A.Miller
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Propositional Logic and the Compactness Theorem

The syntax (grammar) of propositional logic is the following. The logical symbols are $\wedge, \vee, \neg, \rightarrow$, and $\leftrightarrow$. The nonlogical symbols consist of an arbitrary nonempty set $\mathcal{P}$ that we assume is disjoint from the set of logical symbols to avoid confusion. The set $\mathcal{P}$ is referred to as the set of atomic sentences or as the set of propositional letters. For example, $\{P, Q, R\}$, $\{P_0, P_1, P_2, \ldots\}$, or $\{S_r : r \in \mathbb{R}\}$. The set of propositional sentences $\mathcal{S}$ is the smallest set of finite strings of symbols such that $\mathcal{P} \subseteq \mathcal{S}$, and if $\theta \in \mathcal{S}$ and $\psi \in \mathcal{S}$, then $(\neg\theta) \in \mathcal{S}, (\theta \wedge \psi) \in \mathcal{S}, (\theta \vee \psi) \in \mathcal{S}, (\theta \rightarrow \psi) \in \mathcal{S}$, and $(\theta \leftrightarrow \psi) \in \mathcal{S}$.

The semantics (meaning) of propositional logic consists of truth evaluations. A truth evaluation is a function $e : \mathcal{S} \rightarrow \{T, F\}$, that is consistent with the following truth tables:

θ	ψ	$\neg\theta$	$(\theta \wedge \psi)$	$(\theta \vee \psi)$	$(\theta \rightarrow \psi)$	$(\theta \leftrightarrow \psi)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

For example if $e(\theta) = T$ and $e(\psi) = F$, then $e(\theta \rightarrow \psi) = F$. Also $e(\neg\theta) = T$ iff $e(\theta) = F$. For example, if $\mathcal{P} = \{P_x : x \in \mathbb{R}\}$ and we define $e(P_x) = T$ if x is a rational and $e(P_x) = T$ if x is a irrational, then $e((P_2 \wedge \neg P_{\sqrt{2}})) = T$. However if we define $e'(P_x) = T$ iff x is an algebraic number, then $e'((P_2 \wedge \neg P_{\sqrt{2}})) = F$.

A sentence θ is called a validity iff for every truth evaluation e, $e(\theta) = T$.

We say that two sentences θ and ψ are logically equivalent iff for every truth evaluation e, $e(\theta) = e(\psi)$. A set of logical symbols is adequate for propositional logic iff every propositional sentence is logically equivalent to one whose only logical symbols are from the given set.

1. Define $\mathcal{S}_0 = \mathcal{P}$ the atomic sentences and define

$$\mathcal{S}_{n+1} = \mathcal{S}_n \cup \{\neg\theta : \theta \in \mathcal{S}_n\} \cup \{(\theta \# \psi) : \theta, \psi \in \mathcal{S}_n, \# \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}\}$$

Prove that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots$.

2. Prove that for any function $f : \mathcal{P} \to \{T, F\}$ there exists a unique truth evaluation $e : \mathcal{S} \to \{T, F\}$ such that $f = e \upharpoonright \mathcal{P}$. The symbol $e \upharpoonright \mathcal{P}$ stands for the restriction of the function e to $\mathcal{P}$.

3. Let θ and ψ be two propositional sentences. Show that θ and ψ are logically equivalent iff $(\theta \leftrightarrow \psi)$ is a validity.

4. Suppose θ is a propositional validity, P and Q are two of the propositional letters occurring in θ, and ψ is the sentence obtained by replacing each occurrence of P in θ by Q. Prove that ψ is a validity.

5. Can you define $\vee$ using only $\to$? Can you define $\wedge$ using only $\to$?

6. Show that $\{\vee, \neg\}$ is an adequate set for propositional logic.

7. The definition of the logical connective nor ($\circ$) is given by the following truth table:

θ	ψ	$(\theta \circ \psi)$
T	T	F
T	F	F
F	T	F
F	F	T

Show that $\{\circ\}$ is an adequate set for propositional logic.

8. (Sheffer) Find another binary connective that is adequate all by itself.

9. Show that $\{\neg\}$ is not adequate.

10. Show that $\{\vee\}$ is not adequate.

11. How many binary logical connectives are there? We assume two connectives are the same if they have the same truth table.

12. Show that there are exactly two binary logical connectives that are adequate all by themselves. Two logical connectives are the same iff they have the same truth table.

13. Suppose $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$. How many propositional sentences (up to logical equivalence) are there in this language?

