hint: Prove: If $V_\kappa \models \text{ZFC}$, then there exists $\alpha \leq \kappa$ of countable cofinality such that V_α is an elementary substructure of V_κ .

10. (2-22) Define $H_\kappa = \{x : |TC(x)| < \kappa\}$. Show that:

- (a) $H_\kappa \subseteq V_\kappa$ for any infinite cardinal κ .
- (b) for any uncountable regular cardinal κ that $H_\kappa \models \text{ZFC} \setminus \text{PowerSetAxiom}$.
- (c) for any regular $\kappa \geq \omega$ that $V_\kappa = H_\kappa$ iff $\kappa = \omega$ or κ is strongly inaccessible.
- (d) there exists a cardinal $\kappa > \omega$ with countable cofinality such that $V_\kappa = H_\kappa$.

Hint: (10d) If $\kappa_{n+1} = |V_{\kappa_n}|^+$ then $V_{\kappa_n} \in H_{\kappa_{n+1}}$.

11. (2-29) Suppose M and N are standard transitive models of ZFC with the same set of ordinals. Suppose that for every ordinal $\alpha \in M$

$$P(\alpha) \cap M = P(\alpha) \cap N$$

Prove that $M = N$.

Hint: It isn't true for models of ZF.

Show $P(\alpha \times \alpha) \cap M = P(\alpha \times \alpha) \cap N$.

Consider $(X, \in) \simeq (\alpha, E)$ for some $E \subseteq \alpha \times \alpha$.

12. (2-29) For any set X let $[X]^{<\omega} = \bigcup_{n < \omega} [X]^n$ where $[X]^n = \{s \subseteq X : |s| = n\}$.

A cardinal κ is Erdős iff $\kappa \rightarrow (\omega)_2^{<\omega}$ which means for every $f : [\kappa]^{<\omega} \rightarrow 2$ there exists an infinite $X \subseteq \kappa$ such that for every $n < \omega$ the function f is constant when restricted to $[X]^n$. The smallest Erdős cardinal (if any exist) is strongly inaccessible.

Suppose $M \subseteq N$ are standard transitive models of ZFC and $\kappa \in M$ and $N \models \text{"}\kappa \text{ is Erdős"}$.

Prove that $M \models \text{"}\kappa \text{ is Erdős"}$.

Hint: If not, find a well-founded partial order in M which is not well-founded in N . Consider $\{s \in [\kappa]^{<\omega} : \forall n \leq |s| \ f \restriction [s]^n \text{ is constant}\}$.

13. (3-7) Suppose that ZF has a well-founded model. Then prove there exists a countable δ such that

- (a) $L_\delta \models \text{ZFC}$ and for any transitive standard model M of ZF we have that $L_\delta \subseteq M$
- (b) every element of L_δ is definable in L_δ without parameters.

Hint: L has a well-ordering which is definable without parameters. Also if

$$X = \{a \in L_\delta : a \text{ def without para in } L_\delta\}$$

then show X is an elementary substructure of L_δ .

14. (3-7) For X a set of ordinals define the class $L[X]$ as follows:

- (a) $L_0[X] = \emptyset$
- (b) $L_\lambda[X] = \cup_{\alpha < \lambda} L_\alpha[X]$ for λ limit
- (c) $L_{\alpha+1}[X] = \text{Def}(L_\alpha[X], X \cap \alpha, \in)$ (add a new unary predicate to language)
- (d) $L[X] = \cup_{\alpha \in ORD} L_\alpha[X]$

Prove that if $V = L[X]$ for some $X \subseteq \omega_1$ then ZFC+GCH is true in V .

15. (3-21) Prove that if there exists a Souslin tree then there exists a nice Souslin subtree of $\omega^{<\omega_1}$.

Hint: Define a Souslin tree T is normal iff for all $\alpha < \beta < \omega_1 \ \forall s \in T_\alpha \exists t \in T_\beta \ s \leq t$

- (a) Show that every Souslin tree contains a normal subtree which is Souslin.
- (b) Show that if T is a normal Souslin tree then there exist a club $C \subseteq \omega_1$ such that for every $\alpha, \beta \in C$ if $\alpha < \beta$ then $\forall s \in T_\alpha \exists^\infty t \in T_\beta \ s \leq t$.

