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Math 542

OK

36. Let $A, B \in \mathbb{R}^{n \times n}$

Prove A, B are similar in $\mathbb{C}^{n \times n}$ iff A, B are similar in $\mathbb{R}^{n \times n}$

pf: $\Leftarrow$: This is trivial as $\mathbb{R}^{n \times n} \subseteq \mathbb{C}^{n \times n}$

$\Rightarrow$: A, B similar in $\mathbb{C}^{n \times n}$

$\Rightarrow \exists P \in \mathbb{C}^{n \times n}$ s.t.

$A = P^{-1} B P$

Consider the set $\{P : PA = BP\} = V$

Claim: V is a subspace of $\mathbb{C}^{n \times n}$

pf: $0 \in V$ where $0 = \begin{bmatrix} 0 & \cdots \\ \vdots & \end{bmatrix}$

If $P, Q \in V$, we have $PA = BP$ and $QA = BQ$

Then, $(P+Q)A = PA + QA = BP + BQ = B(P+Q)$ using the above

$\Rightarrow P + Q \in V$

Let $c \in \mathbb{C}$, $P \in V$

$PA = BP \Leftrightarrow (cP)A = c(BP) = B(cP)$

So $cP \in V$

Thus, V is a subspace of $\mathbb{C}^{n\times n}$

Let $P \in V$

If $P \in \mathbb{R}^{n\times n}$ we are done

If $P \in \mathbb{C}^{n\times n} \setminus \mathbb{R}^{n\times n}$, take $P = X + iY$

i.e. if $P = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}$ $\quad z_{ij} = a_{ij} + ib_{ij} \quad a_{ij}, b_{ij} \in \mathbb{R}$

then $X = (a_{ij})_{1\le i,j\le n}$, $Y = (b_{ij})_{1\le i,j\le n}$

$\Rightarrow A, B \in \mathbb{R}^{n\times n}$

$PA = BP$

$= XA + iYA = BX + iBY$

Equating real and imaginary parts, we get

$XA = BX, \quad YA = BY$

Since $P \in V$, $\det P \neq 0$ (as P^{-1} exists)

$\Rightarrow \det(A + iB) \neq 0$

$\det(A + xB)$ is a polynomial of degree n in $\mathbb{C}[x]$

Since $x = i$ is not a zero, this is not identically zero and thus has at most n distinct roots in $\mathbb{C}$

Choose $x \in \mathbb{R}$ s.t. x is not a root

Then $A + xB$ is invertible $\Rightarrow$ Since $A, B \in V$, $xB \in V$ and $A + xB \in V$ $\in \mathbb{R}^{n\times n}$ $\square$