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M571
Fall 2008

Homework problems are due in class one week from the day assigned (which is in parentheses).

1. (9-3 W) Draw a parse tree for the propositional sentence:

$$\neg(((P \vee Q) \rightarrow \neg R) \wedge ((R \vee P) \leftrightarrow \neg Q))$$

2. (9-3 W) Write the same sentence in Polish notation.

3. (9-3 W) Given the parse tree drawn on the board in class today, find the propositional sentence (in standard form) and also write the sentence in Polish notation.

4. (9-5 F) Using General Induction prove the unique readability lemma for propositional sentence in standard (with parentheses) form.

5. (9-5 F) Using General Induction prove that for an propositional sentence that the number of left parentheses is equal to the number of right parentheses. Show how this could be used to construct the parse tree.

6. (9-8 M) Compute a truth table for

$$((P \rightarrow Q) \vee (\neg P \leftrightarrow Q)).$$

7. (9-8 M) Compute a truth table for

$$((\neg P \leftrightarrow \neg Q) \rightarrow R).$$

8. (9-8 M) Prove that the set $\{\rightarrow, \neg\}$ is adequate.

9. (9-10 W) Prove or disprove: the set $\{\leftrightarrow, \neg\}$ is adequate.
(Extra credit: for each of the sixteen binary logical connectives $\bowtie$ determine if the set $\{\bowtie, \neg\}$ is adequate.)

10. (9-10 W) How many different ternary logical connectives are there? (Extra credit: How many are adequate? Find a lower bound and upper bound.)

11. (9-12 F) Use truth tables to find a disjunctive normal form for the sentence:

$$(P \to Q) \wedge (\neg R \vee P)$$

12. (9-12 F) For the same sentence find a disjunctive normal form by interchanges of conjuncts and disjuncts.

13. (9-15 M) Give a Tableau proof of

$$((P \wedge Q) \to (P \vee Q))$$

14. (9-15 M) Give a Tableau proof of

$$(P \wedge (Q \vee R)) \leftrightarrow ((P \wedge Q) \vee (P \wedge R))$$

15. (9-17 W) Give a Tableau proof of

$$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$$

16. (9-19 F) Fix distinct propositional letters P and Q in the set $\mathcal{P}$. Let ν_i for $i = 1, 2, 3, 4$ be the four different true evaluations on $\mathcal{P}$ such that $\nu_i(R) = T$ for all $R \in (\mathcal{P} \setminus \{P, Q\})$. Prove that for any sentence θ in $\mathcal{S}(\leftrightarrow, \neg)$ that exactly 0,2, or 4 of the ν_i satisfy $\nu_i(\theta) = T$.

17. (9-19 F) Prove that for any θ in $\mathcal{S}(\leftrightarrow, \neg)$ that there is a θ^* in $\mathcal{S}(\leftrightarrow)$ such that either $\theta \equiv \theta^*$ or $\theta \equiv \neg\theta^*$.

18. (9-22 M) Prove that the random Tableau finishing algorithm always halts.

19. (9-22 M) Construct a finished Tableau for an attempted proof of:

$$(P \wedge (Q \vee R)) \to (P \wedge Q)$$

and use a finished branch to construct a truth evaluation which makes it false.

20. (9-24 W) Suppose $\Sigma \vdash \theta$ and Σ is finitely satisfiable. Without using the Compactness Theorem, prove that $\Sigma \cup \{\theta\}$ is finitely satisfiable.

21. (9-24 W) Define $Th(\Sigma) = \{\theta : \Sigma \vdash \theta\}$. Suppose Σ is finitely satisfiable. Without using the Compactness Theorem, prove that $Th(\Sigma)$ is finitely satisfiable.

22. (9-26 F) Prove that if $\Gamma \subseteq \mathcal{S}$ is complete and finitely satisfiable and ν is the truth evaluation defined on the Propositional letters $\mathcal{P}$ by: for all $P \in \mathcal{P}$

$$\nu(P) = T \text{ iff } P \in \Gamma$$

then for all $\theta \in \mathcal{S}$

$$\nu(\theta) = T \text{ iff } \theta \in \Gamma.$$

23. (9-26 F) Prove that if $\Gamma \subseteq \mathcal{S}$ is complete and finitely satisfiable, then Γ is finished, i.e., Hintikka, for definition see simpson.pdf page 20.

