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Gödel's Completeness Theorem

We only consider countable languages $\mathcal{L}$ for first order logic with equality which have only predicate symbols and constant symbols. We regard the symbols "$\exists x$"' as an abbreviation for "$\neg\forall x\neg$" or vice-versa if you prefer.

Propositional Tautologies

We take all propositional tautologies as Logical Axioms. For example, for any $\mathcal{L}$-formula θ the formula

$$(\theta \vee \neg\theta)$$

is a logical axiom. More generally, for any proposition tautology we replace the propositional letters with $\mathcal{L}$-formulas and get a Logical Axiom.

The other axioms are axiom schemas, i.e, all formulas of a certain syntactical form.

Equality Axioms

All formulas of the form:
$u = u$
$u = v \rightarrow v = u$
$(u = v \wedge v = w) \rightarrow u = w$
$(u_1 = v_1 \wedge u_2 = v_2 \wedge \cdots \wedge u_n = v_n) \rightarrow (R(u_1, u_2, \ldots, u_n) \leftrightarrow R(v_1, v_2, \ldots, v_n))$
are Logical Axioms. Here u, v, w, u_i, v_i are terms, i.e., either variables or constant symbols in any mix and R is an n-ary predicate symbol.

Substitution Axioms

For any formula $\theta(x)$ and constant symbol c the axiom:

$$(\forall x\theta(x)) \rightarrow \theta(c)$$

where $\theta(c)$ is the formula which results from substituting c for all free occurrences of x in $\theta(x)$.

And the axiom:

$$(\forall x\theta(x)) \rightarrow \theta(y)$$

where y is variable that does not occur in $\theta(x)$ at all.

Henkin Axioms

These aren't in Enderton, so we probably don't need them. However they make the proof of Lemma 6 easier so why not add them. For any formula $\theta(x)$ and variable y which doesn't occur in $\theta(x)$ at all we have the axiom:

$$\exists y[(\exists x\theta(x)) \rightarrow \theta(y)]$$

Generalization Axioms

These axioms are used to prove a weak form of the generalization Lemma (see the Claim in the proof of Lemma 6). They are needed because we have not taken Generalization as a proof rule. Which on the other hand makes the Deduction Lemma easier to prove.

For all formulas θ and ψ and variables x the following is a logical axiom:

$$[\forall x(\theta(x) \rightarrow \psi(x))] \rightarrow [(\forall x\theta(x)) \rightarrow (\forall x\psi(x))]$$

and for all formulas ρ and variables y such that y does not occur in ρ at all (free or bound) the axiom:

$$\rho \rightarrow \forall y\rho$$

Finally, (like Enderton) for any Logical Axiom we can and put as many $\forall x_1 \forall x_2..$ as we want in front of it and we get another Logical Axiom. This concludes our list of logical axioms.

Summary of the Logical Axioms

(P) All propositional tautologies

(E1) $u = u$
(E2) $u = v \rightarrow v = u$
(E3) $(u = v \wedge v = w) \rightarrow u = w$
(E4)$(u_1 = v_1 \wedge \cdots \wedge u_n = v_n) \rightarrow (R(u_1, u_2, \ldots, u_n) \leftrightarrow R(v_1, v_2, \ldots, v_n))$
u, v, w, u_i, v_i are terms (constant symbols or variables)

(S1) $(\forall x \theta(x)) \rightarrow \theta(c)$ where c constant symbol
(S2) $(\forall x \theta(x)) \rightarrow \theta(y)$ where y is a variable that does not occur in $\theta(x)$

(H) $\exists y[(\exists x \theta(x)) \rightarrow \theta(y)]$ for any variable y that does not occur in $\theta(x)$

(G1) $[\forall x(\theta(x) \rightarrow \psi(x))] \rightarrow [(\forall x \theta(x)) \rightarrow (\forall x \psi(x))]$
(G2) $\rho \rightarrow \forall y \rho$ where y does not occur ρ
(G3) If θ any Logical Axiom, so is $\forall x \theta$

Definition: For Σ a set of $\mathcal{L}$-sentences and θ a $\mathcal{L}$-formula $\Sigma \vdash \theta$ iff there is a finite sequence of $\mathcal{L}$-formulas $\theta_1,\ \theta_2\ \ldots\ \theta_n$ such that $\theta = \theta_n$ and each θ_k is either a logical axiom or member of Σ or follows from previous θ_i using Modus Ponens.

