

Open Book Open Notes

Prove **three** the following statements. One correct solution is worth more than three wrong ones.

Please use a separate sheet of paper for each proof.

Rewrite your solutions carefully inserting the reasons why what you say is true.

Don't hand in a "solution" which you know is incorrect and say nothing about it. If part of your solution is incorrect and you know it, say so. If you just hand in a proof that you know is incorrect and you say nothing, then I will assume that you don't know what you have written is incorrect.

1. Suppose D is dense in X and U is open in X . Prove that $U \subseteq \overline{U \cap D}$.
2. A point $x \in X$ is a cluster point of $A \subseteq X$ iff every open neighborhood of x contains uncountably many points of A . Suppose X is a Lindelöf space and A is an uncountable closed subspace of X . Prove that at least one point of A , $(x \in A)$ is a cluster point of A .
3. Suppose τ and τ' are topologies on the same set X . Suppose (X, τ) is a Hausdorff topology, $\tau \subseteq \tau'$, and (X, τ') is a compact space. Prove that $\tau = \tau'$.
4. Suppose $C \subseteq X$ is a connected subspace of the connected space X . Suppose $X - C = A \cup B$ where $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Prove that both $C \cup A$ and $C \cup B$ are connected.
5. Suppose X is regular and $Q = \{x_n : n < \omega\}$ is a countable discrete subset of X , i.e., Q is discrete in the subspace topology. Prove there exist open sets $\{U_n : n < \omega\}$ which are pairwise disjoint and separate Q , i.e., $x_n \in U_n$ for each n and $U_n \cap U_m = \emptyset$ for each $n \neq m$.
6. Suppose $X \subseteq \mathbb{R}_l$ (the Sorgenfrey line) and X is compact in the subspace topology. Prove that X is countable.

1. Let $V = U - \overline{U \cap D}$. The set V is open, hence if it is nonempty it must meet the dense set D . But this is a contradiction, since $V \subseteq U$ and so $V \cap D \subseteq U \cap D$, but V is disjoint from $U \cap D$.

2. Closed subspaces of Lindelof spaces are Lindelof. If there is no cluster point in A then A can be covered by countably many open sets U with the property that $U \cap A$ is countable. But then A would be the countable union of countable sets, hence countable.

3. If C is τ' -closed, then it is compact. If it is τ' -compact it is τ -compact, since $\tau \subseteq \tau'$. In a Hausdorff space compact sets are closed, hence C is τ -closed. Since τ' -closed implies τ -closed, by taking complements $\tau' \subseteq \tau$.

4. Suppose $C \cup A$ is not connected. Then there exists nonempty E_1, E_2 such that $C \cup A = E_1 \cup E_2$ and $E_1 \cap \overline{E_2} = \overline{E_1} \cap E_2 = \emptyset$. Since C is connected it cannot be separated by the E_i , so assume $E_1 = C \cup A_1$ and $E_2 = A_2$ where $A_1 \cup A_2 = A$. Then $A_3 = C \cup A_1 \cup B$ and A_2 separate X , since $A_2 \cup A_3 = X$ and $A_3 \cap \overline{A_2} = \overline{A_3} \cap A_2 = \emptyset$.

5. Inductively construct U_n so that $\overline{U_n}$ are pairwise disjoint, $x_n \in U_n$, and $x_m \notin \overline{U_n}$ for all $m > n$. Suppose we already have $U_0, U_1, \dots, U_n$. Since the set Q is discrete there exists W an open neighborhood of x_{n+1} which does not contain any x_m for $m \neq n+1$. Since x_{n+1} is not in the closed set $\bigcup_{m \leq n} \overline{U_m}$, we can shrink W if necessary so that it is disjoint from it. Since the space is regular we can find an open set U_{n+1} with

$$x_{n+1} \in U_{n+1} \subseteq \overline{U_{n+1}} \subseteq W$$

6. First note that if X is an uncountable set of reals it must contain a strictly increasing sequence a_n for $n \in \omega$. Letting a be the sup of this sequence we see that X is covered by $\{(-\infty, a_n) : n \in \omega\} \cup \{[a, \infty)\}$ which has no finite subcover.

To see that an uncountable X must contain a strictly increasing sequence, suppose not. Then for every real a there exists $b < a$ such that (b, a) is disjoint from X . But the reverse Sorgenfrey line $\mathbb{R}_r$ is Lindelof, hence countably many such $(b, a]$ cover $\mathbb{R}_r$ and hence X would be countable.

Another way to see this is to note that an X with no infinite ascending sequence is a reverse well-ordering, hence since it is a subset of the real line, it must be countable.