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Math 542

Assigned: 4/23/14 Due: 4/30/14

QR

Problem 55 Let $K \supseteq \mathbb{Q}$ be the splitting field of $f(x) = (x^2+1)\cdot(x^2-2) = x^4 - x^2 - 2$

(a) Prove that $\mathrm{aut}(K \mid \mathbb{Q}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

proof Let $f_1(x) = x^2+1$ and $f_2(x) = x^2 - 2$

The roots of $f_1(x)$ are $\{\pm i\}$ and the roots of $f_2(x)$ are $\{\pm\sqrt{2}\}$

Thus, $K = \mathbb{Q}(\sqrt{2}, i)$ is the splitting field of $f(x) = f_1(x)\cdot f_2(x)$

We verify $[K:\mathbb{Q}] = 4$. Since $f_2(x) \in \mathbb{Q}[x]$ is 2-Eisenstein, it is irreducible over $\mathbb{Q}$ $\Rightarrow$ $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$

$f_1(x) = x^2+1$ has two non-real roots. But, $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$.
Thus, x^2+1 is irreducible over $\mathbb{Q}(\sqrt{2})$ and $[K : \mathbb{Q}(\sqrt{2})] = 2$

$\therefore$ $[K:\mathbb{Q}] = 4$ and $|\mathrm{aut}(K \mid \mathbb{Q})| = 4$

Hence $\mathrm{aut}(K \mid \mathbb{Q}) \simeq \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$
We note that $\mathrm{aut}(K \mid \mathbb{Q})$ has two elements a and b defined below

$$a: \sqrt{2} \mapsto \sqrt{2}, \quad i \mapsto -i$$
$$b: \sqrt{2} \mapsto -\sqrt{2}, \quad i \mapsto i$$

The elements a and b are distinct elements of order 2
$\Rightarrow$ $\mathrm{aut}(K \mid \mathbb{Q}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

(b) Find an irreducible polynomial $p(x)$ whose splitting field is K.

solution Define $\alpha = i + \sqrt{2}$. Thus, $\alpha^2 = 1 + 2i\sqrt{2}$ $\Rightarrow$ $(\alpha^2 - 1)^2 = -8$

Thus, α satisfies $\boxed{f(x) = (x^2-1)^2 + 8 = x^4 - 2x^2 + 9}$ and $f(x)$ is irreducible over $\mathbb{Q}$.

By definition, the splitting field K is still $\mathbb{Q}(\sqrt{2}, i)$.