

1. Let X be a topological space and $A \subseteq X$. Prove that for every $x \in X$
 $x \in \overline{A}$ iff
for all U an open neighborhood of x we have that $U \cap A \neq \emptyset$.
2. Let X and Y be topological spaces and $f : X \rightarrow Y$ a function. Assume that
for every $A \subseteq X$ that $f(\overline{A}) \subseteq \overline{f(A)}$.
Prove that f is continuous.
3. Suppose $f : X \rightarrow Y$ is a function and A and B are subsets of Y .
Prove that $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.
4. Suppose $\mathcal{B}$ is a basis for a topology τ on X .
Prove that τ is the intersection of all topologies τ' on X such that $\mathcal{B} \subseteq \tau'$.
5. Let X be a Hausdorff space and $A \subseteq X$.
Prove that the set of limit points of A is a closed subset of X .

Answers

1. Thm 17.5a, p.96
2. Thm 18.1, p.104 (2) $\Rightarrow$ (3) $\Rightarrow$ (1)
3. Exer 2d, p.20
4. Exer 5, p.83
5. A' denotes the limit points of A . Suppose $p \in X$ is not in A' . Then
there exists an open neighborhood U of p such that there are no $q \in A \cap U$
which are different from p .
Claim. $U \cap A' = \emptyset$
Suppose $q \in U$. If $q = p$, then we already know that $q \notin A'$. If $q \neq p$, then
 $q \in V = U - \{p\}$. But V is an open set, since in a Hausdorff space, the set $\{p\}$
is closed and so $V = U \cap (X - \{p\})$ is the intersection of two open sets and
hence is open. Since $A \cap V = \emptyset$ we have that $q \notin A'$ (in fact, it is not even in
the closure of A).
The claim proves that for any $p \in X - A'$ there is an open neighborhood
 U of p such that $U \cap A' = \emptyset$ It follows that

$$X - A' = \bigcup \{U^{\text{open}} \subseteq X : U \cap A' = \emptyset\}$$

and so the complement of A' is the union of open sets hence open and so A' is closed.