16. (3-21) Show that for every club $C \subseteq \kappa$ (uncountable regular κ) there exists $f : \kappa \rightarrow \kappa$ such that if $C_f = \{\alpha < \kappa : \forall \beta < \alpha \ f(\beta) < \alpha\}$ then $C_f \subseteq C$.

17. (3-21) Prove there is a stationary subset of ω_1 whose complement is also stationary.

Hint: Given $x_\alpha : \omega \rightarrow 2$ for $\alpha < \omega_1$ and $n < \omega$, consider $\{\alpha < \omega_1 : x_\alpha(n) = 0\}$

18. (3-23) Suppose $S \subseteq \kappa$ is a stationary subset of the regular uncountable cardinal κ . Prove there exists $S' \subseteq S$ a stationary subset of κ such that $\forall \alpha \in S'$ the set $S' \cap \alpha$ is not stationary in α .

Hint: Define S' inductively by $\alpha \in S'$ iff $\alpha \in S$ and $S' \cap \alpha$ not stationary in α .

19. (3-23) Suppose $\diamond_{\omega_1}$. Prove there exists $\langle S_\alpha \subseteq \omega_1 : \alpha < 2^{\omega_1} \rangle$ such that each S_α is stationary in ω_1 and $S_\alpha \cap S_\beta$ is countable for all $\alpha \neq \beta$.

20. (3-28) Suppose $V=L$. Prove that for any regular κ that $L_\kappa = H_\kappa$.

21. (4-4) Suppose T is a Souslin tree and $f : T \rightarrow \mathbb{R}$ is continuous. Prove that the range of f is countable.

Hint: First show for every $\epsilon > 0$ for every $s \in T$ there exists $t \in T$ with $s \leq t$ and the diameter of $\{f(r) : t \leq r \in T\}$ less than ϵ .

22. (4-4) Assume $V = L$. Prove that there is a diamond sequence on every stationary $S \subseteq \omega_1$.

23. (4-4) Assume $V = L[X]$ for some $X \subseteq \omega_1$. Prove that $\diamond_{\omega_1}$ holds.

24. (4-11) Assume M is a transitive standard model of ZFC and $M \models \text{"}\mathbb{P} = (T, \geq) \text{"}$ where T is a Souslin tree". Suppose G is an ω_1 -branch thru T . Prove that G is a $\mathbb{P}$ -generic filter over M .

Hint: Suppose $M \models D \subseteq T$ is dense. Show that
 $M \models \exists \alpha < \omega_1 \ \forall s \in T_\alpha \ \exists t \in D \ t < s$.

25. (4-13) Assume M is a transitive standard model of ZFC and $\mathbb{P} \in M$. Prove that the following are equivalent:

- (a) $M \models \text{"}\{s \in \mathbb{P} : \mathbb{P}_{\leq s} \text{ is ccc}\} \text{ is dense in } \mathbb{P}"$ (where $\mathbb{P}_{\leq s} = \{t \in \mathbb{P} : t \leq s\}$)
- (b) for any G which is $\mathbb{P}$ -generic over M and for any $f, \alpha, \beta \in M[G]$ if

$$M[G] \models f : \alpha \rightarrow \beta, \alpha, \beta \in ORD$$

then there exists $g \in M$ such that

$$M \models g : \alpha \rightarrow [\beta]^{\leq \omega}$$

and $f(\delta) \in g(\delta)$ for all $\delta \in \alpha$.

Hint: (in M) Let $|\mathbb{P}| = \{p_\delta : \delta < \beta\}$ and let $\langle A_\gamma : \gamma < \alpha \rangle$ list all maximal antichains in $\mathbb{P}$. For any G $\mathbb{P}$ -generic over M consider the function:

$$f(\gamma) = \delta \text{ iff } p_\delta \in G \cap A_\gamma.$$

26. (4-18) Assume that ZFC has a transitive model. Let ZFC_n be the first n axioms of ZFC.

