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32) Compute the Galois group of the following polynomials:

a) x^5-1
b) x^4-2
c) x^4-2x^2-2

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a) the roots of x^5-1 are $e^{n2\pi i/n} = \{1, \theta_1, \theta_2, \theta_3, \theta_4\}$

Since $1 \in \mathbb{Q}$, $\sigma \in aut(K|\mathbb{Q}) \to \sigma(1) = 1$

Since $\theta_k = \theta_1^k$, each σ is completely determined by its operation on θ_1

Thus the only possible elements of $aut(K|\mathbb{Q})$ are

$$\sigma_1 = e$$
$$\sigma_2 = (1\,2\,4\,3)$$
$$\sigma_3 = (1\,3\,4\,2)$$
$$\sigma_4 = (1\,4)(2\,3)$$

Since $x^5-1 = (x-1)(x^4+x^3+x^2+1)$ and $x^4+x^3+x^2+1$ is irreducible

So K is the splitting field of $x^4+x^3+x^2+1$ and $|aut(K|\mathbb{Q})| = [K:\mathbb{Q}] = 4$

So $aut(K|\mathbb{Q}) = \{e, (1243), (1342), (14)(23)\} \simeq \mathbb{Z}_4$ ok.

b) the roots of x^4-2 are $\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i = \theta_1, \theta_2, \theta_3, \theta_4$

Since x^4-2 is irreducible we have $|aut(K|\mathbb{Q})| = [K:\mathbb{Q}] =$
$= [\mathbb{Q}[\sqrt[4]{2}, i] : \mathbb{Q}[\sqrt[4]{2}]] \cdot [\mathbb{Q}[\sqrt[4]{2}] : \mathbb{Q}] = 2 \cdot 4 = 8$

Since $\theta_1 = -\theta_2$, $\theta_3 = -\theta_4$,
This means the only possible elements of $aut(K|\mathbb{Q})$ are

e	(12)	$(13)(24)$	$(14)(23)$
(34)	$(12)(34)$	(1324)	(1423)

ok

So this must be the Galois group

c) the roots of x^4-2x^2-2 are $\sqrt{1+\sqrt{3}}, -\sqrt{1+\sqrt{3}}, \sqrt{1-\sqrt{3}}, -\sqrt{1-\sqrt{3}} = \theta_1, \theta_2, \theta_3, \theta_4$

Since x^4-2x^2-2 is irreducible, we have $|aut(K|\mathbb{Q})| = [K:\mathbb{Q}] =$
$= [\mathbb{Q}[\sqrt{3}, \sqrt{1+\sqrt{3}}] : \mathbb{Q}[\sqrt{3}]] \cdot [\mathbb{Q}[\sqrt{3}] : \mathbb{Q}] = 2 \cdot 2 = 4$

We also have $\theta_1 = -\theta_2$, $\theta_3 = -\theta_4$ as in part (b). The only collection of those elements which form a group of order 4 are

e	e	e
(12)	$(12)(34)$	$(12)(34)$
(34)	$(13)(24)$	(1324)
$(12)(34)$	$(14)(23)$	(1423)

?

c) The roots of x^4-2x^2-2 are

$$\theta_1 = \sqrt{1+\sqrt{3}}$$
$$\theta_2 = -\sqrt{1+\sqrt{3}}$$
$$\theta_3 = \sqrt{1-\sqrt{3}}$$
$$\theta_4 = -\sqrt{1-\sqrt{3}}$$

x^4-2x^2-2 is irreducible $\Rightarrow |\mathrm{aut}(K|\mathbb{Q})| = [K:\mathbb{Q}]$

$[K:\mathbb{Q}] = [\mathbb{Q}(\sqrt{1+\sqrt{3}}, \sqrt{1-\sqrt{3}}) : \mathbb{Q}] = [\mathbb{Q}[\sqrt{1+\sqrt{3}}, \sqrt{1-\sqrt{3}}] : \mathbb{Q}[\sqrt{1+\sqrt{3}}]] \cdot [\mathbb{Q}[\sqrt{1+\sqrt{3}}] : \mathbb{Q}] = 2 \cdot 4 = 8$

as in the previous problem, we have

$$\theta_1 = -\theta_2$$
$$\theta_3 = -\theta_4$$

Thus the only possible elements of $\mathrm{aut}(K|\mathbb{Q})$ are

e	(12)	$(13)(24)$	$(14)(23)$
(34)	$(12)(34)$	(1324)	(1423)

Since this set is a group of 8 elements, it must be $\mathrm{aut}(K|\mathbb{Q})$