# Problem 29

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**Problem 1.** *(Fri Apr 4) Prove the following:*
*(a) Suppose $\alpha + \beta$ is algebraic over $F$, then $\alpha$ is algebraic over $F(\beta)$.*
*(b) Suppose $\alpha + \beta$ and $\alpha\beta$ are both algebraic over $F$, then $\alpha$ is algebraic over $F$.*

*Proof.* (a) Since $\alpha + \beta$ is algebraic over $F$, $\alpha + \beta$ is algebraic over $F(\beta)$. Therefore,

$$[F(\beta, \alpha + \beta) : F(\beta)] < \infty.$$

Since $F(\alpha, \beta) = F(\beta, \alpha + \beta)$, $\alpha$ is algebraic over $F(\beta)$.

(b) Since both $\alpha + \beta$ and $\alpha\beta$ are algebraic over $F$,

$$[F(\alpha + \beta, \alpha\beta) : F] < \infty.$$

Consider polynomial $f(x) = x^2 - (\alpha + \beta)x + \alpha\beta \in F(\alpha + \beta, \alpha\beta)[x]$ and extension field $F(\alpha, \beta)$. Since $\alpha, \beta$ are two root of $f$ in $F(\alpha, \beta)$, $[F(\alpha, \beta) : F(\alpha + \beta, \alpha\beta)] \leq 2$, and hence

$$[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\alpha + \beta, \alpha\beta)][F(\alpha + \beta, \alpha\beta) : F] < \infty.$$

Thus, both $\alpha$ and $\beta$ are algebraic over $F$. □