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M571
Homework

(1-23)

(A) Use Venn Diagrams to prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(B) Let $A = \{1, 2, \ldots, n\}$. How many binary relations R on A, (ie. $R \subseteq A^2$), are there such that

(1) (no conditions)
(2) R is reflexive
(3) (A, R) is a linear order
(4) (A, R) is a graph, ie. R is irreflexive and symmetric
(5) (A, R) is an equivalence relation with exactly two equivalence classes.

(1-25)

(A) Suppose $f : A \to B$ and $g : B \to C$ are functions. Prove

1. if f and g are 1-1, then $g \circ f$ is 1-1.
2. if f and g are onto, then $g \circ f$ is onto.
3. Show by examples that neither of the above implications reverse.

(B) Suppose A is a nonempty set. Prove the following are equivalent:

1. there is a 1-1 $g : A \to \omega$
2. there is an onto $f : \omega \to A$.

(1-28)

(A) Prove that for every set X there is no map $f : X \to P(X)$ which is onto.

(B) Let $\mathbb{Q}[x]$ be the polynomials with rational coefficients, i.e.,

$$\mathbb{Q}[x] = \{p \ : \text{ for some } n \in \omega,\ a_i \in \mathbb{Q} \ \ p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\}$$

Prove that $\mathbb{Q}[x]$ is countable.

(C) Let $\mathbb{A} \subseteq \mathbb{C}$ be the set of algebraic numbers. A complex number is algebraic iff it is the root of a nontrivial polynomial $p \in \mathbb{Q}[x]$. Prove that $\mathbb{A}$ is countable.

(1-30)
p.19 : 2,3,5

(2-1)
p.27-29 : 5,9,10,14

(2-4)
(A) Prove or disprove: For every WFF θ there exists a WFF θ^* which is logically equivalent to θ and does not contain the negation symbol, i.e., WFF which are strings of the symbols:

$$\{\vee, \wedge, \rightarrow, \leftrightarrow,), (, A_1, A_2, A_3, \ldots\}$$

(B) For each of the 16 binary logical connectives $\circ$, let $\mathrm{WFF}_\circ$ be the set of well-formed formulas in the language $L = \{\circ, (,), A_1, A_2, \ldots\}$. The connective $\circ$ is called adequate for propositional logic iff every WFF is logically equivalent to one in $\mathrm{WFF}_\circ$. Determine (with proof) all adequate binary logical connectives.

(2-6)
p.53-9 : It is not clear what $(A \leftrightarrow B \leftrightarrow C)$ means. Probably Enderton means $((A \leftrightarrow B) \leftrightarrow C)$ or $(A \leftrightarrow (B \leftrightarrow C))$ which are logically equivalent. Every other mathematician would mean $((A \leftrightarrow B) \wedge (B \leftrightarrow C))$.
p.54-12

(2-8)
proplog handout: 21,22,23

(2-11)
(A). Suppose $\Sigma \subseteq WFF$ is complete and finitely satisfiable. Suppose $F \subseteq \Sigma$ is finite and θ is a WFF such that $F \models \theta$.
Prove that $\theta \in \Sigma$.

(2-13)
(A). Suppose $\Lambda \subseteq WFF$ is complete and finitely satisfiable and θ and ψ are logically equivalent WFFs. Prove that $\theta \in \Lambda$ iff $\psi \in \Lambda$.

(2-15)

proplog handout: 24,25,26,27

(2-18)

proplog handout: 30,32,34

(2-20)

p.65-1,2,3

(2-22)

U and V are unary predicates, x and y distinct variables, and $\equiv$ means logically equivalent.

