Math 571 Midterm
A. Miller
March 5, 1987
Due in class Friday, March 27

The first part of the exam consists of Computer problem 2.5 (except for ORDER6 which is an extra credit optional problem).

The second part consists of the following problems:

Polish notation for propositional logic is defined as follows. The logical symbols are $\{\wedge, \vee, \neg, \Leftrightarrow, \Rightarrow\}$, and the nonlogical symbols or proposition symbols are the elements of an arbitrary set $\mathcal{P}_0$. The well-formed formulas in Polish notation (wffpn) are the members of the smallest set of strings which satisfy:

1. Each $p \in \mathcal{P}_0$ is wffpn;
2. If $\mathbf{A}$ is wffpn, then so is $\neg\mathbf{A}$;
3. If $\mathbf{A}$ is wffpn and $\mathbf{B}$ is wffpn, then $\wedge\mathbf{AB}$ is wffpn, $\vee\mathbf{AB}$ is wffpn, $\Leftrightarrow \mathbf{AB}$ is wffpn, and $\Rightarrow \mathbf{AB}$ is wffpn.

Note that no parentheses or brackets are needed for Polish notation.

1. Put the formula $[p \Leftrightarrow q] \Rightarrow [\neg q \vee r]$ into Polish notation.

2. Construct a parsing sequence for the wffpn

$$\vee\neg \Rightarrow pq \Leftrightarrow rp$$

to verify that it is wffpn. Write this formula in regular notation.

3. State the principle of induction as it should apply to wffpn. Prove using induction that for any wffpn $\mathbf{A}$ that the number of logical symbols of the kind $\{\wedge, \vee, \Leftrightarrow, \Rightarrow\}$ in $\mathbf{A}$ is always exactly one less than the number of nonlogical symbols.

We say that a set Δ of wffs is *closed under equality substitution* iff for every variable free term τ we have that $\tau = \tau \in \Delta$ and whenever $[\tau = \sigma] \in \Delta$, $\mathbf{A} \in \Delta$ and $\mathbf{B}$ is obtained from $\mathbf{A}$ by an equality substitution on $\tau = \sigma$, we have that $\mathbf{B} \in \Delta$.

Use the following definition of Gödel number for the next two problems. Let $U : \mathbf{N} \times \mathbf{N} \mapsto \mathbf{N}$ be a universal partial recursive function and for any partial recursive function $f : \mathbf{N} \mapsto \mathbf{N}$ if $e \in \mathbf{N}$ has the property that

$$\forall m \in \mathbf{N}[f(m) \doteq U(e, m)]$$

then e is a Gödel number for f. [If you prefer you can reason about the specific universal function given by the program UNIV2.GN and the specific Gödel numbers used there.]

6. Define $D(x) \doteq U(x, x) + 1$. Let e be a Gödel number of D. Prove that $D(e) \uparrow$.

7. Define $E(x) \doteq U(x, x)$. Let e be a Gödel number of E. Prove that $E(e) \uparrow$. Prove or disprove: $U(e, e) \downarrow$.

8. Prove using the compactness theorem of propositional logic that for any set X and binary relation $R \subset X \times X$ if

1. for every finite $X' \subset X$ there exists a 1-1 function $f : X' \mapsto X$ such that $\forall x \in X' < x, f(x) > \in R$; and
2. for every $x \in X$

$$\{y \in X \ : < x, y > \in R\}$$

is finite;

then there exists a 1-1 function $f : X \mapsto X$ such that

$$\forall x \in X < x, f(x) > \in R.$$

(Note: I have received from three students a proof that the second item above is necessary; anymore proofs will also be accepted.)

9. Number 9 p.150.

Take home final A. Miller Spring 87
due Tue. May 12 12:25 360 Science
Note: There will be a regular final also.

Do any five problems.

1. Suppose instead of using the URM instruction set we use JN,S,Z,T,H where

JN 1 2 6

would mean Jump to instruction 6 if the contents of register 1 is not equal to the contents of register 2. Prove that every computable function is computable in this new sense.

2. Suppose we consider programs that only use the instructions S,Z,T,H; i.e. no jump instructions at all. Show that not every computable function is computable in this sense.

(Hint: show add cannot be programmed.)

3. Prove Theorem 4.7.1 p.115 (You will convince me if you just state and prove the case n=1 and m=1.)

4. Suppose $P : \mathbf{N} \mapsto \mathbf{N}$ is a partial recursive function and X is the domain of P, i.e.

$$X = \{n \; : \; P(n) \downarrow\}$$

Show that if X is nonempty, then there exists a total recursive function $f : \mathbf{N} \mapsto \mathbf{N}$ such that X is the range of f, i.e.

$$X = \{f(n) \; : \; n \in \mathbf{N}\}.$$

[Recursion theorists would say that X is recursively enumerable.]

5. Prove Lemma 3.4.3 p.93. (Warning the definition has been changed: see p.93 def of $\equiv$, p.93 the = rules, also p.89 = rules. Also in 6.5 p. 182 line 6 should read:

4. Prove by induction that for any two wffpn **A** and **B** if **A** is an initial substring of **B**, then they are the same string. Show how this statement implies unique readability of formulas in Polish notation.

We say that a model $(P, \leq)$ is a *linear order* (where $\leq$ is a binary relation on the universe P) iff

1. $(P, \leq) \models \forall x\, x \leq x$
2. $(P, \leq) \models \forall x \forall y \forall z\, [x \leq y \wedge y \leq z \Rightarrow x \leq z]$
3. $(P, \leq) \models \forall x \forall y\, [x \leq y \wedge y \leq x \Rightarrow x = y]$
4. $(P, \leq) \models \forall x \forall y\, [x \leq y \vee y \leq x]$

If $(P, \leq)$ satisfies only the first three it is called a *partial order.*

5. Find a linear order $(P, \leq)$ which satisfies all of the following:

$$(P, \leq) \models \forall x \exists y\, \neg x \leq y$$

$$(P, \leq) \models \forall x \exists y\, \neg y \leq x$$

$$(P, \leq) \models \forall x \forall y\, [\neg y \leq x \Rightarrow \exists z [\neg x \leq z \wedge \neg y \leq z]$$

6. Show by induction that for any finite partial order $(P, \leq)$ (i.e. P is finite) there is a linear order $(P, \leq^*)$ which extendes $\leq$, i.e. for every $a, b \in P$ if $a \leq b$, then $a \leq^* b$.

7. Use the Compactness Theorem for **Propositional Logic** and the last problem to show that every partial order (finite or infinite) can be extended to a linear order.
Hint: Let $(P, \leq)$ be any partial order. Let $P_0 = \{R_{ab} : a, b \in P\}$. Consider interpreting the symbol R_{ab} as "$a \leq^* b$". Keep in mind that you are not given $\leq^*$; you must show that it exists.

Final Exam M571 A. Miller May 87

① give a tableau proof of
$$\forall x\ (\neg q \rightarrow \neg p(x))$$
from hypothesis set
$$\{\ [\exists x\ p(x)] \rightarrow q\ \}.$$

② find a model of
$$[\forall x\ \exists y\ R(x,y)] \wedge \neg [\exists y\ \forall x\ R(x,y)]$$

③ Construct an URM which computes $f(x) = 2x$. Include a flow chart, URM-program with labels, and finally the actual URM-code.

Summertime reading assignment:

"Gödel, Escher, Bach: an Eternal Golden Braid", Douglas R. Hofstadter
Basic Books 1979.

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④ Do not do this problem.

⑤ Prove that no one will have a perfect score on this exam. (Hint: see previous problem).