14. Show that every propositional sentence is equivalent to a sentence in disjunctive normal form, i.e. a disjunction of conjunctions of atomic or the negation of atomic sentences:

$$\vee_{i=1}^{m} (\wedge_{j=1}^{k_i} \theta_{ij})$$

where each θij is atomic or $\neg$atomic. The expression $\vee_{i=1}^{n}\psi_i$ abbreviates $(\psi_1 \vee (\psi_2 \vee (\cdots \vee (\psi_{n-1} \vee \psi_n))) \cdots)$.

In the following definitions and problems Σ is a set of propositional sentences in some fixed language and all sentences are assumed to be in this same fixed language. Σ is realizable iff there exists a truth evaluation e such that for all $\theta \in \Sigma$, $e(\theta) = T$. Σ is finitely realizable iff every finite subset of Σ is realizable. Σ is complete iff for every sentence θ in the language of Σ either θ is in Σ or $\neg\theta$ is in Σ.

15. Show that if Σ is finitely realizable and θ is any sentence then either $\Sigma \cup \{\theta\}$ is finitely realizable or $\Sigma \cup \{\neg\theta\}$ is finitely realizable.

16. Show that if Σ is finitely realizable and $(\theta \vee \psi)$ is in Σ, then either $\Sigma \cup \{\theta\}$ is finitely realizable or $\Sigma \cup \{\psi\}$ is finitely realizable.

17. Show that if Σ is finitely realizable and complete and if θ and $(\theta \to \psi)$ are both in Σ, then ψ is in Σ.

18. Show that if Σ is finitely realizable and complete, then Σ is realizable.

19. Suppose that the set of all sentences in our language is countable, e.g., $S = \{\theta_n : n = 0, 1, 2, \ldots\}$. Show that if Σ is finitely realizable, then there exists a complete finitely realizable Σ' with $\Sigma \subseteq \Sigma'$.

20. (**Compactness theorem for propositional logic**) Show that every finitely realizable Σ is realizable. You may assume there are only countably many sentences in the language.

A family of sets $\mathcal{C}$ is a chain iff for any X, Y in $\mathcal{C}$ either $X \subseteq Y$ or $Y \subseteq X$. The union of the family $\mathcal{A}$ is

$$\bigcup \mathcal{A} = \{b : \exists c \in \mathcal{A}, b \in c\}.$$

M is a maximal member of a family $\mathcal{A}$ iff $M \in \mathcal{A}$ and for every B if $B \in \mathcal{A}$ and $M \subseteq B$, then $M = B$. A family of sets $\mathcal{A}$ is closed under the unions of chains iff for every subfamily, $\mathcal{C}$, of $\mathcal{A}$ which is a chain the union of the chain, $\bigcup \mathcal{C}$, is also a member of $\mathcal{A}$.

Maximality Principle: Every family of sets closed under the unions of chains has a maximal member.

21. Show that the family of finitely realizable Σ is closed under unions of chains.

22. Show how to prove the compactness theorem without the assumption that there are only countably many sentences. (You may use the Maximality Principle.)

23. Suppose Σ is a set of sentences and θ is some sentence such that for every truth evaluation e if e makes all sentences in Σ true, then e makes θ true. Show that for some finite $\{\psi_1, \psi_2, \psi_3, \ldots \psi_n\} \subseteq \Sigma$ the sentence

$$(\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \cdots \wedge \psi_n) \rightarrow \theta$$

is a validity.

A binary relation R on a set A is a subset of $A \times A$. Often we write xRy instead of $\langle x, y \rangle \in R$. A binary relation $\leq$ on a set A is a partial order iff

a. (reflexive) $\forall a \in A \; a \leq a$;

b. (transitive) $\forall a, b, c \in A \; [(a \leq b \wedge b \leq c) \rightarrow a \leq c]$; and

c. (antisymmetric) $\forall a, b \in A \; [(a \leq b \wedge b \leq a) \rightarrow a = b]$.

Given a partial order $\leq$ we define the strict order $<$ by

$$x < y \leftrightarrow (x \leq y \wedge x \neq y)$$

A binary relation $\leq$ on a set A is a linear order iff $\leq$ is a partial order and

d. (total) $\forall a, b \in A (a \leq b \vee b \leq a)$.

A binary relation R on a set A extends a binary relation S on A iff $S \subseteq R$.

24. Show that for every finite set A and partial order $\leq$ on A there exists a linear order $\leq^*$ on A extending $\leq$.

25. Let A be any set and let our set of atomic sentences $\mathcal{P}$ be:

$$\mathcal{P} = \{P_{ab} : a, b \in A\}$$

For any truth evaluation e define $\leq_e$ to be the binary relation on A defined by

$$a \leq_e b \text{ iff } e(P_{ab}) = T.$$

Construct a set of sentences Σ such that for every truth evaluation e,

e makes Σ true iff $\leq_e$ is a linear order on A.