- (a) Show there is a model of " $\text{ZFC} + \text{Con}(\text{ZFC}) + \text{ZFC}$ has no transitive model + $\forall n \in \omega \ \text{ZFC}_n$ has a transitive model"
- (b) Show there is a model of " $\text{ZFC} + \text{Con}(\text{ZFC}) + \exists n \in \omega \ \text{ZFC}_n$ has no transitive model"

27. (4-25) Say that $\mathbb{P}_1$ is densely embeddable in $\mathbb{P}_2$ iff $\mathbb{P}_1$ is isomorphic to a dense subset of $\mathbb{P}_2$. Suppose that

$$M \models j : \mathbb{P}_1 \rightarrow D \subseteq \mathbb{P}_2 \text{ is a dense embedding.}$$

Prove that $\mathbb{P}_1$ and $\mathbb{P}_2$ are forcing equivalent by showing:

- (a) If G_1 is $\mathbb{P}_1$ -generic over M , then $G_2 = \{q \in \mathbb{P}_2 : \exists p \in G_1 \ q \leq_2 j(p)\}$ is $\mathbb{P}_2$ -generic over M and $M[G_1] = M[G_2]$.
- (b) If G_2 is $\mathbb{P}_2$ -generic over M , then $G_1 = j^{-1}(G_2)$ is $\mathbb{P}_1$ -generic over M and $M[G_1] = M[G_2]$.

28. (4-25) Define $\mathbb{P}$ is trivial beneath $p \in \mathbb{P}$ iff any two conditions extending p are compatible. Show:

- (a) If $\mathbb{P}$ is trivial beneath p , then there is only one G $\mathbb{P}$ -generic over M with $p \in G$. Furthermore, $G \in M$.
- (b) If $\mathbb{P}$ is everywhere nontrivial (i.e., any condition has a pair of incompatible extensions), then for every G which is $\mathbb{P}$ -generic over M we have $G \notin M$.

29. (4-25) Suppose that $\mathbb{P}$ is countable and everywhere nontrivial. Prove that $\omega^{<\omega}$ densely embeds into $\mathbb{P}$ and hence they are forcing equivalent.

30. (4-25) Let M be a cstm ZFC. We say that $x \in \omega^\omega$ is a Cohen real over M iff $G = \{x \restriction n : n \in \omega\}$ is $\omega^{<\omega}$ -generic over M . Suppose x is a Cohen real over M . Show:

(a) $\forall y \in M \cap \omega^\omega \exists^\infty n \ x(n) = y(n)$

(b) $\forall y \in M[x] \cap \omega^\omega \exists u \in M \cap \omega^\omega \exists^\infty n \ u(n) = y(n)$

(c) Let $z \in 2^\omega$ be defined by $z(n) = x(n) \bmod 2$. Show that $\{z \restriction n : n \in \omega\}$ is $2^{<\omega}$ -generic over M .

(d) Let z be as in (c) and let $y = \sum_{n=1}^\infty z(n)/2^n$. Let $\mathbb{P}$ be the partial order of open intervals in $\mathbb{R}$ with rational endpoints. Prove that $G = \{I \in \mathbb{P} : y \in I\}$ is $\mathbb{P}$ -generic over M .

Hint: (b) Let $\omega^{<\omega} = \{p_n : n \in \omega\}$. (In M) construct $u \in \omega^\omega$ so that for each $n \in \omega$ there exists $q \leq p_n$ such that $q \Vdash u(n) = y(n)$.

For each of the **problems 31-35** assume MA_{ω_1} and prove the given statement.

31. (5-2) For any family $\mathcal{C}$ of ω_1 closed sets of positive measure there exists $\mathcal{C}' \in [\mathcal{C}]^{\omega_1}$ such that $\bigcap \mathcal{C}'$ has positive measure.

32. (5-2) $\forall F \in [\omega^\omega]^{\omega_1} \exists g \in \omega^\omega \forall f \in F \exists^\infty n \ f(n) = g(n)$

33. (5-2) Let $P = \{f \in \omega^\omega : \forall n \ f(n) \leq n\}$.

$\forall F \in [P]^{\omega_1} \exists g \in P \forall f \in F \forall^\infty n \ f(n) \neq g(n)$

34. (5-2) $\forall F \in [[\omega]^\omega]^{\omega_1} \exists X \in [\omega]^\omega \forall Y \in F \ X \subseteq^* Y \text{ or } X \cap Y =^* \emptyset$

35. (5-2) A set of reals $X \subseteq \mathbb{R}$ is ω_1 -dense iff it meets every interval in a set of size ω_1 . Any two ω_1 -dense sets of reals are homomorphic.