24. (9-29 M) Problem 24 is from proplog.pdf. See
www.math.wisc.edu/~miller/old/m571-02/proplog.pdf

(24) Show that for every finite set A and partial order $\leq$ on A there exists a linear order $\leq^*$ on A extending $\leq$.

25. (9-29 M) Problem 25 in proplog.pdf.

(25) Let A be any set and let our set of atomic sentences $\mathcal{P}$ be:

$$\mathcal{P} = \{P_{ab} : a, b \in A\}$$

For any truth evaluation e define $\leq_e$ to be the binary relation on A defined by

$$a \leq_e b \text{ iff } e(P_{ab}) = T.$$

Construct a set of sentences Σ such that for every truth evaluation e,

$$e \text{ makes } \Sigma \text{ true iff } \leq_e \text{ is a linear order on } A.$$

26. (9-29 M) Problem 26 in proplog.pdf.

(26) Without assuming the set A is finite prove for every partial order $\leq$ on A there exists a linear order $\leq^*$ on A extending $\leq$.

27. (10-3 F) Suppose $\Sigma_1, \Sigma_2 \subseteq \mathcal{S}$. Prove that $\Sigma_1 \cup \Sigma_2$ is satisfiable iff there does not exist $\theta \in \mathcal{S}$ such that $\Sigma_1 \vdash \theta$ and $\Sigma_2 \vdash \neg\theta$.

28. (10-6 M) Consider the natural deduction system $\vdash_t$ with
Logical Axioms:

$$(A1) \quad \theta \to (\rho \to \theta)$$

Logical Rule: Modus Toadus

$$\frac{\begin{array}{l}\theta \to \psi \\ \psi\end{array}}{\theta}$$

Prove that this proof system is unsound, in fact, show that for every sentence θ that $\vdash_t\ \theta$.

29. (10-6 M) Consider a subset of the natural deduction system from benyaacov.pdf page 18 which has logical axioms A1, A2, (but not A3) and logical rule Modus Ponens.

$$(A2) \qquad (\phi \to (\rho \to \theta)) \to ((\phi \to \rho) \to (\phi \to \theta)).$$

Prove that this system is not complete, i.e., there exists a logical validity θ in $\mathcal{S}(\to, \neg)$ such that it is not true that there is a proof of θ.

Extra credit: Prove or disprove: every validity in $\mathcal{S}(\to)$ can be proved in this system.

30. (10-8 W) Problem 30 in proplog.pdf.

(30) A triangle in a graph (V, E) is a set $\Delta = \{a, b, c\} \subseteq V$ such that aEb, bEc, and cEa. Suppose that every finite subset of V can be partitioned into n or fewer sets none of which contain a triangle. Show that V is the union of n sets none of which contain a triangle.

31. (10-10 F) Let $\mathcal{L} = \{U, V\}$ where U and V are unary predicate symbols. For each of the following Prove or Disprove.

(a) $\exists x\ (U(x) \vee V(x)) \equiv \exists x\ U(x) \vee \exists x\ V(x)$

(b) $\exists x\ (U(x) \wedge V(x)) \equiv \exists x\ U(x) \wedge \exists x\ V(x)$

(c) $\neg\exists x\ U(x) \equiv \exists x\ \neg U(x)$

(d) $\exists x\ (U(x) \to V(x)) \equiv (\exists x\ U(x)) \to (\exists x\ V(x))$

(e) $\exists x\ (U(x) \rightarrow V(x)) \equiv (\forall x\ U(x)) \rightarrow (\exists x\ V(x))$

32. (10-10 F) Let $\mathcal{L} = \{U, V, R\}$ where U and V are unary predicate symbols and R is a binary predicate symbol. For each of the following Prove or Disprove.
(a) $\forall x \exists y\ R(x, y) \equiv \exists y \forall x\ R(x, y)$
(b) $\forall x \exists y\ (U(x) \wedge V(y)) \equiv \exists y \forall x\ (U(x) \wedge V(y))$

33. (10-13 M) Let $\mathcal{L} = \{R, G\}$ where each of R and G are binary relation symbols. Given any real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ consider the $\mathcal{L}$-structure

$$M_f = (\mathbb{R}, <, Graph(f)).$$

For each of the following properties of the function f find an $\mathcal{L}$-sentence θ so that $M_f \models \theta$ iff f has the given property:
(a) f is a constant function.
(b) f is one-to-one.
(c) f is onto.
(d) f^2 is the identity function (f^2 is the composition of f with itself.)
(e) f is strictly increasing.
(f) f is continuous.