Definition: $Th(\Sigma) = \{\theta \text{ an } \mathcal{L}\text{-formula} : \Sigma \vdash \theta\}$.

Note that $Th(\Sigma)$ can also be characterized as the smallest family of $\mathcal{L}$-formulas which contain $\Sigma \cup$Logical Axioms and is closed under Modus Ponens.

The Deduction Lemma only depends on the fact that we have included all propositional tautologies as logical axioms and that our only proof rule is Modus Ponens.

Lemma 1 *(Deduction Lemma) For any set $\Sigma \cup \{\theta\}$ of $\mathcal{L}$-sentences and ψ an $\mathcal{L}$-formula*

$$\Sigma \vdash \theta \rightarrow \psi \quad \textit{iff} \quad \Sigma \cup \{\theta\} \vdash \psi$$

proof:
The direction $\rightarrow$ is easy from Modus Ponens.
For the direction $\leftarrow$ we prove it by showing that the set of all ψ such that

$$\Sigma \vdash \theta \rightarrow \psi$$

contains $\Sigma \cup \{\theta\} \cup$Logical Axioms and is closed under Modus Ponens.

(1) Given ψ note that $\psi \to (\theta \to \psi)$ is a propositional tautology. Hence if $\psi \in \Sigma$ or if ψ is a logical axiom, then $\Sigma \vdash \theta \to \psi$ by using Modus Ponens.

(2) Note that

$$(\theta \to \psi_1) \to [(\theta \to (\psi_1 \to \psi_2)) \to (\theta \to \psi_2)]$$

is a propositional tautology. Hence if $\Sigma \vdash \theta \to \psi_1$ and $\Sigma \vdash \theta \to (\psi_1 \to \psi_2)$ then by two uses of Modus Ponens, $\Sigma \vdash \theta \to \psi_2$.

It follows from (1) and (2) that:

$$Th(\Sigma \cup \{\theta\}) \subseteq \{\psi \ : \ \Sigma \vdash \theta \to \psi\}$$

$\square$

Definition: Σ is an inconsistent set of $\mathcal{L}$-sentences iff $\Sigma \vdash \#$ where $\#$ is some propositional contradiction, for example, $\#$ could be

$$\# \quad = \quad P \wedge \neg P \quad = \quad (\exists x \ \ x = x) \wedge \neg(\exists x \ \ x = x)$$

Definition: Σ is consistent iff Σ is not inconsistent.

Note that since $(\# \to \theta)$ is a propositional tautology for any formula θ, by Modus Ponens, if $\Sigma \vdash \#$ then Σ proves θ. Hence inconsistent Σ prove everything, i.e., $Th(\Sigma)$ is the set of all $\mathcal{L}$-formulas. An equivalent definition of Σ is inconsistent is that for some $\mathcal{L}$-formula θ both $\Sigma \vdash \theta$ and $\Sigma \vdash \neg\theta$. This is because $\theta \to (\neg\theta \to \#)$ is a propositional tautology and therefor by Modus Ponens twice $\Sigma \vdash \#$.

The next couple of lemmas only use the deduction lemma.

Lemma 2 *The following two forms of the completeness theorem are equivalent:*
(a) For every set $\Sigma \cup \{\theta\}$ of $\mathcal{L}$-sentences
if every model of Σ is a model of θ, then $\Sigma \vdash \theta$.
(b) Every consistent set Γ of $\mathcal{L}$-sentences has a model.

proof:

Suppose (a). If Γ has no model, then every model of Γ is a model of $\#$. But then $\Gamma \vdash \#$, hence it is inconsistent.