26. Without assuming the set A is finite prove for every partial order $\leq$ on A there exists a linear order $\leq^*$ on A extending $\leq$.

In the next problems n is an arbitrary positive integer.

27. If $X \subseteq A$ and R is a binary relation on A then the restriction of R to X is the binary relation $S = R \cap (X \times X)$. For a partial order $\leq$ on A, a set $B \subseteq A$ is an $\leq$-chain iff the restriction of $\leq$ to B is a linear order. Show that given a partial order $\leq$ on A:

the set A is the union of less than n $\leq$-chains iff every finite subset of A is the union of less than n $\leq$-chains.

28. A partial order $\leq$ on a set A has dimension less than $n+1$ iff there exists n linear orders $\{\leq_1, \leq_2, \leq_3, \ldots, \leq_n\}$ on A (not necessarily distinct) such that:

$$\forall x, y \in A\ [x \leq y \text{ iff } (x \leq_i y \text{ for } i = 1, 2, \ldots, n)].$$

Show that a partial order $\leq$ on a set A has dimension less than $n+1$ iff for every finite X included in A the restriction of $\leq$ to X has dimension less than $n+1$.

29. A binary relation E (called the edges) on a set V (called the vertices) is a graph iff

a. (irreflexive) $\forall x \in V \neg xEx$; and

b. (symmetric) $\forall x, y \in V\ (xEy \rightarrow yEx)$.

We say x and y are adjacent iff xEy. (V', E') is a subgraph of (V, E) iff $V' \subseteq V$ and E' is the restriction of E to V'. For a graph (V, E) an n coloring is a map $c : V \rightarrow \{1, 2, \ldots, n\}$ satisfying $\forall x, y \in V(xEy \rightarrow c(x) \neq c(y))$, i.e. adjacent vertices have different colors. A graph (V, E) has chromatic number $\leq n$ iff there is a n coloring on its vertices. Show that a graph has chromatic number $\leq n$ iff every finite subgraph of it has chromatic number $\leq n$.

30. A triangle in a graph (V, E) is a set $\Delta = \{a, b, c\} \subseteq V$ such that aEb, bEc, and cEa. Suppose that every finite subset of V can be partitioned into n or fewer sets none of which contain a triangle. Show that V is the union of n sets none of which contain a triangle.

31. (Henkin) A transversal for a family of sets $\mathcal{F}$ is a one-to-one choice function. That is a one-to-one function f with domain $\mathcal{F}$ and for every $x \in \mathcal{F}$ $f(x) \in x$. Suppose that $\mathcal{F}$ is a family of finite sets such that for every finite $\mathcal{F}' \subseteq \mathcal{F}, \mathcal{F}'$ has a transversal. Show that $\mathcal{F}$ has a transversal. Is this result true if $\mathcal{F}$ contains infinite sets?

32. Let $\mathcal{F}$ be a family of subsets of a set X. We say that $\mathcal{C} \subseteq \mathcal{F}$ is an exact cover of $Y \subseteq X$ iff every element of Y is in a unique element of $\mathcal{C}$. Suppose that every element of X is in at most finitely many elements of $\mathcal{F}$. Show that

there exists an exact cover $\mathcal{C} \subseteq \mathcal{F}$ of X iff for every finite $Y \subseteq X$ there exists $\mathcal{C} \subseteq \mathcal{F}$ an exact cover of Y. Is it necessary that every element of X is in at most finitely many elements of $\mathcal{F}$?

33. If $\mathcal{F}$ is a family of subsets of X and $Y \subseteq X$ then we say Y splits $\mathcal{F}$ iff for any $Z \in \mathcal{F}$, $Z \cap Y$ and $Z \setminus Y$ are both nonempty. Prove that if $\mathcal{F}$ is a family of finite subsets of X then $\mathcal{F}$ is split by some $Y \subseteq X$ iff every finite $\mathcal{F}' \subseteq \mathcal{F}$ is split by some $Y \subseteq X$. What if $\mathcal{F}$ is allowed to have infinite sets in it?

34. Given a set of students and set of classes, suppose each student wants one of a finite set of classes, and each class has a finite enrollment limit. Show that if each finite set of students can be accommodated, they all can be accommodated.

35. Show that the compactness theorem of propositional logic is equivalent to the statement that for any set I, the space 2^I, with the usual Tychonov product topology is compact, where $2 = \{0, 1\}$ has the discrete topology. (You should skip this problem if you do not know what a topology is.)