34. (10-15 W) Let $\mathcal{M}(\theta)$ be the models of θ. Let Σ_1 and Σ_2 be sets of $\mathcal{L}$-sentences. Prove that
$\mathcal{M}(\Sigma_1) \cap \mathcal{M}(\Sigma_2) = \emptyset$ iff
there is an $\mathcal{L}$-sentence θ with $\mathcal{M}(\Sigma_1) \subseteq \mathcal{M}(\theta)$ and $\mathcal{M}(\Sigma_2) \subseteq \mathcal{M}(\neg\theta)$.

35. (10-17 F) proplog.pdf number 27
(27) If $X \subseteq A$ and R is a binary relation on A then the restriction of R to X is the binary relation $S = R \cap (X \times X)$. For a partial order $\leq$ on A, a set $B \subseteq A$ is an $\leq$-chain iff the restriction of $\leq$ to B is a linear order. Show that given a partial order $\leq$ on A:
the set A is the union of less than n $\leq$-chains iff every finite subset of A is the union of less than n $\leq$-chains.

36. (10-20 M) Give a Tableau proof of

$$(\exists y \forall x\ \ \theta(x, y)) \rightarrow (\forall x \exists y\ \ \theta(x, y))$$

37. (10-20 M) Give a Tableau proof of

$$[\forall x(\theta(x) \wedge \psi)] \leftrightarrow [(\forall x\ \theta(x)) \wedge \psi]$$

38. (10-22 W) For the sentence θ below decide if θ is a logical validity or not and either give a Tableau proof of θ or find a model M of $\neg\theta$.

$$\forall x \exists y\ (R(x,y) \vee \neg R(x,y))$$

39. (10-22 W) For the sentence θ below decide if θ is a logical validity or not and either give a Tableau proof of θ or find a model M of $\neg\theta$.

$$(\forall x \exists y\ R(x,y)) \vee (\forall x \exists y\ \neg R(x,y))$$

40. (10-22 W) For the sentence θ below decide if θ is a logical validity or not and either give a Tableau proof of θ or find a model M of $\neg\theta$.

$$(\forall x \exists y\ R(x,y)) \vee (\exists x \forall y\ \neg R(x,y))$$

In the next six problems Σ is a set of $\mathcal{L}$-sentences and θ is an $\mathcal{L}$-sentence. For each of them first state whether the statement is True or False. If it is true, say why, (give a proof). If it is false, give an example for which it fails. Recall that

$$Th(\Sigma) = \{\theta\ :\ \Sigma \vdash \theta\}$$

and

$$\mathcal{M}(\Sigma) = \{M\ :\ \text{ for every } \theta \in \Sigma\ \ M \models \theta\}$$

41. (10-24 F) Prove or disprove:

$$\theta \in Th(\Sigma) \text{ iff } \mathcal{M}(\Sigma) \subseteq \mathcal{M}(\theta)$$

42. (10-24 F) Prove or disprove:

$$\Sigma_1 \subseteq Th(\Sigma_2) \text{ iff } \mathcal{M}(\Sigma_1) \subseteq \mathcal{M}(\Sigma_2)$$

43. (10-24 F) Prove or disprove:

$$\mathcal{M}(\Sigma_1)\subseteq\mathcal{M}(\Sigma_2) \rightarrow \Sigma_1\subseteq\Sigma_2$$

44. (10-24 F) Prove or disprove:

$$\theta \in Th(\Sigma) \text{ iff } \text{ for some finite } \Sigma_0\subseteq\Sigma \quad \theta \in Th(\Sigma_0)$$

45. (10-24 F) Prove or disprove:

$$\mathcal{M}(\Sigma) = \emptyset \text{ iff } \text{ for some finite } \Sigma_0\subseteq\Sigma \quad \mathcal{M}(\Sigma_0) = \emptyset$$

46. (10-24 F) Prove or disprove:

$$\text{for any } \Sigma \text{ there exists } \theta \text{ such that } \mathcal{M}(\Sigma) = \mathcal{M}(\theta)$$

47. (10-29 W) proplog.pdf number 28.