Suppose (b) and every model of Σ is a model of θ. Then $\Gamma = \Sigma \cup \{\neg\theta\}$ has no model and so is inconsistent by (b). Hence $\Gamma = \Sigma \cup \{\neg\theta\} \vdash \#$.

Now according to the deduction lemma:

$$\Sigma \vdash (\neg\theta \rightarrow \#)$$

But $(\neg\theta \rightarrow \#) \rightarrow \theta$ is a propositional tautology. So by Modus Ponens

$$\Sigma \vdash \theta$$

□

The above proof also shows that:

Lemma 3 *If $\Sigma \cup \{\theta\}$ is an inconsistent set of $\mathcal{L}$-sentences, then $\Sigma \vdash \neg\theta$.*

proof:
If $\Sigma \cup \theta \vdash \#$ then by the Deduction Lemma, $\Sigma \vdash \theta \rightarrow \#$. But

$$(\theta \rightarrow \#) \rightarrow \neg\theta$$

is a propositional tautology and so by Modus Ponens, $\Sigma \vdash \neg\theta$.
□

It also follows from the Deduction Lemma that:

Lemma 4 *For any consistent Σ and sentence θ either $\Sigma \cup \{\theta\}$ is consistent or $\Sigma \cup \{\neg\theta\}$ is consistent.*

proof:
Otherwise by the Deduction Lemma $\Sigma \vdash \theta \rightarrow \#$ and $\Sigma \vdash \neg\theta \rightarrow \#$. But

$$(\neg\theta \rightarrow \#) \rightarrow (\theta \rightarrow \#)$$

is a propositional tautology and so by Modus Ponens twice $\Sigma \vdash \#$.
□

The next lemma shows that adding new constant symbols can't hurt.

Lemma 5 *Suppose Σ is a consistent set of $\mathcal{L}$-sentences and c is a new constant symbol not appearing in $\mathcal{L}$. Then Σ is a consistent set of $\mathcal{L} \cup \{c\}$-sentences.*

proof:
Suppose

$$\theta_1(c), \theta_2(c), \ldots, \theta_n(c)$$

is a proof from Σ of #. Let y be any variable that does not occur in any of the $\theta_i(c)$. Then we claim

$$\theta_1(y), \theta_2(y), \ldots, \theta_n(y)$$

is a proof of # from Σ in $\mathcal{L}$. This is because

(1) if $\theta_i(c) \in \Sigma$ then c does not occur in $\theta_i(c)$, so $\theta(c) = \theta(y)$.

(2) If $\theta_i(c)$ is a logical axiom and y is any variable that does not occur in $\theta_i(c)$, then $\theta_i(y)$ is a logical axiom. (Note that an instance of S1 may turn into an instance of S2.)

(3) Modus Ponens transfers over: e.g.
From $\theta(c)$ and $\theta(c) \to \psi(c)$ infer $\psi(c)$.
From $\theta(y)$ and $\theta(y) \to \psi(c)$ infer $\psi(y)$.
□

Lemma 6 *Suppose Σ is a consistent set of $\mathcal{L}$-sentences and $\theta(x)$ is an $\mathcal{L}$-formula with one free variable x. Let c be a new constant symbol not appearing $\mathcal{L}$. Then $\Sigma \cup \{(\exists x \theta(x)) \to \theta(c)\}$ is a consistent set of $\mathcal{L} \cup \{c\}$-sentences.*

proof:

The following claim is why we need the generalization axioms. It says basically that proving some statement about an arbitrary new constant is the same as proving for all y the statement holds.

Claim. Suppose that $\rho(c)$ is any $\mathcal{L} \cup \{c\}$-sentence such that $\Sigma \vdash \rho(c)$. Then for all but finitely many variables y we have that $\Sigma \vdash \forall y \rho(y)$.
proof:

It is enough to prove that the set of such ρ which satisfy the claim contain Σ and the Logical Axioms, and they are closed under Modus Ponens.

Case 1. $\rho(c)$ is a logical axiom. In this case $\rho(y)$ is also a logical axiom provided that y does not occur in ρ. By (G3) closure under universal quantification, $\forall y \rho(y)$ is also a logical axiom.

