A.Miller
M571
Spring 2002
Midterm Exam

Do not hand in this exam, so do not write your answers on it.
You can take as long as you want on this exam.
You will need your own paper.

1. Let H be the set of all integers which are either divisible by 3 or negative. Prove that H is countable by constructing a map from $\mathbb{N} = \{1, 2, 3, \ldots\}$ onto H.

(Be sure and prove that the map you construct is onto.)

2. Which of the following sets are countable?
(a) $\mathbb{R}$
(b) $\mathbb{Q}$
(c) $\{0, 3, 5\}$
(d) $\{\mathbb{R}, \mathbb{C}, \sqrt{2}\}$
(e) the empty set
(f) $\{x \in \mathbb{R} : x^2 = -1\}$
(g) the set of points in the plane $\mathbb{R}^2$.
(h) the set of all subsets of $\mathbb{N}$.
(i) the set of all finite subsets of $\mathbb{N}$.
(j) $\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$

(For this problem just answer countable or uncountable (ctble, unctble), you need not justify your answer.)

3. Give the truth table for the propositional sentence

$$((A \to B) \vee C)$$

4. Let $\Sigma = \{A, (A \to B)\}$. For each of these prove or disprove:
(a) $\Sigma \models B$
(b) $\Sigma \models ((\neg A) \vee (\neg B))$

5. Find a sentence in disjunctive normal form which is logically equivalent to

$$\neg(A \leftrightarrow B)$$

6. Prove or disprove:

Every well-formed formula of propositional logic is logically equivalent to one in which the only symbols are $\rightarrow$, $\neg$, parentheses, and propositional letters.

7. Suppose that Σ is a set of formulas of propositional logic which is finitely satisfiable. Show that for any formula θ of propositional logic that either $\Sigma \cup \{\theta\}$ is finitely satisfiable or $\Sigma \cup \{\neg\theta\}$ is finitely satisfiable.

(Do not use the compactness theorem or any lemma used to prove compactness in your proof.)

8. A square in a graph (V, E) is a set $\square = \{a, b, c, d\} \subseteq V$ of four distinct vertices such that aEb, bEc, cEd, dEa, $\neg aEc$, and $\neg bEd$. Let n be a fixed positive integer.

Suppose that every finite subgraph of V is the union of n sets none of which contain a square. Show that V is the union n sets none of which contain a square.

(You may use the compactness theorem without proof.)

9. R is a binary predicate symbol. Prove or disprove:

$(\forall y \exists x\ R(x, y)) \rightarrow (\exists x \forall y\ R(x, y))$ is a logical validity.

10. Let $\mathcal{L}_R$ be the language containing the binary relation symbol R. Write down a first order $\mathcal{L}_R$-sentence θ such that for any $\mathcal{L}_R$-structure $\mathfrak{A}$

$\mathfrak{A} \models \theta$ iff $\mathfrak{A}$ is a linear order with no greatest element.

Answers

1. Define $g(n) = 3n$ then g maps $\mathbb{N}$ onto the positive multiples of 3. Define $h(n) = -n + 1$ then h maps $\mathbb{N}$ onto the integers $\{0, -1, -2, \ldots\}$. Combine them into one map $f : \mathbb{N} \to H$ by the rule $f(2n) = g(n)$ and $f(2n-1) = h(n)$.

2. ucccccuucu

3. There is only one false line in the truth table. It is ABC - TFF.

4. a is true and b is false.

5. $(\neg A \wedge B) \vee (A \wedge \neg B)$

6. This is true. Note that $(\theta \vee \psi)$ is logically equivalent to $(\neg\theta) \to \psi$. Use induction to prove that every WFF equivalent to one using only $\to$, $\neg$.

7. (Do not use that Σ is satisfiable in your proof. We only know this by the compactness theorem.)

Suppose for contradiction that neither $\Sigma \cup \{\theta\}$ is finitely satisfiable nor $\Sigma \cup \{\neg\theta\}$ is finitely satisfiable. Then there must exists finite sets $\Sigma_0, \Sigma_1 \subseteq \Sigma$ such that $\Sigma_0 \cup \{\theta\}$ is not satisfiable and $\Sigma_1 \cup \{\neg\theta\}$ is not satisfiable.

But $\Sigma_0 \cup \Sigma_1 \subseteq \Sigma$ is finite. So by assumption it is satisfiable. Let ν be a truth evaluation which makes every ψ in $\Sigma_0 \cup \Sigma_1$ true.

If $\nu(\theta) = T$, then $\Sigma_0 \cup \{\theta\}$ is satisfiable by ν, which is a contradiction.

If $\nu(\theta) = F$, then $\nu(\neg\theta) = T$ and therefore $\Sigma_1 \cup \{\neg\theta\}$ is satisfiable by ν, which is a contradiction.

Since either way leads to a contradiction the result is proved.

8. Let the set of propositional letters be the set of all P_v^k for $v \in V$ and $k = 1, 2, \ldots, n$.

Define

$\Sigma_1 = \{(P_v^1 \vee P_v^2 \vee \cdots \vee P_v^n) \ : \ v \in V\}$

$\Sigma_2 = \{\neg(P_a^k \wedge P_b^k \wedge P_c^k \wedge P_d^k) \ : \ \{a, b, c, d\} \text{ is a square in } V, 1 \leq k \leq n\}$

$\Sigma = \Sigma_1 \cup \Sigma_2$.

Claim. Σ is finitely satisfiable.

proof: Given a finite $\Sigma_0 \subseteq \Sigma$. Let $V' \subseteq V$ be the set of all vertices mentioned in Σ_0. Since it is finite by assumption V' is the union of n square free sets, so let $V' = A_1 \cup A_2 \cup \ldots \cup A_n$ where each A_i contains no square. Define a truth evaluation by

$\nu(P_v^k) = T$ iff $v \in A_k$.

But then it is easy to check that $\nu(\psi) = T$ for each $\psi \in \Sigma_0$.

By the compactness theorem Σ is satisfiable. So let ν be a truth evaluation such that $\nu(\Sigma) = T$. For each k define the set A_k by

$$A_k = \{v \ : \ \nu(P_v^k) = T\}$$

By the axioms in Σ_1 we see that for every $v \in V$ there exists a k such that $v \in A_k$. By the axioms in Σ_2 we see that no A_k contains a square. Hence V is the union of n sets, none of which contain a square.

9. This is not a logical validity. To show this exhibit a single specific structure in which it is false - don't natter on and on about this or that.

Counterexample: Let $A = \{0, 1\}$ and $R_A = \{(0,0), (1,1)\}$. Then

$$\mathfrak{A} \models (\forall y \exists x \ R(x, y))$$

since given y taking $x = y$ always works. But

$$\mathfrak{A} \models \neg(\exists x \forall y \ R(x, y))$$

since neither $x = 0$ nor $x = 1$ works for all y.

10. Remember a sentence of first-order logic has no free variables.

$\forall x \ \ R(x, x)$
$\forall x \forall y \ \ (R(x, y) \wedge R(y, x)) \rightarrow x = y$
$\forall x \forall y \forall z \ \ (R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$
$\forall x \forall y \ \ R(x, y) \vee R(y, x)$

These are the axioms of a linear order. To say that there is no greatest element: $\neg \exists x \forall y \ \ R(y, x)$ or $\forall x \exists y \ \ (R(x, y) \wedge x \neq y)$