(28) A partial order $\leq$ on a set A has *dimension* less than $n+1$ iff there exists n linear orders $\{\leq_1, \leq_2, \leq_3, \ldots, \leq_n\}$ on A (not necessarily distinct) such that:

$$\forall x, y \in A\; [x \leq y \text{ iff } (x \leq_i y \text{ for } i = 1, 2, \ldots, n)].$$

Show that a partial order $\leq$ on a set A has dimension less than $n+1$ iff for every finite X included in A the restriction of $\leq$ to X has dimension less than $n+1$.

48. (11-05 W) Let Σ_1 and Σ_2 be sets of $\mathcal{L}$-sentences. Prove

$$\mathcal{M}(\Sigma_1) = \mathcal{M}(\Sigma_2) \text{ iff } Th(\Sigma_1) = Th(\Sigma_2)$$

49. (11-05 W) Let Σ be a set of $\mathcal{L}$-sentences. Prove

$$\mathcal{M}(Th(\Sigma)) = \mathcal{M}(\Sigma)$$

50. (11-05 W) For $\mathcal{K}$ a class of $\mathcal{L}$-structures, define $Th(\mathcal{K})$ to be the set of all $\mathcal{L}$-sentence true in every structure in $\mathcal{K}$, i.e.,

$$Th(\mathcal{K}) = \{\theta \;:\; \text{ for all } M \in \mathcal{K} \;\; M \models \theta\}.$$

Let Σ be a set of $\mathcal{L}$-sentences and let $\mathcal{K} = \mathcal{M}(\Sigma)$. Prove

$$Th(\mathcal{K}) = Th(\Sigma).$$

51. (11-05 W) For $\mathcal{K}$ a class of $\mathcal{L}$-structures we say that $\mathcal{K}$ is finitely axiomatizable iff there is an $\mathcal{L}$-sentence θ such that $\mathcal{K} = \mathcal{M}(\theta)$. Suppose Σ is a set of $\mathcal{L}$-sentences and $\mathcal{K} = \mathcal{M}(\Sigma)$. Prove that if $\mathcal{K}$ is finitely axiomatizable, then there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\mathcal{K} = \mathcal{M}(\Sigma_0)$.

52. (11-07 F) proplog.pdf number 32

(32) Let $\mathcal{F}$ be a family of subsets of a set X. We say that $\mathcal{C} \subseteq \mathcal{F}$ is an *exact cover* of $Y \subseteq X$ iff every element of Y is in a unique element of $\mathcal{C}$. Suppose that every element of X is in at most finitely many elements of $\mathcal{F}$. Show that there exists an exact cover $\mathcal{C} \subseteq \mathcal{F}$ of X iff for every finite $Y \subseteq X$ there exists $\mathcal{C} \subseteq \mathcal{F}$ an exact cover of Y. Is it necessary that every element of X is in at most finitely many elements of $\mathcal{F}$?

53. (11-10 M) Find prenex normal form for

$$[\exists x \forall y\ (R(x,y) \vee \neg R(y,z))] \wedge \neg[\forall y\ (R(y,z) \vee \neg U(z))]$$

54. (11-10 M) Use Skolem functions to find a universal sentence which is satisfiable iff the following sentence is satisfiable:

$$\forall x \exists y \forall u \forall v \exists w\ [R(x,y,u) \wedge \neg S(u,v) \wedge U(w)]$$

55. (11-12 W) Let $\mathcal{L} = \{R\}$ be the language consisting of a single binary relations symbol R. Consider the $\mathcal{L}$-structure $M = (\mathbb{R}, \leq)$. Prove that for every $\mathcal{L}$-formula $\theta(x)$ with one free variable x that the set

$$\{a \in \mathbb{R}\ :\ M \models \theta(a)\}$$

is either the empty set or $\mathbb{R}$.

56. (11-17 M) cmpthy.pdf exercise 1.3

(1.3) Prove that the greatest common divisor function $d = gcd(n,m)$ is UR-Basic computable. Or if you prefer the function $f(n) =$ the n^{th} prime. Or you can prove that your favorite function is UR-Basic computable.