Case 2. $\rho(c) \in \Sigma$. In this case c does not appear in $\rho = \rho(c)$. And so

$$\rho \to \forall y \rho$$

is a logical axiom and by Modus Ponens $\Sigma \vdash \forall y \rho$.

Case 3. $\rho(c)$ is obtained by Modus Ponens from formulas, $\psi(c) \to \rho(c)$ and $\psi(c)$ which Σ proves. By induction we assume that for all but finitely many variables y that

$$\Sigma \vdash \forall y \ (\psi(y) \to \rho(y))$$

and

$$\Sigma \vdash \forall y \ \psi(y)$$

Now using the generalization axiom $G1$ and Modus Ponens we get that $(\forall y(\psi(y)) \to (\forall y \rho(y)))$ and so by Modus Ponens again we get $\forall y \rho(y)$. This ends the proof of the Claim.

To prove the Lemma assume for contradiction that $\Sigma \cup \{(\exists x \theta(x)) \to \theta(c)\}$ is inconsistent. Then by Lemma 3 we have that

$$\Sigma \vdash \rho(c)$$

where $\rho(c) = \neg[(\exists x \theta(x)) \to \theta(c)]$. By the claim for all but finitely many variables y

$$\Sigma \vdash \forall y \rho(y)$$

But this is exactly the negation of the Axiom H:

$$[\exists y[(\exists x \theta(x)) \to \theta(y)]$$

is the same as

$$[\neg \forall y \neg[(\exists x \theta(x)) \to \theta(y)]$$

which is the same as

$$\neg \forall y \rho(y)$$

Thus Σ is inconsistent as a $\mathcal{L} \cup \{c\}$-theory and hence by Lemma 5 an inconsistent $\mathcal{L}$-theory.
□

Finally we prove part (b) of Lemma 2.

Theorem 7 *(Gödel's Completeness Theorem) Any consistent set Σ of $\mathcal{L}$-sentences has a model.*

proof:

The first step is to add to $\mathcal{L}$ infinitely many new constant symbols. Let $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \omega\}$. By an induction on N Lemma 5 shows that Σ is a consistent set of $\mathcal{L} \cup \{c_n : n < N\}$-sentences. Since proofs are finite it must be that Σ is a consistent set of $\mathcal{L}'$-sentences.

Now let $\{\psi_n : n \in \omega\}$ be the set of all $\mathcal{L}'$-sentences and let the set of all $\mathcal{L}'$-formulas with exactly one free variable be $\{\theta_n(x_{k_n}) : n \in \omega\}$. These sets are countable because $\mathcal{L}'$ is. Construct an increasing sequence Σ_n of consistent $\mathcal{L}'$-sentences as follows.

Set $\Sigma_0 = \Sigma$.

For even $n = 2m$ put Σ_{n+1} to be either $\Sigma_n \cup \{\psi_m\}$ or $\Sigma_n \cup \{\neg\psi_m\}$ whichever is consistent. One of the two must be consistent by Lemma 4.

For odd $n = 2m + 1$. Let c be a constant not appearing in any of the sentences in Σ_n or in $\theta_m(x)$ and let

$$\Sigma_{n+1} = \Sigma_n \cup \{(\exists x_m \theta_m(x)) \rightarrow \theta_m(c)\}$$

This is consistent by Lemma 6.

Now let

$$\Gamma = \cup_{n\in\omega} \Sigma_n$$

Since the notion of proof is finite the union of an increasing sequence of consistent sets of sentences must be consistent. By construction Γ satisfies:

(1) Γ is consistent

(2) Γ is complete, i.e., for every $\mathcal{L}'$-sentence ψ either $\psi \in \Gamma$ or $\neg\psi \in \Gamma$

(3) Γ has the constant witness property: for any $\mathcal{L}'$-formula $\theta(x)$ with one free variable x, there is a constant c such that $(\exists x \theta(x)) \rightarrow \theta(c)$ is in Γ.

