Bk

Math 542 Exercise 32

Joe Timmerman

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Compute the Galois groups of the following polynomials:

$p(x) = x^5 - 1$:

Note that the roots of p are simply the fifth roots of unity: numbers of the form ζ^k, for $\zeta = e^{\frac{2\pi i}{5}}$ and $0 \leq k \leq 4$. Since $\zeta^0 = 1$, the identity, any automorphism of the Galois field must fix 1. Thus, there are 4 possible permutations, each induced by $\sigma_k : \zeta \mapsto \zeta^k$ for $1 \leq k \leq 4$. Note that since σ is a field homomorphism, this completely characterizes each permutation, i..e, $\sigma_k(\zeta^\ell) = \sigma_k(\prod^\ell \zeta) = \prod^\ell \sigma_k(\zeta)$. Then we have the following:

$$\sigma_2(\zeta) = \zeta^2 \implies \sigma_2^2(\zeta) = \sigma_2(\zeta^2) = \zeta^4 = \sigma_4(\zeta) \implies \sigma_2^2 = \sigma_4$$

$$\sigma_2^3(\zeta) = \sigma_2(\zeta^4) = \zeta^8 = \zeta^3 = \sigma_3(\zeta) \implies \sigma_2^3 = \sigma_3$$

$$\sigma_2^4(\zeta) = \sigma_2(\zeta^3) = \zeta = \sigma_1(\zeta) \implies \sigma_2^4 = \sigma_1$$

Thus, every element of the four-element Galois group is generated by a signal element, so the Galois group is the cyclic group of four elements, i.e., isomorphic to $\mathbb{Z}_4$.

$p(x) = x^4 - 2$:

We have $p(x) = (x^2 + \sqrt{2})(x^2 - \sqrt{2})$, so it has roots $\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}$. Letting $\alpha = \sqrt[4]{2}$, we can rewrite these roots as $\alpha, -\alpha, i\alpha, -i\alpha$. We know $G \subseteq S_4$. Further, since p is irreducible (by Eisenstein), G must be transitive, i.e., for every pair of elements, there's a permutation that swaps them (I got this from Artin, chapter 16 section 9). The only transitive subgroups of S_4 are $S_4, A_4, D_4, \mathbb{Z}_4, D_2$.

Let $F = \mathbb{Q}[\alpha, i\alpha]$. Since p is irreducible and $deg(p) = 4$, $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 4$. Further, since i is the root of a quadratic ($x^2 + 1 = 0$), we get that $[F : \mathbb{Q}] = 8 = |G|$, by theorem 10.7. Note that of the above transitive subgroups of S_4, D_4 is the only one with 8 elements, so it must be that the Galois group of p is isomorphic to D_4.

$p(x) = x^4 - 2x^2 - 2$:

The roots of p are of the form $\pm\sqrt{1 \pm \sqrt{3}}$. Let $\alpha = \sqrt{1+\sqrt{3}}, \alpha' = \sqrt{1-\sqrt{3}}$, so our roots are $\alpha, -\alpha, \alpha', -\alpha'$. Let F be the Galois field for p. Since p is irreducible and degree 4, $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 4$. We once again need i, so as in the previous part we get $[F : \mathbb{Q}] = 8 = |G|$. By the same reasoning as in the previous part (we need an eight element transitive subgroup of S_4), it must be that $G = D_4$.

Note:

I also used Sage to verify parts (b) and (c):

```
sage: G = NumberField(x^4-2, 'theta').galois_group(type="pari")
sage: H = G.group(); H
PARI group [8, -1, 3, "D(4)"] of degree 4
sage: G = NumberField(x^4 - 2*x^2 - 2, 'theta').galois_group(type="pari")
sage: H = G.group(); H
PARI group [8, -1, 3, "D(4)"] of degree 4
```