57. (11-17 M) See, for example, the FOR-NEXT LOOP on p.5 cmpthy.pdf. Similarly show that the two programming construction below can be "mimicked" in UR-BASIC :

DO
  S_1
  S_2
  $\vdots$
  S_n
LOOP UNTIL $X \leq Y$

DO WHILE $X \leq Y$
  S_1
  S_2
  $\vdots$
  S_n
LOOP

58. (11-19 W) proplog.pdf problem number 33.

(33) If $\mathcal{F}$ is a family of subsets of X and $Y \subseteq X$ then we say Y *splits* $\mathcal{F}$ iff for any $Z \in \mathcal{F}$, $Z \cap Y$ and $Z \setminus Y$ are both nonempty. Prove that if $\mathcal{F}$ is a family of finite subsets of X then $\mathcal{F}$ is split by some $Y \subseteq X$ iff every finite $\mathcal{F}' \subseteq \mathcal{F}$ is split by some $Y \subseteq X$. What if $\mathcal{F}$ is allowed to have infinite sets in it?

59. (11-21 F) Show that every finite subset and every cofinite subset of ω is definable in the structure $(\omega, \leq)$.

Extra credit. Show there are exactly four subsets of $(\mathbb{Q}, +)$ which are definable.

60. (11-21 F) Exercise 2.5 in cmpthy.pdf

(2.5) Let $r(n) = n^{th}$ digit of $\sqrt{2} = 1.4142136\ldots$, so $r(0) = 1$, $r(1) = 4$, and so on. Prove that r is primitive recursive. If you prefer you may use $e = 2.7182818\ldots$ instead of $\sqrt{2}$. Does every naturally occurring constant in analysis have this property?

61. (11-24 M) UR-Java is really just C. UR-C or URC is similar to UR-Basic except it has the following three types of statements:

(1) $X++;$
(2) $X--;$
(3) While $X \leq Y$ $\{S_1, S_2, \ldots, S_n\};$

X and Y can be any variables and take on values in ω. The ++ and $--$ mean to increment of decrement the value of X. The "While" statements means to repeatedly execute the code block of statements $S_1, S_2, \ldots, S_n$ which may alter the values X and Y, until and if $X > Y$.

Prove that the assignment statement

$$X = Y;$$

is URC-computable, i.e., input X, Y and output $X = Y =$ input value of Y.

62. (11-24 M) Prove that every primitive recursive function is URC-computable.

63. (11-26 W) Exercise 4.4 in cmpthy.pdf

(4.4) Another way to code finite sequences of arbitrary length is to use prime factorization.

(a) Define: nextprime$(x) = y$ to be the smallest prime $y > x$. Prove that nextprime(x) is primitive recursive.

(b) Define: $p_0 = 2$ and p_n is the n^{th} odd prime. Prove that the function $n \mapsto p_n$ is primitive recursive.

(c) Define $c(x, i) = k$ iff k is the least integer such that p_i^{k+1} does not divide x. Prove that c is primitive recursive and for any finite sequence $x_0, x_1, \ldots, x_n$ there exists x such that $c(x, k) = x_k$ for all $k \leq n$.

64. (11-26 W) Exercise 4.5 in cmpthy.pdf

(4.5) Suppose that $f : \omega \to \omega$ is UR-Basic computable by a program P and there exists a primitive recursive function $s : \omega \to \omega$ such that for every x the program P computes $f(x)$ in $\leq s(x)$ steps. Prove that f is primitive recursive.

65. (12-01 M) Exercise 5.2 in cmpthy.pdf

(5.2) Prove that there exists a (total) $h : \omega \to \omega$ whose graph is a primitive recursive predicate but h is not a primitive recursive function. Hint: consider $h(x) = \mu z\, Q(e, x, z)$.

66. (12-05 F) Suppose Σ is a computably enumerable set of sentences in the language of arithmetic $\mathcal{N}$.

(a) Prove that

$$Th(\Sigma) = \{\theta \ : \ \Sigma \vdash \theta\}$$

is computably enumerable.

(b) Prove that there exists a computable set Γ such that

$$Th(\Gamma) = Th(\Sigma).$$