Since Γ is a complete consistent theory it must contain all Logical Axioms which are sentences and it must contain exactly one of θ or $\neg\theta$ for each sentence θ.

Now we build the canonical model $\mathfrak{A}$ from Γ and prove that for every $\mathcal{L}'$-sentence θ that

$$\mathfrak{A} \models \theta \text{ iff } \theta \in \Gamma$$

Let $\mathcal{C}$ be the set of constant symbols in the language $\mathcal{L}'$. Define a binary relation on $\mathcal{C}$ by $c \approx d$ iff $c = d \in \Gamma$. Notice that the equality axioms E1,E2,E3 imply that $\approx$ is an equivalence relation. Since Γ is consistent and complete it must contain all Logical axioms which are sentences, in particular the E1,E2,E3,E4 when the terms involved are constant symbols. It follows from E1,E2,E3 that $\approx$ is an equivalence relation. We define the universe A of the canonical model $\mathfrak{A}$ to be set of equivalence classes of $\approx$:

$$A = \mathcal{C}/\approx$$

For each constant symbol c we define

$$c_A = [c] = \{d \in \mathcal{C} \ : \ c \equiv d\}$$

the equivalence class containing c.

It follows from E4 and that of Γ that
if $c_1 \approx d_1$, $c_2 \approx d_2$, ..., and $c_n \approx d_n$,
then $R(c_1, c_2, \ldots, c_n) \in \Gamma$ iff $R(d_1, d_2, \ldots, d_n) \in \Gamma$.
Thus we may define the relation R_A on A^n by

$$([c_1], [c_2], \ldots, [c_n]) \in R_A \text{ iff } R(c_1, c_2, \ldots, c_n) \in \Gamma$$

This definition of the canonical model guarantees that for any atomic $\mathcal{L}'$-sentence θ:

$$\mathfrak{A} \models \theta \text{ iff } \theta \in \Gamma$$

Inductive steps

$\neg\theta$:

$$\mathfrak{A} \models \neg\theta \text{ iff } \text{ not } \mathfrak{A} \models \theta \text{ iff } \theta \notin \Gamma \text{ iff } \neg\theta \in \Gamma$$

The last "iff" requires proof. If $\theta \notin \Gamma$ then $\neg\theta \in \Gamma$ since Γ is complete. If $\neg\theta \in \Gamma$, then $\theta \notin \Gamma$ because otherwise Γ is inconsistent.

$(\theta_1 \vee \theta_2)$:

$$\mathfrak{A} \models \theta_1 \vee \theta_2 \text{ iff } (\mathfrak{A} \models \theta_1 \text{ or } \mathfrak{A} \models \theta_2) \text{ iff } (\theta_1 \in \Gamma \text{ or } \theta_2 \in \Gamma) \text{ iff } (\theta_1 \vee \theta_2) \in \Gamma)$$

The last "iff" is because Γ is complete and consistent, otherwise one would get a propositional contradiction in Γ.

$\exists x\theta(x)$:
If $\mathfrak{A} \models \exists x\theta(x)$, then for some constant symbol c we have that $\mathfrak{A} \models \theta(c)$. By inductive hypothesis $\theta(c) \in \Gamma$. Since Γ is complete either $\exists x\theta(x) \in \Gamma$ or $\neg\exists x\theta(x) \in \Gamma$. If that latter is the case, then $\neg\neg\forall x\neg\theta(x) \in \Gamma$ and we may drop the double negation, since $\neg\neg A \rightarrow A$ is propositional tautology.

Buy by the substitution axiom S1 we would have $\neg\theta(c)$ provable from Γ and therefore it would be inconsistent.

Conversely suppose that $\exists x\theta(x) \in \Gamma$. Now by the constant witness property $(\exists x\theta(x)) \rightarrow \theta(c) \in \Gamma$ for some constant c. So by Modus ponens $\theta(c) \in \Gamma$ and by inductive hypothesis $\mathfrak{A} \models \theta(c)$ and so $\mathfrak{A} \models \exists x\theta(x)$.

This completes the proof of the completeness theorem.