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Math 542 4/21 HW

OK

34 $\sigma, \tau \in A_5$ are conjugate in A_5 iff $\exists\ \rho \in A_5$ such that $\sigma = \rho^{-1}\tau\rho$. Prove that every element of A_5 except the identity is conjugate to one of the following

(a) $(1,2,3)$
(b) $(1,2)(3,4)$
(c) $(1,2,3,4,5)$
(d) $(2,1,3,4,5)$

3-cycles conjugate to $(1,2,3) \doteq \sigma$:

Let $\tau = (a_1, a_2, a_3)$ be a 3-cycle in A_5 which fixes a_4 and a_5.

Define $\rho_1, \rho_2 \in S_5$ by $\rho_1(i) = a_i$ and $\rho_2(i) = \begin{cases} a_i & \text{if } i=1,2,3 \\ a_4 & \text{if } i=5 \\ a_5 & \text{if } i=4 \end{cases}$. Then $\sigma = \rho_1^{-1}\tau\rho_1 = \rho_2^{-1}\tau\rho_2$. The counting numbers The crossing numbers $\#\rho_1$ and $\#\rho_2$ differ by 1 since ρ_2 has one additional crossing added, so one is even mod 2 and thus in A_5.

Disjoint 2-cycles conjugate to $(1,2)(3,4) \doteq \sigma$:

Let $\tau = (a_1, a_2)(a_3, a_4)$ be a disjoint 2-cycle in A_5 which fixes a_5. Define $\rho_1, \rho_2 \in S_5$ by $\rho_1(i) = a_i$ and $\rho_2(i) = \begin{cases} a_2 & \text{if } i=1 \\ a_1 & \text{if } i=2 \\ a_i & \text{if } i \notin \{1,2\} \end{cases}$. Then $\sigma = \rho_1^{-1}\tau\rho_1 = \rho_2^{-1}\tau\rho_2$, and as before $\#\rho_2 - \#\rho_1 = 1$, so one of ρ_1 or ρ_2 is in A_5.

5-cycles

First note $(1,2)(2,1,3,4,5)(1,2) = (1,2,3,4,5)$.
All 5-cycles are conjugate in S_5, so for any 5-cycle σ $\exists\ \rho$ such that $\rho^{-1}(1,2,3,4,5)\rho = \sigma$. If $\#\rho = 0 \mod 2$, σ is conjugate to $(1,2,3,4,5)$ in A_5. If not, let $\rho' = (1,2)\rho$. Then $(\rho')^{-1} = \rho^{-1}(1,2)$ and

$$\begin{aligned}(\rho')^{-1}(2,1,3,4,5)\rho' &= \rho^{-1}(1,2)(2,1,3,4,5)(1,2)\rho \\ &= \rho^{-1}(1,2,3,4,5)\rho \\ &= \sigma\end{aligned}$$

So σ is conjugate to $(2,1,3,4,5)$ since $\#\rho' = \#\rho = 0 \mod 2$ if $\#\rho = 1 \mod 2$.

continued on back

So every 5-cycle in A_5 is conjugate to $(1,2,3,4,5)$ or $(2,1,3,4,5)$ (or both). To show an element cannot be conjugate to both, it suffices to show $\sigma = (1,2,3,4,5)$ and $\tau = (2,1,3,4,5)$ are not conjugate.

Suppose $\exists\ \rho_0 \in S_5$ such that

$$\sigma = \rho_0^{-1} \tau \rho_0 \Rightarrow$$

$$\rho_0 \sigma = (1,2) \sigma (1,2) \rho_0$$

$$(1,2) \rho_0 \sigma = \sigma (1,2) \rho_0$$

$$\rho \sigma = \sigma \rho \text{ , where } \rho = (1,2) \rho_0$$

Let $\rho(1) = a \in \{1,2,3,4,5\}$. Then $\forall\ 1 \le i \le 4$,

$\rho(i+1) = \rho\sigma(i) = \sigma(\rho(i)) = \rho(i) + 1$

And in particular $\forall\ b \in \{1,2,3,4,5\}$ $\rho(b) - b = \rho(1) + (b-1) - b$
$= \rho(1) - 1$

So ρ shifts an element b up by a constant amount, i.e. it is either the identity or a 5-cycle since 5 is prime.

Now $\#\rho$ is even, so $\#\rho_0 = \#\rho - 1$ is odd, and thus $\rho_0 \notin A_5$. ρ_0 was arbitrary, so $(1,2,3,4,5)$ and $(2,1,3,4,5)$ are not conjugate in A_5.