Prove or disprove

(A) $(\exists x U(x)) \wedge (\exists x V(x)) \equiv \exists x (U(x) \wedge V(x))$

(B) $(\forall x U(x)) \vee (\forall x V(x)) \equiv \forall x \forall y (U(x) \vee V(y))$

(2-25)

p. 79 - 1,2,5

(2-27)

p. 100-104 : 9,16,18,27

(3-13)

p. 146 - 6,8,9

(3-20)

p. 145 - 3,7

(4-3)

p.100- 11,12,15

(4-8)

p.180- 1,4,5

Hint (1)

$$(\mathbb{R}, \mathbb{Q}, <, ..) \models \forall x \forall y \; (x < y \rightarrow \exists q \; \mathbb{Q}(q) \wedge x < q < y)$$

Hint (4) If A is finite,then say $A = \{a_1, a_2, \ldots, a_n\}$.

$$(\mathbb{R}, A, ..) \models \forall x \; (A(x) \leftrightarrow (x = a_1 \vee x = a_2 \vee \cdots \vee x = a_n)$$

If A is infinite, then A is either unbounded or contains a limit point. Or you can use that A contains an infinite sequence.

(4-12)

(A) In the language with one binary relation symbol and one unary relation (say $\mathcal{L} = \{\leq, U\}$) prove that the following two structures are elementarily equivalent

$$(\mathbb{R}, \leq, \mathbb{Q}) \equiv (\mathbb{Q}, \leq, D_2)$$

where D_2 is the set of diyadic rational numbers:

$$D_2 = \{\frac{m}{2^n} \ : \ m \in \mathbb{Z}, \ n = 1, 2, 3, \ldots\}$$

Prove that

$$(\mathbb{R}, \leq, \mathbb{Q}) \not\equiv (\mathbb{R}, \leq, \mathbb{Z})$$

(B) In the language of one binary relation let
$\mathcal{A} = (\mathbb{R}^*, \approx)$ where $x \approx y$ iff x and y are infinitesimally close.
$\mathcal{B} = (\mathbb{R}, \approx_{\mathbb{Q}})$ where $x \approx_{\mathbb{Q}} y$ iff $\exists q \in \mathbb{Q} \quad x = y + q$
$\mathcal{C} = (\mathbb{Q}, \approx_{\mathbb{Z}})$ where $q \approx_{\mathbb{Z}} r$ iff $q - r \in \mathbb{Z}$
Prove that $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$

(C) In the same language let $\mathcal{D}_k = (\mathbb{Z}, \equiv_k)$ for $k = 2, 3, \ldots$ where $m \equiv_k n$ iff $m - n$ is divisible by k (i.e. $m = n \bmod k$).
Prove that for every k, $\mathcal{D}_k \not\equiv \mathcal{C}$.
Prove that for every sentence θ if $\mathcal{C} \models \theta$ then there exists N such that for all $k \geq N$ $\mathcal{D}_k \models \theta$.

Hint (A). Suppose the language of these structures is $\{\leq, U\}$ where U is a unary predicate symbol. Let DLO* be the theory of dense linear orders without end points plus

$$\forall x \forall y (x < y) \rightarrow \exists u \exists v (U(u) \wedge \neg U(v) \wedge x < u < y \wedge x < v < y)$$

Use the Łos-Vaught Test to prove that DLO* is a complete theory.

(B). Write down axioms Σ which say that the binary relation is an equivalence relation, with infinitely many equivalence classes, and all equivalence classes are infinite. Prove the Σ is complete by using the Łos-Vaught Test.

(4-22)

Prove that the definable subsets of $(\omega, S, 0)$ are the finite and cofinite subsets of ω.

Hint: $(\omega, S, 0) \equiv (\omega, S, 0) + (\mathbb{Z}, S)$ or you can use elimination of quantifiers as in book.

(4-24)

Prove that $\mathrm{Th}(\omega, +, |)$ is undecidable where $|$ is the binary predicate $n|m$ iff n divides m.

Hints: Show multiplication is definable in this structure. $n^2 + 1$ is the least common multiple of n and $n + 1$, $(a + b)^2 = a^2 + 2ab + b^2$

(5-1)

(A) An integer x is square-free iff $x \geq 2$ and no integer $y \geq 2$ exists such that y^2 divides x. Let $S(n)$ be the sum of the first n square-free integers. Prove that $S : \omega \to \omega$ is primitive recursive.

(B) Prove there exists an integer n such that 5 divides n, 7 divides n+1, 9 divides n+2, 11 divides n+3, and 13 divides n+4.